Minimal Surfaces

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1 Introduction

These notes outline De Giorgi's theory of minimal surfaces. They are based on part of a course given by Ovidiu Savin during Fall 2011. The sections on BV functions, and the regularity of Caccioppoli sets mostly follow Giusti [2] and Evans-Gariepy [1]. We give a small perturbations proof of De Giorgi's "improvement of flatness" result, a technique pioneered by Savin in [3], among other papers of his. We would like to thank Prof. Savin for his clear and insightful lectures, and acknowledge that whatever typos or errors these notes contain are solely due to the author.

Finally, most of the arguments in these notes are very geometric and translated from picture form. We urge any readers to translate back to pictures to get a stronger intuitive understanding of the results we present.

2 BV Functions

In this section we develop the technical tools necessary to understand De Giorgi's theory of minimal surfaces.

2.1 Caccioppoli Sets

Definition 1 (**Perimeter**). *Let $E \subset \mathbb{R}^n$ be a measurable set. We define the perimeter of E as*

$$P(E) = \sup_{|g| \leq 1} \int_E div(g)$$

over all vector fields $g \in C_0^1(\mathbb{R}^n)$ with $|g| \leq 1$.

Note that if E is smooth then by the divergence theorem we have

$$\int_E \mathrm{div} g = \int_{\partial E} g \cdot \nu d\mathcal{H}^{n-1}$$

where ν is the outer unit normal. In particular, we may take g to be an extension of ν to recover the usual notion of perimeter. We can also pick out the perimeter of E in some fixed open set Ω by restricting g to have compact support in Ω.

Definition 2. *Let $E \subset \mathbb{R}^n$ be a measurable set and $\Omega \subset \mathbb{R}^n$ be some open set. We define the perimeter of E in Ω to be*

$$P(E, \Omega) = \sup_{|g| \leq 1} \int_E div(g)$$

over all vector fields $g \in C_0^1(\Omega)$ with $|g| \leq 1$.

A simple consequence of this definition is that if $|E \Delta F| = 0$ then $P(E, \Omega) = P(F, \Omega)$. An extremely important property of perimeter is lower semicontinuity.

Proposition 1 (**Lower Semicontinuity**). *If $E_n \to E \in L^1_{loc}(\Omega)$ then*

$$\liminf P(E_n, \Omega) \geq P(E, \Omega).$$

Proof. For any g,

$$\int_\Omega \chi_E \mathrm{div}(g) = \lim \int_\Omega \chi_{E_n} \mathrm{div}(g) \leq \liminf P(E_n, \Omega).$$

□

The intuition is that the "wiggles" of E_n can only lose mass in the limit. For example, one could approximate the unit circle in $\mathbb{R}^2$ by very oscillatory curves contained in $B_{1+\epsilon} - B_{1-\epsilon}$, so that the lengths remain very large but the difference in L^1 between their interiors is small.

Sets with finite perimeter are called Caccioppoli sets. In general, Caccioppoli sets can behave wildly. Take for example

$$E = \cup_{i=1}^{\infty} B_{2^{-i}}(x_i)$$

where x_i is an enumeration of $\mathbb{Q}^n$. Then for each k, $E_k = \cup_{i=1}^k B_{2^{-i}}(x_i)$ are piecewise smooth and have uniformly bounded perimeter. Since $E_k \to E$ in L^1 we obtain by lower semicontinuity that

$$P(E) < C.$$

However, $|E| < C$ so $|\partial E| = \infty$. This illustrates that for rough sets, the boundary could be much larger than the perimeter.

2.2 Bounded Variation

The preceding discussion can be viewed in a more general context.

Definition 3 (BV). *Let $f \in L^1(\Omega)$. We define the total variation*

$$\int_\Omega |\nabla f| = \sup_{|g| \leq 1} \int_\Omega f \, div(g)$$

over all vector fields $g \in C_0^1(\Omega)$.

We say that f is of class BV if $\int_\Omega |\nabla f| < \infty$. Observe that for $W^{1,1}$ functions this coincides with the usual definition. However, we would like to emphasize the key differences between BV and $W^{1,1}$.

First, the derivatives of BV functions are measures, not necessarily in L^1. For example, a smooth bounded set E. Then $\chi_E \in BV$ since

$$\int |\nabla \chi_E| = P(E) < \infty$$

but any L^1 weak derivative of χ_E would have to be 0.

If we define

$$L(g) = \int_\Omega f \mathrm{div} g.$$

Then $|L(g)| \leq (\int_\Omega |\nabla f|) \|g\|_{L^\infty}$, so L extends to a bounded linear functional on $C_0(\Omega)$. By Riesz representation we conclude that

$$L(g) = \int_\Omega g \cdot d\mu$$

for some vector-valued Radon measure μ with total variation $\int_\Omega |\nabla f|$. Denoting $d\mu$ by ∇f we obtain a weak integration by parts formula. Observe that for a Caccioppoli set E, the measure $\nabla \chi_E$ is supported on ∂E by the divergence theorem.

Another way to understand this difference is that BV is closed under L^1 limits by lower semicontinuity, whereas $W^{1,1}$ is not. For example, let $E \subset \mathbb{R}^n$ be a smooth bounded set and let f_ϵ be mollifications of χ_E. Then f_ϵ converge to E in L^1 but $E \in BV$ and not $W^{1,1}$.

Definition 4. *Let $f \in L^1(\Omega)$. We define*

$$\|f\|_{BV(\Omega)} = \|f\|_{L^1} + \int_\Omega |\nabla f|.$$

Proposition 2. *$BV(\Omega)$ is a Banach space.*

Proof. We just check completeness. If $\{f_k\}$ is Cauchy in $BV(\Omega)$ then $f_k \to f$ in L^1 for some f. By lower semicontinuity,

$$\int_\Omega |\nabla(f_n - f)| \le \liminf_{m\to\infty} \int_\Omega |\nabla(f_n - f_m)|$$

and the right side converges to 0 as $n \to \infty$. $\square$

2.3 Approximation by Smooth Functions and Applications

Theorem 1 (**Approximation**). *Let $f \in BV(\Omega)$. Then there exist $f_n \in C^\infty(\Omega)$ such that $f_n \to f$ in L^1 and*

$$\int_\Omega |\nabla f_n| \to \int_\Omega |\nabla f|.$$

We don't expect to have $\int_\Omega |\nabla(f_n - f)| \to 0$ because then the only possible limits would be in $W^{1,1}$. The obvious strategy for this theorem is to mollify, but near the boundary the derivative of the mollifications become a poor approximation to $|\nabla f|$. For example if $f = \chi_{(0,1)}$ then $\int_0^1 |f'| = 0$ but $\rho_\epsilon * f$ goes from $1/2$ to 1 on $(0, \epsilon)$ and thus $\int_0^1 |(\rho_\epsilon * f)'|$ does not converge to 0.

The strategy is to instead divide Ω into rings and take mollifications of f which are more and more precise towards the boundary.

Proof. (Sketch). Let Ω_k be the points further than $1/k$ from the boundary. Since $|\nabla f|$ is a finite measure we know $\int_\Omega |\nabla f|$ is well approximated by $\int_{\Omega_M} |\nabla f|$ for M large. The differences of Ω_i for $i \ge M$ divide the rest of Ω into rings. Using a partition of unity we can write

$$f = \sum_i f_i$$

where f_i are just f cut off everywhere except for a few rings. We define an approximation

$$\tilde{f} = \sum_i \rho_{\epsilon_i} * f_i = \sum_i \tilde{f}_i$$

where ϵ_i are chosen so small that the supports of $\tilde{f}_i$ are in just a few rings each and so that f_i and the derivatives of partition cutoffs are approximated geometrically better with i.

It is easy to show that $\|\tilde{f} - f\|_{L^1}$ is as small as we like. Lower semicontinuity gives the $\ge$ direction. For the other direction apply the definition of total variation to $\tilde{f}$. $\square$

Since the BV and $W^{1,1}$ norms agree where both are defined, approximation shows that BV inherits many of the important Sobolev estimates. The first is compactness:

Theorem 2 (**Compactness**). *If $\{f_n\}$ is bounded in $BV(\Omega)$ and Ω is Lipschitz then there is an L^1-convergent subsequence.*

Proof. We can find smooth approximations $\tilde{f}_n$ to f_n with $\|\tilde{f}_n - f_n\|_{L^1} < 1/n$ and $\int_\Omega |\nabla f_n|$ uniformly bounded. By compactness in $W^{1,1}$ we have $\tilde{f}_n \to f$ in L^1. It follows easily that $f_n \to f$ in L^1. $\square$

To see why we need $\partial\Omega$ Lipschitz take for example a jagged domain with height 1 so that the distance between consecutive peaks is 2^{-k}. Let T_k be the triangle in Ω bounded by $\partial\Omega$ and the $k, k+1$ peaks. Finally, let $f_k = 2^k \chi_{T_k}$. Then $\|f_k\|_{L^1} = 1/2$ and $\int_\Omega |\nabla f_k| = 1$ for all k, so $\|f_k\|_{BV(\Omega)}$ are uniformly bounded. However, $\{f_k\}$ converges to 0 almost everywhere, so it has no L^1 convergent subsequence.

Theorem 3 (**Sobolev and Poincaré Inequalities**). *If f has compact support in $\mathbb{R}^n$ then*

$$\left(\int |f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C(n) \int |\nabla f|$$

and if $f \in BV(B_1)$ then

$$\left(\int_{B_1} |f - f_{B_1}|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C(n) \int_{B_1} |\nabla f|.$$

Proof. Take smooth approximations f_k to f. Then by the Sobolev inequality f_k are uniformly bounded in $L^{\frac{n}{n-1}}$, and thus converge weakly to f in $L^{\frac{n}{n-1}}$. We conclude that

$$\|f\|_{L^{\frac{n}{n-1}}} \leq \liminf \|f_n\|_{L^{\frac{n}{n-1}}} \leq C \lim \int |\nabla f_n| = \int |\nabla f|.$$

The Poincaré inequality is similar. $\square$

Applying these to characteristic functions gives the useful isoperimetric and Poincaré inequalities.

Theorem 4 (**Isoperimetric Inequalities**). *If E is a Caccioppoli set then*

$$|E|^{\frac{n-1}{n}} \leq C(n)P(E)$$

and

$$\min\{|E \cap B_1|, |E^c \cap B_1|\}^{\frac{n-1}{n}} \leq C(n)P(E, B_1).$$

Proof. Apply Sobolev to get the isoperimetric inequality. For the second, observe that $|\chi_E - \frac{|E \cap B_1|}{|B_1|}|$ exceeds $\frac{1}{2}$ either in $E \cap B_1$ or in $E^c \cap B_1$ (whichever one has less measure). Then apply the Poincaré inequality. $\square$

The Poincaré inequality says in a scale-invariant way that perimeter controls the smaller volume.

2.4 The Coarea Formula and Applications

Theorem 5 (Coarea Formula). *Let $f \in BV(\Omega)$ and $f \geq 0$. Then*

$$\int_\Omega |\nabla f| = \int_0^\infty P(\{f \geq t\})dt.$$

Proof. Computing

$$\begin{aligned}
\int_\Omega f \operatorname{div} g &= \int_\Omega \int_0^{f(x)} \operatorname{div} g \, dt \, dx \\
&= \int_0^\infty \int_{\{f \geq t\}} \operatorname{div} g \, dx \, dt \\
&\leq \int_0^\infty P(\{f \geq t\})dt
\end{aligned}$$

for all g gives us one direction. For the other direction, observe that

$$\int_\Omega |f_n - f| = \int_\Omega \left| \int_{f(x)}^{f_n(x)} dt \right| dx = \int_0^\infty |\{f_n \geq t\} \Delta \{f \geq t\}| dt,$$

which implies that for a subsequence we have $|\{f_n \geq t\} \Delta \{f \geq t\}| \to 0$ for almost every t. We conclude by lower semicontinuity and Fatou that

$$\begin{aligned}
\int_0^\infty P(\{f \geq t\})dt &\leq \int_0^\infty \liminf P(\{f_n \geq t\})dt \\
&\leq \liminf \int_0^\infty P(\{f_n \geq t\})dt \\
&= \lim \int_\Omega |\nabla f_n| \\
&= \int_\Omega |\nabla f|.
\end{aligned}$$

□

As a consequence we can approximate Caccioppoli sets by smooth sets.

Theorem 6. *Let E be a bounded Caccioppoli set. Then there exist smooth sets E_n converging to E in L^1 such that $P(E_n) \to P(E)$.*

The usual mollification procedure doesn't give approximation by characteristic functions, so we must modify this approach by taking level sets of mollifiers.

Proof. The first step is to approximate E with smooth functions. Let $f_\epsilon = \rho_\epsilon * \chi_E$. Then it is easy to see that $f_\epsilon \to \chi_E$ in L^1. Furthermore, we have

$$\int f_\epsilon \operatorname{div} g = \int \chi_E \operatorname{div}(\rho_\epsilon * g)$$

and $|\rho_\epsilon * g| \leq 1$. This combined with lower semicontinuity gives

$$\int |\nabla f_\epsilon| \to P(E).$$

The next step is to show that the level sets of f_ϵ approximate E in L^1. To see this, fix $0 < t < 1$. On $\{f_\epsilon \geq t\}\Delta E$ we have that $|\chi_E - f_\epsilon| > \min\{t, 1-t\}$. Chebyshev's inequality with $|\chi_E - f_\epsilon|$ gives the result.

In the third step we apply lower semicontinuity and the coarea formula to show that the perimeters of level sets approach $P(E)$. Lower semicontinuity gives

$$P(E) \leq \liminf P(\{f_\epsilon \geq t\}).$$

Integrating from 0 to 1 and applying Fatou gives

$$P(E) \leq \liminf \int_0^1 P(\{f_\epsilon \geq t\})dt = \lim \int |\nabla f_\epsilon| = P(E),$$

showing that $P(E) = \liminf P(\{f_\epsilon \geq t\})$ for almost every t.

Finally, using Sard's theorem we know that the level sets of f_ϵ are smooth for almost every t. We can thus choose some t and some subsequence $\epsilon_i \to 0$ such that $\{f_{\epsilon_i} \geq t\}$ are smooth and approximate E in the desired way.

$\square$

2.5 Traces of BV Functions

The Lebesgue differentiation theorem says that L^1 functions are well-approximated by their averages $\mathcal{H}^n$ almost everywhere. In this section we would like to show that BV functions are well-approximated $\mathcal{H}^{n-1}$ almost everywhere, and in particular that there is a good notion of boundary values for BV. In the following B_1^{n-1} denotes the unit ball in the hyperplane $\{x_n = 0\}$ and for $x' \in B_1^{n-1}$ we let $Q_r(x')$ be the cylinder $B_r^{n-1}(x') \times (0, r)$.

Theorem 7 (**Existence of Traces**). *Let $f \in BV(B_1^+)$. Then there exists a function $f^+ \in L^1(B_1^{n-1})$ such that*

$$\frac{1}{r^n}\int_{Q_r(x_0')} |f - f^+(x_0')|dx \to 0$$

for $\mathcal{H}^{n-1}$ almost every $x_0' \in B_1^{n-1}$, and

$$\int_{B_1^+} f\, div(g) = -\int_{B_1^+} g \cdot \nabla f - \int_{B_1^{n-1}} f^+ g_n d\mathcal{H}^{n-1}.$$

For a general L^1 function like $\log(x_n)$ which is not in $W^{1,1}$ there is clearly no such notion of boundary value; having a locally integrable derivative is the key.

Proof. Assume that $f \in C^\infty(B_1^+) \cap BV(B_1^+)$. The general case follows easily by approximation. For the first step we use the integrability of ∇f to show that f restricted to the slices $\{x_n = \epsilon\}$ are close in L^1. Indeed,

$$|f(x', \epsilon) - f(x', \rho)| \leq \int_0^\epsilon |\nabla f(x', t)|dt,$$

and integrating over B_1^{n-1} we obtain

$$\|f(x',\epsilon)-f(x',\rho)\|_{L^1(B_1^{n-1})} \leq \int_{B_1^+\cap\{x_n\leq\epsilon\}} |\nabla f|.$$

Since ∇f is a finite measure the right side goes to 0 with ϵ, so we conclude that $f(x',\epsilon)$ converge to a function $f^+(x')$ in $L^1(B_1^{n-1})$.

In the next step we show the first claim. Using the previous inequality we obtain

$$\begin{aligned}\frac{1}{r^n}\int_{Q_r(x_0')} |f-f^+(x_0')| &\leq \frac{1}{r^n}\left(\int_{Q_r(x_0')} |f(x',x_n)-f^+(x')| + \int_{Q_r(x_0')} |f^+(x')-f^+(x_0')|\right)\\ &\leq \frac{1}{r^{n-1}}\left(\int_{Q_r(x_0')} |\nabla f| + \int_{B_r^{n-1}(x_0')} |f^+(x')-f^+(x_0')|\right).\end{aligned}$$

The second term goes to 0 almost everywhere $\mathcal{H}^{n-1}$ by Lebesgue differentiation. For the first term we use a standard covering argument. Let

$$A_\delta = \{x' : \limsup_{r\to 0} \frac{1}{r^{n-1}}\int_{Q_r(x')} |\nabla f| > \delta\}.$$

Fix ϵ small. For each $x'\in A_\delta$ we may find a ball $B_r^{n-1}(x')$ with $r<\epsilon$ and

$$\int_{Q_r(x')} |\nabla f| > \frac{\delta}{2} r^{n-1}.$$

Take a Vitali subcover $\{B_{r_i}^{n-1}(x_i')\}$ of disjoint balls whose five-times dilations cover A_δ. We then have

$$\sum_i (5r_i)^{n-1} \leq \frac{C}{\delta}\sum_i \int_{Q_{r_i}(x_i')} |\nabla f| \leq \frac{C}{\delta}\int_{x_n\leq\epsilon} |\nabla f|.$$

Since the right side is as small as we like by taking ϵ small, we obtain that

$$\mathcal{H}^{n-1}(A_\delta)=0,$$

giving the result.

To show the second part just apply the divergence theorem on the slices $B_1^+\cap\{x_n\geq\epsilon\}$ and let $\epsilon\to 0$, using that $f(x',\epsilon)$ converge to f^+ in $L^1(B_1^{n-1})$. $\square$

We remark that any L^1 function ϕ on B_1^{n-1} is the trace of some $W^{1,1}$ function. The idea is to approximate ϕ in $L^1(B_1^{n-1})$ by smooth functions ϕ_k and define f close to $\{x_n=0\}$ by interpolating between ϕ_k. For a sequence $t_k\to 0$ let f be the linear interpolation in the x_n direction between ϕ_{k+1} and ϕ_k for $t_{k+1}\leq x_n\leq t_k$. It is clear from the above proof that if f is BV then ϕ is its trace. To verify BV note that

$$|\nabla f| \leq |\nabla_{x'} f| + |f_n| \leq |\nabla\phi_k| + |\nabla\phi_{k+1}| + |\phi_k-\phi_{k+1}|(t_k-t_{k+1})^{-1}$$

for $t_{k+1}\leq x_n\leq t_k$. It follows that

$$\int |\nabla f| \leq \sum_i \left((\|\nabla\phi_k\| + \|\nabla\phi_{k+1}\|)(t_k-t_{k+1}) + \|\phi_k-\phi_{k+1}\|\right),$$

where the norms are in $L^1(B_1^{n-1})$. Choosing a subsequence of ϕ_k so that the last term converges and t_k so that the first converges does the trick.

An important observation is that the trace sees "jumps" of a BV function f across the boundary.

Theorem 8. *If $f_1 \in BV(B_1^+)$ and $f_2 \in BV(B_1^-)$ with traces $tr(f_i)$ then the function*

$$f(x) = \begin{cases} f_1(x), & x \in B_1^+ \\ f_2(x) & x \in B_1^- \end{cases}$$

is BV and we have

$$\int_{B_1^{n-1}} |tr(f_1) - tr(f_2)| = \int_{B_1^{n-1}} |\nabla f|.$$

Proof. Note that

$$\int_{B_1} f\mathrm{div}(g) = -\int_{B_1^+} g \cdot \nabla f - \int_{B_1^-} g \cdot \nabla f + \int_{B_1^{n-1}} (tr(f_2) - tr(f_1)) g_n d\mathcal{H}^{n-1}.$$

The right side is uniformly bounded for $|g| \leq 1$, so f is BV, and furthermore the last term must be

$$-\int_{B_1^{n-1}} g \cdot \nabla f.$$

We conclude that $\nabla f = (tr(f_1) - tr(f_2)) e_n d\mathcal{H}^{n-1}$ on $\{x_n = 0\}$ and the result follows. $\square$

The preceding analysis can be carried out for Lipschitz domains Ω with analogous results, replacing $f^+(x)$ by the trace

$$\phi(x) = \lim_{r \to 0} \frac{1}{|B_r(x) \cap \Omega|} \int_{B_r(x) \cap \Omega} f$$

which exists for $\mathcal{H}^{n-1}$ almost every $x \in \partial\Omega$, and replacing $-e_n$ by the outer unit normal $\nu(x)$, i.e.

$$\int_\Omega f\mathrm{div}(g) = -\int_\Omega g \cdot \nabla f + \int_{\partial\Omega} \phi g \cdot \nu d\mathcal{H}^{n-1}.$$

To prove this one could flatten the boundary locally (lipschitz transformations change $|\nabla f|$ by at most a constant and thus preserve BV), take the resulting trace of the flat boundary, project to obtain a local trace on $\partial\Omega$ and use a partition of unity.

An especially important situation is to look at the traces of a BV function f on the balls B_r, with f^- denoting the trace of f in B_r and f^+ denoting the trace of f in $B_1 - \overline{B_r}$. Theorem 8 shows that

$$\int_{\partial B_r} |f^+ - f^-| d\mathcal{H}^{n-1} = \int_{\partial B_r} |\nabla f|,$$

and in fact that

$$\nabla f = (f^+ - f^-) \nu d\mathcal{H}^{n-1}$$

on ∂B_r, where u is the outward unit normal.

Since $|\nabla f|$ is a finite measure, $\int_{\partial B_r} |\nabla f|$ can be positive for at most countably many r (indeed, the collection of radii where it is larger than $1/k$ is finite for all k). We conclude that

$$f^+ = f^- = f$$

for almost every r. In particular, if $f = \chi_E$ for some Caccioppoli set E we see that the trace $tr(E)$ of χ_E in B_r coincides with $E \cap \partial B_r$ for almost every r.

An example to have in mind for "jumps" in trace is to take $E = \mathbb{R}^n_+ - B_1$. Then the trace of E in B_1 is 0, but the trace outside B_1 is 1 on the lower hemisphere.

3 Regularity of Caccioppoli Sets

In this section we show that the boundary of a Caccioppoli has a tangent plane at most of its boundary points.

3.1 Measure Theoretic vs Topological Boundary

When we consider a Caccioppoli set χ_E we would like to view its boundary points as those points x such that

$$0 < |E \cap B_r(x)| < |B_r(x)|$$

for all $r > 0$, i.e. the measure theoretic boundary. This is easy to arrange as follows. If $x \in E$ and $|B_r(x) \cap E| = 0$ for some $r > 0$ then discard x (notice that we discard all points in E in $B_r(x)$ as well). Likewise, if $x \in E^c$ and $|B_r(x) \cap E| = |B_r|$ for some $r > 0$ then include it in E. Both of these sets have measure 0 by Lebesgue differentiation.

Having made these modifications to get an equivalent set $\tilde{E}$, suppose that $|B_r(x) \cap E| = 0$ for some $r > 0$. Then x is in the interior of E^c by construction; a similar conclusion holds if $|B_r(x) \cap E| = |B_r|$. It easily follows that the topological boundary and measure-theoretic boundary agree for $\tilde{E}$. For the remainder of these notes we will assume we are working with this modification.

3.2 The Reduced Boundary

We now identify a subset of ∂E that has a well-defined notion of normal vector.

Definition 5 (**Reduced Boundary**). *We say a point $x_0 \in \partial E$ belongs to the reduced boundary $\partial^* E$ if*

$$\lim_{r \to 0} \frac{\int_{B_r(x)} \nabla \chi_E}{\int_{B_r} |\nabla \chi_E|} = \nu(x_0)$$

for some unit vector $\nu(x_0) \in S^{n-1}$.

To get a feel for this let's discuss a few examples. If E is a C^1 bounded set then by the trace theorems above $\nabla \chi_E = \nu d\mathcal{H}^{n-1}$ on the boundary of E. (It is easy to see by integration by parts that the measure $|\nabla \chi_E|$ is always supported on ∂E). Then the limit above is the average of ν along ∂E in small regions around x_0 which converge to $\nu(x_0)$ by continuity.

If on the other hand E has a corner, like a square, the normal directions cancel each other and we obtain in the limit a vector with $|\nu| = \frac{1}{\sqrt{2}}$. Even more extreme, if E has a cusp at 0 then the normal directions have average 0 in the limit. It is easy to see that $|\nu| \leq 1$ in general, and heuristically the closer $|\nu|$ is to 1 the flatter E is on average.

We now recall the Lebesgue-Besicovitch differentiation theorem:

Theorem 9. *If μ and ν are finite Borel measures then*

$$\lim_{r \to 0} \frac{\nu(B)r(x)}{\mu(B_r(x))}$$

exists μ almost-everywhere, and if we denote the limit by $D_\mu \nu$ then

$$d\nu = D_\mu \nu(x) d\mu + \tilde{\nu}$$

for some measure $\tilde{\nu}$ which is singular with respect to μ.

The proof is by a covering argument around the points where the limit does not exist. This theorem implies, in particular, that $\partial E - \partial^* E$ has $|\nabla \chi_E|$ measure 0.

3.3 Uniform Density Estimates

The fact that the normals don't oscillate too much around a boundary point gives a very important estimate that prevents E and E^c from disappearing under blowup (i.e. cusp-like behavior can't happen).

Proposition 3 (**Uniform Density Estimate**). *Assume that $0 \in \partial E$ and that*

$$\frac{\left|\int_{B_r} \nabla \chi_E\right|}{\int_{B_r} |\nabla \chi_E|} \geq \frac{1}{2}$$

for all $r \in (0, \rho)$. Then

$$\frac{|E \cap B_r|}{|B_r|}, \frac{|E^c \cap B_r|}{|B_r|} \geq \delta(n) > 0$$

and

$$c(n) r^{n-1} \leq \int_{B_r} |\nabla \chi_E| \leq C(n) r^{n-1}$$

for all $r \in (0, \rho)$.

The hypothesis is heuristically that the normals around 0 don't oscillate too much; the closer the ratio is to 1 the "flatter" we expect E to be on average.

Proof. The key point is that the perimeter of E in B_1 is controlled by its trace on ∂B_r. Indeed, for some direction, say e_n, we have

$$\begin{aligned}
\int_{B_r} |\nabla \chi_E| &\leq 2 \int_{B_r} \nabla \chi_E \cdot e_n \\
&= 2 \left(- \int_{B_r} \chi_E \mathrm{div}(e_n) + \int_{\partial B_r} tr(E) e_n \cdot \nu \right) \\
&\leq 2 |E \cap \partial B_r|_{\mathcal{H}^{n-1}}
\end{aligned}$$

for almost every r, using that $\mathrm{div}(e_n) = 0$ and that $tr(E) = E \cap \partial B_r$ for almost every r. In particular, this gives

$$\int_{B_r} |\nabla \chi_E| \leq C(n) r^{n-1}.$$

Let $V(r) = |E \cap B_r| = \int_0^r |E \cap \partial B_\rho|_{\mathcal{H}^{n-1}}$. Then $V'(r) = |E \cap \partial B_\rho|_{\mathcal{H}^{n-1}}$. Using the isoperimetric inequality and the control on the perimeter of E in B_r by the trace we obtain the differential inequality

$$\begin{aligned} c(n)V(r)^{\frac{n-1}{n}} &\leq P(E \cap B_r, \mathbb{R}^n) \\ &= \int_{B_r} |\nabla \chi_E| + \int_{\partial B_r} tr(E) \\ &\leq 3|E \cap \partial B_r|_{\mathcal{H}^{n-1}} \\ &= 3V'(r) \end{aligned}$$

for almost every r. We conclude that

$$\left(V^{\frac{1}{n}}\right)' > \delta(n)$$

giving a lower bound on the density of E.

The same argument works for E^c using that $\nabla E^c = -\nabla E$. (Indeed,

$$\int_\Omega \chi_{E^c} \mathrm{div}(g) = \int_\Omega (1 - \chi_E)\mathrm{div}(g) = -\int_\Omega \chi_E \mathrm{div}(g)$$

for $g \in C_0^1(\Omega)$.)

Finally, since the density of both E and its complement are large the Poincaré inequality gives

$$\int_{B_r} |\nabla \chi_E| \geq c(n) r^{n-1}.$$

□

An important remark is that the conclusion above also holds around any boundary point of a set of minimal perimeter E (defined in a later section). Indeed, we obtain competitors for E in B_r by including or removing B_r, giving control on $\int_{B_r} |\nabla \chi_E|$ by $\mathcal{H}^{n-1}(E \cap \partial B_r)$ and $\mathcal{H}^{n-1}(E^c \cap \partial B_r)$. The proof then proceeds the same way.

3.4 Blowup of the Reduced Boundary

In this section we define

$$E_t = \frac{1}{t}E = \left\{\frac{x}{t} : x \in E\right\}$$

to be the blow-up of E by factor $\frac{1}{t}$.

Theorem 10. *[**Tangent Planes at Reduced Boundary Points**] Assume that E is a Caccioppoli set, that $0 \in \partial^* E$ and that $\nu(0) = e_n$. Then the rescalings $E_t = \frac{1}{t}E$ converge in L^1 to the half-space $\{x_n \leq 0\}$ as $t \to 0$, and furthermore*

$$\int_{B_1} |\nabla \chi_{E_t}| \to |B_1^{n-1}|_{\mathcal{H}^{n-1}}.$$

This theorem says that at points in the reduced boundary, there is a nice tangent plane (L^1_{loc} convergence) and quantatively that the perimeters of rescalings go to the perimeter of the tangent plane.

Proof. For $g \in C^1_0(B_1)$ let $\tilde{g}(tx) = g(x)$. By change of variable

$$\begin{aligned}\int_{B_1} \chi_{E_t}(x)\mathrm{div}(g)(x)dx &= t\int_{B_1} \chi_E(tx)\mathrm{div}(\tilde{g})(tx)d(x)\\ &= \frac{1}{t^{n-1}}\int_{B_t} \chi_E \mathrm{div}(\tilde{g}),\end{aligned}$$

giving that

$$\int_{B_1} \chi_{E_t} = \frac{1}{t^{n-1}}\int_{B_t} \chi_E, \quad \int_{B_1} |\chi_{E_t}| = \frac{1}{t^{n-1}}\int_{B_t} |\chi_E|$$

and in particular that

$$\frac{\int_{B_r} \nabla\chi_{E_t}}{\int_{B_r} |\nabla\chi_{E_t}|} \to e_n.$$

For t small the uniform density hypothesis holds, and we have

$$\int_{B_r} |\nabla\chi_{E_t}| < C(n)$$

uniformly in t. By compactness and we can extract an L^1_{loc} convergent subsequence converging to a (by lower semicontinuity) Caccioppoli set E^*; by uniform density, $0 \in \partial E^*$ and E^* has uniform density in and outside of B_r.

By L^1_{loc} convergence it is easy to see that

$$\nabla\chi_{E_t} \rightharpoonup \nabla\chi_{E^*}$$

since

$$\int_{B_r} \nabla\chi_{E^*} \cdot g = \int_{B_r} \chi_{E^*}\mathrm{div}(g) = \lim \int_{B_r} \chi_{E_t}\mathrm{div}(t) = \lim \int_{B_r} \nabla\chi_{E_t} \cdot g$$

for any $g \in C^1_0(B_r)$.

It follows easily that as long as $\int_{\partial B_r} |\nabla\chi_{E^*}| = 0$, which holds for all but countably many r, we have

$$\int_{B_r} \nabla\chi_{E_t} \to \int_{B_r} \nabla\chi_{E^*}.$$

(The rigorous proof would involve taking cutoff functions between $B_{r\pm\epsilon}$ and B_r and taking limits). We conclude using lower semicontinuity and that 0 is on the reduced boundary that

$$\begin{aligned}\int_{B_r} |\nabla\chi_{E^*}| &\leq \liminf \int_{B_r} |\nabla\chi_{E_t}|\\ &= \lim \int_{B_r} \nabla\chi_{E_t} \cdot e_n\\ &= \int_{B_r} \nabla\chi_{E^*} \cdot e_n.\end{aligned}$$

This gives that

$$\int_{B_r} |\nabla \chi_{E^*}| = \int_{B_r} \chi_{E^*} \cdot e_n,$$

and in particular that $\nabla E^* \cdot e_i = 0$ for $i < n$, whence, E^* can be written $\{x_n \leq c\}$. (For instance, mollifying we obtain

$$\partial_i(\rho_\epsilon * E^*) = \rho_\epsilon * (\nabla E^* \cdot e_i) = 0$$

which is constant in the x' directions).

By uniform density $c = 0$. Finally, the above inequalities tell us that

$$\lim \int_{B_r} |\nabla \chi_{E_t}| = \int_{B_r} |\nabla \chi_{E^*}| = |B_1^{n-1}|_{\mathcal{H}^{n-1}}.$$

□

This theorem has some important consequences for the structure of the reduced boundary. The first is that

$$|\nabla \chi_E| = \mathcal{H}^{n-1}|_{\partial^* E}.$$

Remark 1. *This is not immediate to see, but "$\geq$" direction (up to a constant) is easy since around points in A the perimeter in small balls is close to the $\mathcal{H}^{n-1}$ measure of the tangent plane. Indeed, if $A \subset \partial^* E$ then for each $x \in A$, there is some $r_0(x)$ small enough that for $r < r_0$ we have*

$$\int_{B_r(x)} |\nabla \chi_E| \geq c|B_r^{n-1}|_{\mathcal{H}^{n-1}}.$$

(This follows from the previous theorem). We can extract a Vitali subcollection $\{B_{r_i}(x_i)\}$ of disjoint balls whose 3-times dilations cover A. Then

$$\sum_i (3r_i)^{n-1} \leq C \sum_i \int_{B_{r_i}} |\nabla \chi_E| \leq C \int_B |\nabla \chi_E|$$

where $B = \cup_i B_{r_i}(x_i)$. By taking $r_0(x)$ small enough at each x we can make the right integral as close as we like to

$$\int_A |\nabla \chi_E|$$

which gives this direction.

Another important consequence is that

$$\partial^* E \subset (\cup_i \Gamma_i) \cup N$$

where Γ_i are C^1 hypersurfaces and N has $|\nabla \chi_E|$ measure 0. A key idea in the proof of this structure theorem is the Whitney extension theorem, which is heuristically a converse to Taylor expansion: If normals ν are defined on a set A so that points in A separate from the hyperplane perpendicular to $\nu(x)$ like $o(|y - x|)$ then we can show A is the level set of a C^1 function.

(See the appendix for proofs of these consequences).

4 Existence and Compactness of Minimal Surfaces

The De Giorgi approach to a theory for minimal surfaces views them as the boundaries of Caccioppoli sets:

Definition 6. *Fix a Caccioppoli set E_0. We say that a set E has minimal perimeter in Ω if it minimizes*

$$\{P(F) : F = E_0 \text{ outside } \Omega\}.$$

The fixed set $E_0 - \Omega$ acts like "boundary values" for the Plateau problem.

Using compactness and lower semicontinuity, it is an easy matter to prove existence of sets of minimal perimeter:

Existence of Minimal Surfaces: Let $\{E_k\}$ be a minimizing sequence. The perimeters are uniformly bounded since E_0 itself is a competitor. Moreover, since Ω is bounded we know all E_k agree outside some B_r, so we may assume without loss that $|E_k|$ are bounded. By compactness we may extract an L^1 convergent subsequence converging to some set E, and the claim follows from lower semicontinuity. □

An important observation is that sets of minimal perimeter are closed under L^1 convergence:

Theorem 11 (**Compactness of Minimal Surfaces**). *If E_n are sets of minimal perimeter in B_1 converging in $L^1(B_1)$ to a set E, then E has minimal perimeter and*

$$P(E_n, B_r) \to P(E, B_r)$$

for all $r < 1$.

Proof. Let F be a compact perturbation of E, say in B_r for $r < 1$. Let

$$F_n = \begin{cases} F \text{ in } B_{r+\delta}, \\ E_n \text{ outside } B_{r+\delta} \end{cases}$$

for some small δ. Then F_n are competitors for E_n, giving

$$P(E_n, B_1) \leq P(F_n, B_1).$$

Furthermore, we have

$$P(F_n, B_1) = P(F, B_{r+\delta}) + \int_{\partial B_{r+\delta}} |tr(E) - tr(E_n)| + \int_{B_1 - \overline{B_{r+\delta}}} |\nabla \chi_{E_n}|.$$

Note that

$$\int_{B_1} |\chi_{E_n} - \chi_E| = \int_0^1 \int_{\partial B_r} |tr(E) - tr(E_n)| d\mathcal{H}^{n-1} dr,$$

and the key point is that by the L^1 convergence of E_n to E, for almost every δ we have

$$\int_{\partial B_{r+\delta}} |tr(E) - tr(E_n)| \to 0.$$

We conclude taking limits that

$$P(E, B_r) \leq \liminf P(E_n, B_r) \leq P(F, B_{r+\delta})$$

for almost every $\delta > 0$, giving minimality of E.

By lowersemicontinuity we can only lose perimeter in the limit. The fact that E_n are minimal allows us to compare with the limit and say that we in fact cannot lose any mass. Indeed, taking $F = E$ in B_r gives

$$P(E, B_r) \leq \liminf P(E_n, B_r) \leq \limsup P(E_n, B_r) \leq P(E, \overline{B_r})$$

and using that

$$\int_{\partial B_r} |\nabla \chi_E| = 0$$

for almost every r we obtain convergence of the perimeters. $\square$

5 Uniform Density Estimates

The uniform density estimates for minimal surfaces rule out "cusp-like" behavior.

Theorem 12. *If E has minimal perimeter in B_1 and $0 \in \partial E$ then*

$$|E \cap B_r|, |E^c \cap B_r| \geq c(n)r^{n-1}$$

and

$$cr^{n-1} \leq P(E, B_r) \leq Cr^{n-1}$$

for all r small.

Proof. Let $V(r) = |E \cap B_r|$. The key point is that $P(E, B_r)$ is controlled by $|E \cap \partial B_r|_{\mathcal{H}^{n-1}}$ for every r, since by removing B_r from E we obtain a competitor. Using this, along with the isoperimetric inequality (captures the change in scaling class between volume and perimeter) gives us

$$\begin{aligned} V(r)^{\frac{n-1}{n}} &\leq C(n)P(E \cap B_r, \mathbb{R}^n) \\ &\leq C(n)(P(E, B_r) + |E \cap \partial B_r|_{\mathcal{H}^{n-1}}) \\ &\leq C|E \cap \partial B_r|_{\mathcal{H}^{n-1}} \\ &= CV'(r) \end{aligned}$$

for almost every r. Rewriting this as

$$\left(V^{\frac{1}{n}}\right)'(r) \geq c$$

gives the estimate for $V(r)$. We can get a competitor by including B_r as well, which along with the previous steps gives the density of E^c. The Poincarè inequality concludes the proof. □

The uniform density estimates allow us to improve L^1 convergence of minimal surfaces to a sort of uniform convergence. Heuristically, the converging minimal surfaces can't have "spikes" into or away from the set they converge to.

Proposition 4. *If E_k have minimal perimeter in B_1 and $E_k \to E$ in $L^1(B_1)$ then ∂E_k come within an ϵ-neighborhood of ∂E in $B_{1/2}$ for any ϵ and $k(\epsilon)$ sufficiently large.*

Proof. Suppose not. Then for some ϵ_0 there is a sequence E_k with points $x_k \in \partial E_k$ a distance at least ϵ_0 from ∂E. The uniform density estimates give

$$\int_{B_1} |\chi_{E_k} - \chi_E| \geq \min\{|E_k \cap B_{\epsilon_0}(x_k)|, |E_k^c \cap B_{\epsilon_0}(x_k)|\} > c\epsilon_0^n$$

which contradicts L^1 convergence of E_k to E. □

Finally, a natural geometric question is whether the density of E, E^c in $B_r(x)$ for $x \in \partial E$ is scattered or concentrates somewhere. The following proposition says that E always contains a ball of size comparable to r in $B_r(x)$:

Proposition 5. *If E has minimal perimeter in B_1 and $0 \in \partial E$ then there exists some $\delta(n)$ small and $x, y \in B_1$ such that*

$$B_\delta(x) \subset E \cap B_1, \quad B_\delta(y) \subset E^c \cap B_1.$$

Proof. Find a Vitali collection $\{B_\delta(x_i)\}_{i=1}^{N_\delta}$ of disjoint balls centered at points in $E \cap B_1$ so that $B_{3\delta}(x_i)$ cover $E \cap B_{1/2}$. We will estimate the number N_δ of balls.

On one hand, N_δ must be large because these balls cover E, which has uniform density in B_1:

$$c < |E \cap B_{1/2}| \leq \sum_{i=1}^{N_\delta} |E \cap B_{3\delta}(x_i)| \leq C N_\delta \delta^n,$$

giving

$$N_\delta \geq c\delta^{-n}.$$

On the other hand, if each ball $B_{\delta/2}(x_i)$ contained a point in ∂E then N_δ could not be large because by minimality E has bounded perimeter in B_1, and by uniform density E would have perimeter at least $c\delta^{n-1}$ in $B_\delta(x_i)$:

$$C \geq P(E, B_1) \geq \sum_{i=1}^{N_\delta} P(E, B_\delta(x_i)) \geq c N_\delta \delta^{n-1}$$

giving

$$c\delta^{-n} \leq N_\delta \leq C\delta^{1-n}.$$

For δ small this is a contradiction, so some $B_{\delta/2}(x_i)$ must lie in $E \cap B_1$. $\square$

6 Monotonicity Formulae

Let E be a caccioppoli set in B_1, with $0 \in \partial E$. The monotonicity formula for minimal surfaces is extremely useful for studying the blow-up limits of E. To begin we define

$$\phi_E(r) = \frac{\int_{B_r} |\nabla \chi_E|}{r^{n-1}}.$$

This quantity has the following scaling invariance:

$$\phi_E(r) = \phi_{\frac{1}{r}E}(1).$$

Indeed, for $h(rx) = g(x)$ any vector field in $C_0^1(B_1)$ we have

$$\begin{aligned}\int_{B_1} \chi_{E_r}(x)\mathrm{div}(g)(x)dx &= r\int_{B_1} \chi_E(rx)\mathrm{div}(h)(rx)dx \\ &= \frac{1}{r^{n-1}}\int_{B_r} \chi_E \mathrm{div}(h)(x)dx.\end{aligned}$$

In particular, if E is invariant under dilations (if E is a cone) then $\phi_E(r)$ is constant in r. To find monotonicity formulae this is the philosophy: find a quantity invariant under the natural scaling of the problem which is constant for solutions with a certain radial homogeneity.

6.1 Monotonicity for Minimal Surfaces

The main theorem is that this quantity is monotonic:

Theorem 13 (**Monotonicity Formula**). *If E has minimal perimeter in B_1 and $0 \in \partial E$ then*

$$\phi_E(r) \nearrow \text{ in } r,$$

and furthermore ϕ_E is constant if and only if E is a cone.

Heuristically this theorem says that as we zoom close to a boundary point for a minimal surface, it looks more and more like a cone.

Before we give the rigorous proof we give the heuristic idea. Let

$$\tilde{E}(x) = \begin{cases} (1-\epsilon)E, & x \in B_{1-\epsilon} \\ \text{radial extension}, & x \in B_1 - B_{1-\epsilon}. \end{cases}$$

Notice that the connector between $\partial B_{1-\epsilon}$ and ∂B_1 for $\tilde{E}$ is cheaper than for E itself; roughly,

$$P(\tilde{E}, B_1 - B_{1-\epsilon}) = \epsilon|\partial B_1 \cap \partial E|_{\mathcal{H}^{n-2}} \leq \epsilon \int_{\partial B_1 \cap \partial E} \frac{1}{\sqrt{1-(x\cdot\nu)^2}} d\mathcal{H}^{n-1} = P(E, B_1 - B_{1-\epsilon})$$

up to order $O(\epsilon^2)$ where ν is the normal to E. However, $\tilde{E}$ is a competitor for E since they agree on ∂B_1. We conclude using minimality that

$$P(E, B_{1-\epsilon}) \leq P(\tilde{E}, B_{1-\epsilon}) = (1-\epsilon)^{n-1}P(E, B_1),$$

which is the monotonicity. Note further that the inequality becomes equality when E is radially constant, i.e. when $x \cdot \nu = 0$.

The differential version of this follows from comparing E to the radial surface with the same trace on B_r, for every r. Let $S(r)$ be the area of E in B_r. Then

$$S'(1) \geq \mathcal{H}^{n-2}(\partial E \cap B_1)$$

with equality when E is a cone. Furthermore, if $A(r)$ is the area of the surface obtained by connecting points on $\partial E \cap \partial B_1$ to 0 we have

$$(n-1)A(1) = (n-1)\int_0^1 r^{n-2}\mathcal{H}^{n-2}(\partial E \cap B_1)dr = \mathcal{H}^{n-2}(\partial E \cap B_1).$$

Since A is a competitor for E we conclude that

$$S'(1) \geq (n-1)S(1).$$

To get the rescaled version of this let $\tilde{S}(\rho) = S(r\rho)$. Then by the above inequality we have

$$S'(r) = \frac{1}{r}\tilde{S}'(1) \geq \frac{n-1}{r}\tilde{S}(1) = \frac{n-1}{r}S(r).$$

We then obtain

$$\left(\frac{S(r)}{r^{n-1}}\right)' = \frac{S'(r) - \frac{n-1}{r}S(r)}{r^n} \geq 0$$

as desired.

To prove the monotonicity formula rigorously we approximate E with smooth functions and apply the preceding observation.

Proof of Monotonicity Formula. Let f be a smooth approximation to E in B_1 and let

$$a(r) = \left|\int_{B_r} |\nabla f| - \int_{B_r} |\nabla \chi_E|\right|, \quad b(r) = \int_{\partial B_r} |tr(f) - tr(E)|.$$

We now show that if a, b are small then f is almost minimizing for its trace in B_r. Indeed, let $\tilde{f}$ be any function with $tr(\tilde{f}) = tr(f)$ on ∂B_r for $r < 1$ and extend $\tilde{f}$ to be χ_E outside B_r. Then by minimality and the coarea formula we have

$$\int_{B_1} |\nabla \chi_E| \leq \int_0^1 P(\{\tilde{f} \geq t\})dt \leq \int_{B_1} |\nabla \tilde{f}|,$$

giving

$$\int_{B_r} |\nabla \chi_E| \leq \int_{B_r} |\nabla \tilde{f}| + e(r).$$

We conclude that

$$\int_{B_r} |\nabla f| \leq \int_{B_r} |\nabla \tilde{f}| + a(r) + e(r).$$

Let $J(r) = \int_{B_r} |\nabla f|$. Let $\tilde{h}(x) = f\left(\frac{x}{|x|}\right)$ be the radial extension of f inwards from ∂B_1. It is easy to compute as in the discussion preceding this proof that

$$(n-1)\int_{B_1} |\nabla h| = \int_{\partial B_1} |\nabla_T f|,$$

where $\nabla_T f$ is the tangential component of ∇f. We conclude that

$$J'(1) = (n-1)\int_{B_1} |\nabla h(x)| + \left(\int_{\partial B_1} |\nabla f| - |\nabla_T f|\right).$$

Since h is a competitor for f, this gives

$$J'(1) \geq (n-1)J(1) + \left(\int_{\partial B_1} |\nabla f| - |\nabla_T f|\right) - C(n)(a+b)(1).$$

Using that $|\nabla f|^2 = |\nabla_T f|^2 + (\nabla f \cdot x)^2$ and that $1 - \sqrt{1-t} \geq \frac{t}{2}$ for $0 \leq t \leq 1$ we get

$$J'(1) - (n-1)J(1) \geq \frac{1}{2}\int_{\partial B_1} \frac{(\nabla f \cdot x)^2}{|\nabla f|} - C(n)(a+b)(1).$$

It is easy to see that for $\phi(r) = \frac{\int_{B_r} |\nabla f|}{r^{n-1}}$, the left side is $\phi'(1)$. Doing the rescaling as in the discussion preceding the proof we obtain that

$$\phi'(r) \geq \frac{1}{2}\int_{\partial B_r} \frac{(\nabla f \cdot x)^2}{|\nabla f| r^{n+1}} - C(n,r)(a+b)(r).$$

Finally, integrating from r_1 to r_2 we obtain via Cauchy-Schwartz that

$$\phi(r_2) - \phi(r_1) \geq \frac{\left(\int_{B_{r_2}-B_{r_1}} \frac{\nabla f \cdot x}{|x|^n}\right)^2}{\int_{B_{r_2}-B_{r_1}} \frac{|\nabla f|}{|x|^{n-1}}} - C(n,r_1,r_2)\int_{r_1}^{r_2} (a+b)(r)dr.$$

We can now take limits. If $\int_{B_1} |\nabla f_n| \to \int_{B_1} |\nabla \chi_E|$ then by lower semicontinuity

$$\int_{B_r} |\nabla f_n| \to \int_{B_r} |\nabla \chi_E|$$

for all r such that $\int_{\partial B_r} |\nabla \chi_E| = 0$, which is all but countably many. Thus, $a_n(r)$ goes to 0 for almost every r. The L^1 convergence of f_n to χ_E, together with the fact that $f_n = tr(f_n)$ and $E = tr(E)$ for almost every r implies that $b_n(r)$ goes to 0 for almost every r.

We can thus take limits above to conclude that

$$\phi_E(r_2) - \phi_E(r_1) \geq \frac{\left(\int_{B_{r_2}-B_{r_1}} \frac{\nabla \chi_E \cdot x}{|x|^n}\right)^2}{\int_{B_{r_2}-B_{r_1}} \frac{|\nabla \chi_E|}{|x|^{n-1}}}$$

for almost every r_1, r_2, and in fact for every pair by noticing that

$$|\nabla \chi_E|(B_r) = \lim |\nabla \chi_E|(B_{r_j})$$

for sequences $r_j \nearrow r$ for which the above inequality holds.

It is clear from this inequality that ϕ_E is constant only for radially constant E (i.e. $\nabla \chi_E \cdot x = 0$). $\square$

6.2 Blowup Limits of Minimal Surfaces

The monotonicity formula is a powerful tool for understanding the local behavior of minimal surfaces by taking rescaling "blowup" limits:

Theorem 14 (Blowup to Minimal Cones). *Let E be a set of minimal perimeter in B_1 and $0 \in \partial E$. Then there is a sequence $t_j \to 0$ and a minimal cone C such that $0 \in \partial C$ and*

$$\frac{1}{t_j} E \to C$$

in $L^1_{loc}(\mathbb{R}^n)$.

Proof. Recall the scaling invariance of ϕ_E:

$$\phi_{\frac{1}{t}E}(r) = \phi_E(tr).$$

Monotonicity implies that for each r, $P\left(\frac{1}{t}E, B_r\right)$ is uniformly bounded for $t < 1$. By compactness and a diagonal argument there is a subsequence $\frac{1}{t_j}E$ converging in $L^1_{loc}(\mathbb{R}^n)$ to a limit C. Moreover, by compactness of minimal surfaces, C is minimal, and by uniform density estimates, $0 \in \partial C$.

It remains to show that C is a cone. By the monotonicity formula and the compactness theorem for minimal surfaces,

$$\phi_C(r) = \lim \phi_{\frac{1}{t_j}E}(r) = \lim \phi_E(t_j r) = \phi_E(0^+).$$

Since this is independent of r, we conclude that C is in fact a cone. $\square$

We remark that $\phi_E(0^+) \geq \omega_{n-1}$. Indeed, if $0 \in \partial^* E$ then we know $\phi_E(0^+) = \omega_{n-1}$, and by monotonicity $\phi_E(r) \geq \omega_{n-1} r^{n-1}$ for all $r > 0$. If not, take a sequence $x_k \in \partial^* E$ converging to 0. We have for every ϵ that

$$\int_{B_{r+\epsilon}} |\nabla \chi_E| \geq \limsup \int_{B_r(x_k)} |\nabla \chi_E| \geq \omega_{n-1} r^{n-1}.$$

In particular, for every r such that $\int_{\partial B_r} |\nabla \chi_E| = 0$ (all but countably many) we have

$$\phi_E(r) \geq \omega_{n-1}.$$

From the other direction, by taking the competitors $E \cup B_r$ and $E - B_r$ we see

$$\phi_E(0^+) \leq \frac{1}{2} \mathcal{H}^{n-1}(\partial B_1).$$

See the appendix for further discussion of important monotonicity formulae, especially Almgren's monotonicity for harmonic functions.

7 Improvement of Flatness

Our goal in this section is to prove the following regularity theorem of De Giorgi:

Theorem 15 (**Regularity of Flat Minimal Surfaces**). *Assume that E has minimal perimeter in B_1. There exists some $\epsilon_0(n)$ small such that if*

$$\partial E \subset \{|x_n| < \epsilon_0\}$$

then ∂E is an analytic hypersurface in $B_{1/2}$.

If we blowup to a half-space (for example at points in $\partial^* E$) then by the uniform density estimates, once we zoom in enough we satisfy the hypotheses of De Giorgi's theorem, so around such points minimal surfaces are smooth. We break the proof into several steps.

7.1 ABP Estimate

The following is the key estimate that carries pointwise information on ∂E (balancing of principal curvatures) to information in measure.

Proposition 6. *Assume E has minimal perimeter in B_1. Slide the balls $B_r(x', t)$ from below ∂E for $x' \in A' \subset B_1'$ and assume they touch ∂E at points $B \subset B_1$. Let $\Pi_{e_n}(B)$ be the projection of B onto B_1'. Then*

$$|\Pi_{e_n}(B)|_{\mathcal{H}^{n-1}} \geq c|A'|_{\mathcal{H}^{n-1}}.$$

Heuristically, ∂E cannot have "sharp corners" which would correspond to lots of centers with few touching points.

Proof. For points $x \in B$ let y be a corresponding center, so that

$$y(x) = x - r\nu$$

where ν is the outward unit normal (imagine the balls we slide are in E). The key point is that at x the principal curvatures are at least $-\frac{1}{r}$, so by solving the minimal surface equation in the viscosity sense we know the principal curvatures at touching points are at most $\frac{C}{r}$. In particular, since

$$D_x y = I + rII$$

where II is the second fundamental form, we have

$$\det(D_x y) \leq C$$

and infinitesimally $dx \geq c dy$ in $\mathcal{H}^{n-1}$. Since the y form a parallel surface to B, this remains true under projection. $\square$

7.2 Harnack Inequality

We begin with the following measure estimate, which when applied at all scales gives the Harnack inequality.

Lemma 1. *If E has minimal perimeter in B_1 and $\partial E \subset \{x_n \geq 0\}$, there exist $\epsilon, c(n)$ such that if $\epsilon e_n \in \partial E$ then $\partial E \cap \{x_n \leq 1\}$ projects to a set of $\mathcal{H}^{n-1}$ measure at least c_0 in $B'_{3/4}$.*

Proof. By the ABP estimate, we must just show that balls slid from below with centers in $B_{1/2}$ touch ∂E at points where $x_n \leq 1$. To that end construct the barrier

$$\varphi(x) = \begin{cases} |x|^{-\alpha} - 1, & x \in B_1 - B_{1/4} \\ 0 \text{ outside}, & 2^\alpha - 1 \text{ inside}. \end{cases}$$

For α large, φ is strictly subharmonic in $B'_1 - B'_{1/4}$ and flat outside. We can thus take $\epsilon << 1$ small so that the graph Φ_ϵ of $\epsilon\varphi$ has nonnegative (upward) mean curvature.

Slide balls of radius $1/4$ centered at points x with $x' \in B'_{1/2}$ upward until they touch ∂E. If one with center x_0 touches at a point y with $y_n > 1$, then we know ∂E lies above $x_n = 1/2$ in $B'_{1/4}(x'_0) \subset B'_{3/4}$. But then, if we lifted the graph Φ_ϵ translated so it is centered at x'_0 from below until it touches ∂E, it must touch outside of $B'_{1/4}(x'_0)$ (using construction of φ and that $\partial E \subset \{x_n \geq 0\}$) which contradicts minimality of ∂E. $\square$

By using variants of this measure estimate and covering arguments, one can show:

Proposition 7. *If E has minimal perimeter in B_1 and $\partial E \subset \{x_n \geq 0\}$ then for every δ there are some $\epsilon(\delta), C(\delta)$ so that if $\epsilon < \epsilon(\delta)$ and $\epsilon e_n \in \partial E$ then*

$$|\Pi_{e_n}(\partial E \cap \{x_n \leq C\epsilon\})|_{\mathcal{H}^{n-1}} \geq (1-\delta)|B'_1|_{\mathcal{H}^{n-1}}.$$

Notice that in the preceding arguments all we used about ∂E was that it satisfies mean curvature 0 in the viscosity sense. In particular, we haven't used that ∂E is minimizing. We use this in a relatively weak way (i.e. avoiding the monotonicity formula) in the following argument to prove the Harnack inequality.

Theorem 16. *There is some $\epsilon_0(n)$ and $\eta(n)$ small such that if*

$$\partial E \cap B_1 \subset \{|x_n| < \epsilon_0\}$$

then up to a translation in the x_n direction

$$\partial E \cap B_{1/2} \subset \{|x_n| < (1-\eta)\epsilon_0\}.$$

The idea is that if ∂E is close to both the top and bottom at points in $B_{1/2}$ then it must have pieces with perimeter close to $|B'_1|_{\mathcal{H}^{n-1}}$ close to the top and bottom, like a very squeezed catenoid. However, this would violate that E is minimizing.

Proof. Let ϵ_0 be so small that the vertical side of the cylinder

$$Q = \{|x'| < 1\} \times (-\epsilon_0, \epsilon_0)$$

has area less than $\frac{1}{2}|B_1'|_{\mathcal{H}^{n-1}}$. If ∂E is very close (say $< \eta\epsilon_0$-close) to $x_n = -\epsilon_0$ at some point in $B'_{1/2}$, we can apply Proposition 7 to conclude that

$$|\Pi_{e_n}(\partial E \cap \{x_n < 0\})|_{\mathcal{H}^{n-1}} \geq \frac{7}{8}|B_1'|_{\mathcal{H}^{n-1}}.$$

On the other hand, if it is also close to $x_n = \epsilon_0$ in $B'_{1/2}$ we have

$$|\Pi_{e_n}(\partial E \cap \{x_n > 0\})|_{\mathcal{H}^{n-1}} \geq \frac{7}{8}|B_1'|_{\mathcal{H}^{n-1}}.$$

In particular, we must have that

$$P(E, Q) > \frac{3}{2}|B_1'|_{\mathcal{H}^{n-1}}$$

which contradicts that E is minimizing (if we include Q we get a competitor for E with less area). □

7.3 Improvement of Flatness

The Harnack inequality roughly says that we have C^α estimates for ∂E. We can improve this to $C^{1,\alpha}$ regularity via a compactness/small perturbations argument. The philosophy is that if an equation is translation-invariant and has a Harnack inequality, we should get derivative estimates (by comparing a solution to translations of itself).

Theorem 17. *There is some $\epsilon_0(n)$ small such that if $\epsilon < \epsilon_0$ and*

$$\partial E \subset \{|x_n| < \epsilon\}$$

then

$$\partial E_\eta \subset \{|x \cdot \nu| < \epsilon/2\}$$

for some η small universal and $|\nu| = 1$.

This theorem says that if E is trapped by planes ϵ apart in B_1, then by zooming in a universal amount we can trap it by (possibly rotated) planes $\epsilon/2$ apart, i.e. the flatness is improved. If we look at the normals ν_k from the theorem at each stage, note that

$$|\nu_{k+1} - \nu_k| \leq \frac{\epsilon}{\eta}2^{-k}.$$

Thus, the normals converge to some ν^*. Up to replacing ν_k by ν^* at each stage with k large, we see that ∂E is locally a $C^{1,\alpha}$ graph. We can then bootstrap to obtain C^∞ estimates for ∂E since the linearized equation has Hölder coefficients.

We proceed with the proof of Theorem 17.

Proof. Assume there are $\epsilon_k \to 0$ and E_k such that the hypotheses hold but the conclusion fails for the E_k. Let A_k be the $\frac{1}{\epsilon_k}$-dilations of E_k in the e_n direction. Let ϵ_1 be from the Harnack inequality. Up to taking a subsequence we may assume that $\epsilon_k < 2^{-k}\epsilon_1$, so that

$$osc_{B_r} A_k < Cr^\alpha$$

for $r > 2^{-k}$ (oscillation in the e_n direction around any point in $B'_{1/2}$). By Arzela-Ascoli we may extract a subsequence so that ∂A_k converge uniformly to some C^α graph w in the e_n direction.

We claim that w is harmonic. Heuristically, if $\partial A_k = u_k$ were graphs then $\epsilon_k u_k$ would solve

$$0 = div\left(\frac{\nabla \epsilon_k u_k}{\sqrt{1+\epsilon_k^2|\nabla u_k|^2}}\right) = \epsilon_k \Delta u_k + O(\epsilon_k^3),$$

so in the limit we expect the solutions to be harmonic. (The linearization of MSE is Laplace when the solution is very flat).

Do do this rigorously compare w with quadratic polynomials P. If P touches w by below at x, then it touches ∂A_k by below at a nearby point and at this point we have

$$\Delta P + O(\epsilon_k^2) \leq 0,$$

giving the desired result in the viscosity sense. By solving the Dirichlet problem we see that viscosity harmonic functions are classical.

Finally, we conclude that

$$|w - L|_{B_r} < Cr^2$$

for the tangent L to w at 0, so w is as flat as we like under linear rescaling. In particular, for $r = \eta$ small we have

$$|w - L|_{B_\eta} < \frac{\eta}{4}.$$

Using uniform convergence of ∂A_k to w we obtain that

$$|\partial E_k - \epsilon_k L|_{B_\eta} < \eta\frac{\epsilon_k}{2}$$

for k large, a contradiction. $\square$

8 Minimal Cones

8.1 Energy Gap

One important consequence of the improvement of flatness is the energy gap for minimal cones. The improvement of flatness result says that if a minimal surface ∂E is close enough to flat, then it is regular. This implies that if a minimal cone C is sufficiently flat, it must be a half-plane. A sufficient condition for C to be very flat is for $\Phi_C(1)$ to be close to ω_{n-1}:

Lemma 2. *If C is a minimal cone and $\Phi_C(1) = \omega_{n-1}$ then C is a half-space.*

Proof. Take a regular point on ∂C, say x_0. Monotonicity says that $\Phi_{C,x_0}(r)$ decreases to ω_{n-1}. We claim it is in fact constant. To see this, blow down around that point. The blow-downs converge to C itself and by the compactness theorem, the perimeters of the blow-downs in B_1 approach ω_{n-1}. Monotonicity then implies that C is a cone around x_0, and since x_0 was regular C is a half-space. $\square$

We conclude by compactness that if a tangent cone C is not regular, then

$$\Phi_C(1) \geq \omega_{n-1} + \delta_0$$

for some universal δ_0.

8.2 First and Second Variation of Area

The obvious question to ask, knowing that points in ∂E blow up to minimal cones, is whether minimal cones are in fact half-spaces. De Giorgi's improvement of flatness gives regularity in a neighborhood of such points.

Minimizing cones are stable under small perturbations. Thus, the first step in the investigation is to get the variations of area. Given a surface S, make the small perturbation $S + \epsilon\phi\nu(x)$ for $\phi \in C_0^\infty(S)$ and normal ν. We expand the area of the perturbation in ϵ.

Note that if we have a curve Γ with curvature κ at a point, then the circle agreeing with Γ to second order has radius $r = \frac{1}{\kappa}$, and it is easy to see that perturbing by ϵ in the normal direction corresponds to infinitesimal area change

$$\frac{r-\epsilon}{r} = 1 - \epsilon\kappa.$$

More generally, the change of area infinitesimally is

$$\Pi_{i=1}^n (1 - \epsilon\kappa_i)$$

where κ_i are the principal curvatures.

However, ϕ is not necessarily constant, and generally the area infinitesimally stretches by including the factor $\sqrt{1+\epsilon^2|\nabla\phi|^2}$. The n dimensional change in area is thus

$$\sqrt{1+\epsilon^2|\nabla\phi|^2}\,\Pi_{i=1}^n (1 - \epsilon\phi\kappa_i).$$

Expanding, we obtain
$$1 - \epsilon\phi H + \frac{\epsilon^2}{2}\left(|\nabla\phi|^2 - c^2\phi^2\right) + O(\epsilon^3)$$
where H is the mean curvature and $c^2 = \sum_k \kappa_i^2$. (The c^2 term comes from taking the quadratic contribution from the second term in the product, which involves $\sum_{i<j}\kappa_i\kappa_j = \frac{1}{2}(H^2 - c^2)$). We have assumed that S is minimal, which from the first variation means $H = 0$.

We say that S is stable if it is minimal and small perturbations locally increase area, which means
$$\int_S (|\nabla\phi|^2 - c^2\phi^2) \geq 0$$
for all test functions ϕ. (Note that the area will always decrease if we took ϕ constant).

8.3 Bernstein-Type Results

A fundamental and useful fact in $\mathbb{R}^2$ is that
$$\inf\{\int_{B_1^c} |\nabla\phi|^2 : \phi \in C_0^1(\mathbb{R}^2), \phi|_{\partial B_1} = 1\} = 0,$$
a fact which is false in higher dimensions. This comes from the unboundedness of the fundamental solution in $\mathbb{R}^2$. Indeed, if we take $\phi_\epsilon(x) = 1 - \epsilon\log(x)$ cut off at 1 in B_1 and 0 when it hits 0, the angle it hits ∂B_1 goes to 0, so
$$\int |\nabla\phi_\epsilon|^2 = \int_{\partial B_1} (\phi_\epsilon)_\nu \to 0.$$
In higher dimensions, if ϕ is supported in B_R then the Dirichlet energy is at least that of the harmonic function which is 1 on ∂B_1 and 0 on ∂B_R, which is asymptotically the fundamental solution $|x|^{2-n}$ which has positive Dirichlet energy.

This fact combined with the second variation inequality immediately gives that global 2-dimensional minimal surfaces are planes, since we would have
$$\int_{B_1} c^2$$
as small as we like.

The same result is true for dimensions up to $n = 7$, a result obtained by making a perturbation proportional to c itself and applying an inequality of Simons on the Laplacian of c^2:
$$\frac{1}{2}\Delta c^2 \geq -c^4 + |\nabla c|^2 + 2\frac{c^2}{|x|^2}.$$
Note that since C is a (minimal) cone and the principal curvatures are like inverse of radius, c^2 has the homogeneity
$$c^2(x) = \lambda^2 c^2(\lambda x)$$
and the above inequality respects this rescaling.

Let $\phi = c\psi$ be the test function. We compute

$$0 \leq \int_{\partial C} (|\nabla\phi|^2 - c^2\phi^2) = \int_{\partial C} \left(|\nabla c|^2\psi^2 + c^2|\nabla\psi|^2 + \frac{1}{2}\nabla c^2 \cdot \nabla\psi^2 - c^4\psi^2 \right).$$

Integrating by parts and applying Simons inequality we have

$$\int_{\partial C} \frac{1}{2}\nabla c^2 \cdot \nabla\psi^2 \leq \int_{\partial C} \left(c^4 - |\nabla c|^2 - 2\frac{c^2}{|x|^2} \right) \psi^2,$$

and substituting we obtain

$$0 \leq \int_{\partial C} c^2 \left(|\nabla\psi|^2 - 2\frac{\psi^2}{|x|^2} \right).$$

As before, we choose ψ to be constant on B_1. Extend it to $|x|^{-\alpha}$ outside. The inequality becomes

$$0 \leq \int_{\partial C} c^2 r^{-2\alpha-2}(\alpha^2 - 2).$$

To get that C is flat we want $\alpha^2 \leq 2$, but for this to make sense we need the integrand to decay faster than r^{n-1}. By the homogeneity of c^2 it decays like $-2\alpha - 4$, so we want

$$\alpha^2 \leq 2, \quad n < 2\alpha + 5.$$

The largest dimension for which this is possible comes from taking $\alpha = \sqrt{2}$, giving

$$n \leq 7$$

as claimed. (Also, we integrate the term involving ψ^2 near 0, so we want r^{-2} to be locally integrable, i.e. we need $n > 3$, but the case of $2 - D$ surfaces is discussed above).

This is in fact sharp (see the discussion of the important Simons cone in dimension 8, below).

8.4 Singular Set

The discussion on stability tells us there are no nontrivial minimal cones in dimensions $n \leq 7$. Using this and the following dimension reduction result it is not hard to show the the singular set has Hausdorff dimension at most $n - 8$:

Proposition 8 (**Dimension Reduction**). *If C is a minimal cone in $\mathbb{R}^n$ which is not smooth outside the origin, we can find a nontrivial minimal cone in $\mathbb{R}^{n-1}$.*

The idea of the proof is that if e_n is a singular point of ∂C, we can blow up at e_n to obtain a cone

$$K \times \mathbb{R}$$

(using that C is a cone) where K is minimal in $\mathbb{R}^{n-1}$.

From here, the idea goes as follows. We first show the result for minimal cones. The Bernstein result gives that minimal cones in dimension 8 are smooth outside the origin. We proceed inductively. In dimension 9 we must show that if Σ is the singular set for a minimal cone C,

$$\mathcal{H}^s(\Sigma \cap \partial B_1) = 0$$

for any $s > 0$. By compactness and dimension reduction, for any ϵ, if $x_0 \in \Sigma \cap \partial B_1$ we can dilate around x_0 by $\frac{1}{\delta(x_0,\epsilon)}$ for δ so small that

$$\sum_i r_i^s < \epsilon$$

for a covering of the singular set in $B_1(x_0) \cap \partial B_{1/\delta}$. Taking $\epsilon = \frac{1}{2}$ and scaling back we have

$$\sum_i r_i^s < \frac{1}{2}\delta^s.$$

If we could choose δ uniformly large for all $x \in \Sigma \cap \partial B_1$, we can repeat this procedure in the smaller balls B_{r_i} and iterate k times to get

$$\sum_k \tilde{r}_k^s < 2^{-k}\delta^s$$

in $B_\delta(x)$ for each x, giving the desired result. In general look at the points where $\delta(x, 1/2) > \frac{1}{k}$ for each k and apply the above procedure. In general we obtain

$$\mathcal{H}^s(\Sigma) = 0$$

for all $s > n - 8$ in dimension n.

For a general minimal surface one uses compactness and proceeds similarly.

8.5 The Simons Cone

Bombieri, De Giorgi and Giusti showed that the cone

$$C = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

is locally minimizing in $\mathbb{R}^8$.

Let $x = (x{\cdot}1, ..., x_n)$ and $y = (y_1, ..., y_n)$, and denote $(x, y) \in \mathbb{R}^{2n}$. The reason the cone $|x|^2 = |y|^2$ is a good candidtate for a minimal cone is that is mean curvature away from 0 is easily seen to by 0. Indeed, if we view the surface from either "side" then it looks the exact same, and since the curvature changes sign when we flip sides it must be 0.

If one would rather compute explicitly, C is the 0 level set of

$$f = \frac{1}{2}(|x|^2 - |y|^2).$$

For a level set the normal is

$$\nu = \frac{\nabla f}{|\nabla f|},$$

so the second fundamental form is

$$II = -D_{x,y}\nu = -\frac{D^2 f}{|\nabla f|}$$

when restricted to the tangent vectors of $\{f = 0\}$. In particular,

$$-H = \Delta f - f_{\nu\nu}.$$

For our specific f it is easy to compute $D^2 f = I_x - I_y = M$ giving

$$\Delta f = 0.$$

Furthermore, for $(x, y) \in C$ we have $\nabla f(x, y) = (x, -y)$ and hence

$$f_{\nu\nu} = (x, -y)^T \cdot M \cdot (x, -y) = |x|^2 - |y|^2 = 0.$$

It is easy to see that minimal cones in (ambient) $\mathbb{R}^2$ are lines, since it would be cheaper to cut the cone off near 0 and connect the pieces with a line segment. Similarly, in (ambient) $\mathbb{R}^3$ minimal cones are planes since in the radial direction they have curvature 0, so their trace on the unit sphere must be a great circle to avoid picking up curvature tangentially. Philosophically, $2n = 8$ is the first dimension where it is cheaper to pass through the origin rather than cut off and connect.

The proof that the Simons cone is minimizing is based on the strong maximum principle, which holds under very general hypotheses (see notes on basic elliptic PDE): If u_t are solutions to an elliptic PDE

$$F(D^2 u_t, Du_t) = 0$$

then they can for no t touch a solution u from one side without agreeing with u exactly. Roughly, the difference satisfies a linear equation, and it is easy to write down subsolutions to this which intersect horizontal planes at a positive angle (Hopf-type barriers).

The key step is to construct a family of minimal surfaces that foliate the regions that ∂C disconnects and approach ∂C. Then heuristically if we solved the minimal surface equation with boundary data on ∂C (solving it in one lower dimension) and it didn't agree with C one of the foliating minimal surfaces would touch it from one side, giving a contradiction.

Let $s = |x|$ and $t = |y|$. An ODE approach to this problem would be to look for some surface $t = f(s)$ that is asymptotic to the Simons cone, with $f(s) < s$, and mean curvature in the direction outside of $\{t < f(s)\}$ is positive. The mean curvature is a second-order differential expression in f that is not hard to compute. The principal curvature in the direction tangential to the surface as s moves is $\kappa = f''/(1+f'^2)^{3/2}$. The sphere tangent to the surface from inside $\{t < f(s)\}$ centered on $t = 0$ has radius d_s satisfying $f/d_s = 1/(1+f'^2)^{1/2}$ by similar triangles, and the sphere tangent on the other side centered on $s = 0$ has radius d_t satisfying $d_t/s = d_s/(ff')$. We thus want mean curvature

$$H = \kappa + (n-1)(1/d_t - 1/d_s) > 0,$$

giving the differential inequality

$$\frac{f''}{1+f'^2} > (n-1)\left(\frac{1}{f} - \frac{f'}{s}\right).$$

It turns out that we can make this work only if $n \geq 4$. A heuristic for this is to look as the functions $f = s - s^{-\alpha}$ for s large and $\alpha > 0$. Computing both sides and Taylor expanding, this gives the condition

$$\alpha(\alpha+1) < 2(n-1)(\alpha-1).$$

The line tangent to $x(x+1)$ of the form $a(x-1)$ has slope $a = 3+2\sqrt{2}$, giving the condition

$$n > 5/2 + \sqrt{2},$$

so in particular $n \geq 4$. The solutions for $n = 2, 3$ to the ODE with zero rhs probably oscillate around $f(s) = s$, this would be interesting to check.

For the rigorous proof let

$$u = (|x|^2 - |y|^2)(|x|^2 + |y|^2).$$

Then the level surfaces of u lie in the regions separated by ∂C and approach ∂C asymptotically. It is easy to compute

$$\nabla u = 4(|x|^2 x, |y|^2 y)$$

and

$$D^2 u = 4(|x|^2 I_x + 2x \otimes x - |y|^2 I_y - 2y \otimes y).$$

We conclude that

$$\Delta u = (4n+8)(|x|^2 - |y|^2)$$

and since

$$\nu = \frac{(|x|^2 x, |y|^2 y)}{|x|^6 + |y|^6}$$

we compute

$$u_{\nu\nu} = 12 \frac{|x|^8 - |y|^8}{|x|^6 + |y|^6}.$$

Hence,

$$-H = 4(|x|^2 - |y|^2)\left(n + 2 - 3\frac{|x|^4 + |y|^4}{|x|^4 - |x|^2|y|^2 + |y|^4}\right).$$

Since

$$|x|^4 + |y|^4 - |x|^2|y|^2 \geq \frac{1}{2}\left(|x|^4 + |y|^4\right)$$

we conclude that $H < (>) 0$ in $\{|x|^2 > (<) |y|^2\}$ if

$$n + 2 - 6 = n - 4 \geq 0.$$

Thus, if

$$n \geq 4$$

we obtain a family of sub-minimal surfaces in C that hug ∂C closer and closer. One can conclude from this that given any perturbation of C into itself we can cut out chunks closer and closer to ∂C to obtain sets with smaller perimeter until we get C itself. A similar argument in the exterior of C shows that the Simons cone is minimizing in $\mathbb{R}^{2n}$ with $n \geq 4$.

Remark 2. *Understanding minimal cones is closely connected to the Bernstein problem: Are all globally defined minimal graphs in $\mathbb{R}^{n-1} \times \mathbb{R}$ necessarily planar?*

The answer is yes for $n \leq 8$, and no for $n \geq 9$. The idea is as follows. If we have linear growth then taking a blow-down by linearly rescaling and using the interior gradient estimate one concludes that the graph is a plane. If not, then the blow-down will converge to $\pm\infty$ in regions cut out by a conical shape in $\mathbb{R}^{n-1}$, which is minimal. (Think for example we take the saddle-type function $|x|^2 - |y|^2$ and linearly blow down, obtaining a function which is ∞ in the Simons cone and $-\infty$ in the exterior).

This is ruled out for $n \leq 8$ by Simons theorem. Bombieri, De Giorgi and Giusti constructed a global minimal graph in $\mathbb{R}^9 = \mathbb{R}^8 \times \mathbb{R}$ which blows up cubically towards infinity.

9 Appendix

9.1 Whitney Extension Theorem

The Whitney extension theorem says that if a function has Taylor expansions around points in some closed set, we can extend it to have the prescribed values and derivatives on this set:

Theorem 18 (Whitney Extension Theorem). *If $A \subset \mathbb{R}^n$ is closed and there is a continuous map $\nu : A \to S^{n-1}$ such that*

$$\nu(x) \cdot \frac{y-x}{|y-x|} \to 0$$

as $y \to x$ for $x, y \in A$ uniformly in A, then we can construct a C^1 function f such that

$$f|_A = 0, \quad \nabla f|_A = \nu.$$

In particular, since f is C^1 we know $\nabla f \neq 0$ in a neighborhood of A, so A is contained in a C^1 hypersurface.

Proof. The first step is to get a partition of unity near A which captures the geometry of A nearby A. Cover $\mathbb{R}^n - A$ with balls which are separated from A, say if x has distance $r(x)$ from A take $B_{r/20}(x)$, and take a Vitali five times subcover. The point of doing this is that the ten-times dilations overlap a bit, but uniformly finitely many times. Indeed, for any x, the number of x_i such that $x \in B_{10r_i}(x_i)$ is uniformly bounded, since the r_i will all be comparable to $r(x)$.

We can now take cutoff functions u_i supported in $B_{10r_i}(x_i)$ so that $\sum_i u_i$ is well-defined and strictly positive with derivative at x going like $\frac{1}{r(x)}$, and let

$$\varphi_i(x) = \frac{u_i(x)}{\sum_k u_k(x)}.$$

It is easy to verify that

$$\sum_i \varphi_i = 1, \quad \sum_i \nabla\varphi_i = 0, \quad \sum_i |\nabla\varphi_i| \leq \frac{C}{r(x)}.$$

We now define f as some weighted average of linear functions with slope ν at points nearby x in A. Let y_j be points where the balls around x_j first touch A when we expand them out, and let

$$f(x) = \sum_i \varphi_i(x)\nu(y_i) \cdot (x - y_i).$$

We verify that f satisfies the desired properties. Let $x_0 \in A$ and let x be close by. We compute

$$\begin{aligned} |f(x) - \nu(x_0) \cdot (x - x_0)| &\leq \sum_i \varphi_i |\nu(y_i) \cdot (x - y_i) - \nu(x_0) \cdot (x - x_0))| \\ &\leq C \sum_i |(\nu(y_i) - \nu(x_0)) \cdot (x - x_0)| + C \sum_i |\nu(y_i) \cdot (x_0 - y_i)|. \end{aligned}$$

It is easy to see that $|x_0 - y_i|$ has order $|x - x_0|$, so by continuity of ν for the first term and uniform closeness of nearby points on A to the perpendicular plane to ν we see that

$$|f(x) - \nu(x_0) \cdot (x - x_0)| = o(|x - x_0|),$$

proving that $\nabla f = \nu$ on A.

The final step is to show that f is C^1. It is smooth away from A and $\nabla f = \nu$ is continuous on A, so we just need to test when we have a point $x_0 \in A$ and $x \notin A$. Furthermore, to make the geometry more uniform we may assume that x_0 is the closest point in A to x since if $\tilde{x}$ is the closest point then

$$|\nabla f(x_0) - \nabla f(x)| \leq |\nabla f(x_0) - \nabla f(\tilde{x})| + |\nabla f(\tilde{x}) - \nabla f(x)|$$

and the first term goes to 0. (The point is that now $|y_j - x_0|$ have order $|x - x_0|$).

We compute using that $\sum_i \nabla\varphi_i = 0$ the following:

$$\begin{aligned}
|\nabla f(x) - \nu(x_0)| &\leq \left| \sum_i (\nabla\varphi_i)(\nu(y_i) \cdot (x - y_j)) + \varphi_i(\nu(y_i) - \nu(x_0)) \right| \\
&\leq \sum_i |\nabla\varphi_i|(|\nu(y_i) \cdot (x_0 - y_j)| + |(\nu(y_i) - \nu(x_0)) \cdot (x - x_0)|) \\
&+ C \sum_i |\nu(y_i) - \nu(x_0)|.
\end{aligned}$$

The first term goes to 0 by the hypothesis of uniform closeness of points in A to the perpendicular plane to ν and the fact that $\sum_i |\nabla\varphi_i| < \frac{C}{|x - x_0|}$, the second by the continuity of ν on A and the previous remark and the last by the continuity of ν on A. □

9.2 Structure Theorems for Caccioppoli Sets

Theorem 19. *If E is a Caccioppoli set, then*

$$\partial * E \subset (\cup_i \Gamma_i) \cup N$$

where Γ_i are C^1 hypersurfaces and N has $|\nabla\chi_E|$ measure 0.

We note that by Remark 1, N also has $\mathcal{H}^{n-1}$ measure zero. In the following, $T^+(x)$ will denote the half-space that the blowup of E around x converges to for $x \in \partial^* E$. Likewise, T^- denotes the complementary half-space.

Lemma 3. *Assume that $A \subset \partial^* E$ has the following properties:*

1. *$\nu(x)$ is continuous on A,*
2. *$\frac{|E \cap B_r(x) \cap T^-(x)|}{r^n}$, $\frac{|E^c \cap B_r(x) \cap T^+(x)|}{r^n}$ converge to 0 as $r \to 0$ uniformly for $x \in A$.*

Then A is contained in a C^1 hypersurface.

Proof. Take $\epsilon > 0$ small. Up to a dilation of E we may assume that

$$\frac{|E \cap B_r(x) \cap T^-(x)|}{r^n}, \quad \frac{|E^c \cap B_r(x) \cap T^+(x)|}{r^n} < \epsilon$$

for all $x \in A$ and $r \leq 1$. Suppose that $0 \in A$ and that $\nu(0) = e_n$. Take $y \in A \cap B_{1/2}$, with $|y| = r$. Rescaling by $\frac{1}{2r}$ we may assume that $|y| = \frac{1}{2}$.

If $y_n < 0$ then $B_{|y_n|}(y) \subset T^-(0)$, so

$$|B_{|y_n|}(y) \cap E| \leq C\epsilon.$$

However, since y is in A we know E has some uniform density in $B_{|y_n|}(y)$, i.e.

$$|E \cap B_{|y_n|}(y)| \geq |E \cap B_{|y_n|}(y) \cap T^+(y)| \geq c|y_n|^n.$$

This gives

$$|y_n| \leq C\epsilon^{1/n},$$

or scaling back, that

$$\frac{|y \cdot \nu(0)|}{r} \leq C\epsilon^{1/n}.$$

As similar argument can be used if $y_n > 0$ to obtain the same estimate. Since ϵ was arbitrary, the hypotheses of the Whitney embedding theorem are satisfied. $\square$

Proof of Structure Theorem. We identify subsets of $\partial^* E$ with uniform density properties and continuous normals. First, since the densities of interest converge to 0 everywhere $|\nabla \chi_E|$, by Egoroff there exist $B_i \subset \partial^* E$ such that

$$|\nabla \chi_E|(\partial^* E - B_i) < \frac{1}{i}$$

and the convergence is uniform on B_i. Furthermore, since $\nu(x)$ is $|\nabla \chi_E|$ measurable (by Lebesgue-Besicovitch), Lusin's theorem says there are $A_i \subset B_i$ with

$$|\nabla \chi_E|(B_i - A_i) < \frac{1}{i}$$

and ν continuous on A_i. By the lemma, A_i are contained in C^1 hypersurfaces, and since

$$|\nabla \chi_E|(\partial^* E - \cup_i A_i) \leq |\nabla \chi_E|(\partial^* E - B_M) + |\nabla \chi_E|(B_M - A_M) \leq \frac{2}{M}$$

for any M the conclusion follows. $\square$

An easy consequence of the structure theorem is that

$$|\nabla \chi_E||_{\partial^* E} = \mathcal{H}^{n-1}|_{\partial^* E}.$$

Indeed, it is clear that $\mathcal{H}^{n-1}|_{\Gamma_i \cap \partial^* E}$ agrees with $|\nabla \chi_E|$ since for $x \in \Gamma_i \cap \partial^* E$ we have

$$\frac{\mathcal{H}^{n-1}(B_r(x) \cap \Gamma_i)}{\omega_{n-1} r^{n-1}} \to 1$$

by regularity of Γ_i and

$$\frac{\int_{B_r(x)} |\nabla \chi_E|}{\omega_{n-1} r^{n-1}} \to 1$$

by Theorem 10.

Furthermore, if $B \subset \partial^* E$ then we may ignore $B \cap N$ by Remark 1, and since B may be decomposed into disjoint parts contained in Γ_i, i.e.

$$B = (B \cap \Gamma_1) \cup (B \cap \Gamma_2 - \Gamma_1) \cup ... = \cup_i B_i,$$

the preceding discussion gives that

$$\mathcal{H}^{n-1}(B) = \sum_i \mathcal{H}^{n-1}(B_i) = \sum_i |\nabla \chi_E|(B_i) = |\nabla \chi_E|(B).$$

9.3 Monotonicity Formulae for Harmonic Functions

To get a feel for monotonicity formulae, one may try to look for energy quantities for harmonic functions invariant under rescaling by a chosen homogeneity. Indeed, let

$$E(u, r) = \frac{\int_{B_r} |\nabla u|^2}{r^{n-2+2\alpha}}.$$

It is easy to check that under the rescaling

$$u_r(x) = \frac{u(rx)}{r^\alpha},$$

we have

$$E(u, r) = E(u_r, 1).$$

We perform a heuristic computation similar to the one for minimal surfaces, where we created a competitor which was radially constant (hence cheaper than E) in a thin annulus near the boundary, allowing us to compare E to a rescaling of itself on $B_{1-\epsilon}$. Now our competitor for u is

$$\tilde{u}(x) = \begin{cases} (1-\epsilon)^\alpha u\left(\frac{x}{1-\epsilon}\right), & x \in B_{1-\epsilon}, \\ |x|^\alpha u\left(\frac{x}{|x|}\right), & \text{otherwise }. \end{cases}$$

Note that $\tilde{u}$ is radially homogeneous of degree α on $B_1 - B_{1-\epsilon}$, giving $\tilde{u}_\nu = \alpha u$ on ∂B_1. Furthermore, we have by scaling invariance that

$$E(\tilde{u}, 1-\epsilon) = E(u, 1)$$

which gives heuristically that

$$E(\tilde{u}, 1) = (1-\epsilon)^{n-2+2\alpha} E(u, 1) + \epsilon \int_{\partial B_1} (|\nabla_T u|^2 + \alpha^2 u^2) + O(\epsilon^2).$$

On the other hand,

$$E(u, 1) = (1-\epsilon)^{n-2+2\alpha} E(u, 1-\epsilon) + \epsilon \int_{\partial B_1} (|\nabla_T u|^2 + u_\nu^2) + O(\epsilon^2),$$

and since $\tilde{u}$ is a competitor for u we obtain

$$E'(u,1) \geq \int_{\partial B_1} (u_\nu^2 - \alpha^2 u^2).$$

To obtain monotonicity the trick is to modify by the scale-invariant quantity

$$F(u,r) = \frac{\int_{\partial B_r} u^2}{r^{n-1+2\alpha}}.$$

We compute easily that

$$F'(u,1) = \int_{\partial B_1} 2(uu_\nu - \alpha u^2),$$

from which we see that

$$(E - \alpha F)'(u,1) = \int_{\partial B_1} (u_\nu - \alpha u)^2 \geq 0.$$

By scaling invariance of E and F we thus have

$$\begin{aligned}(E-\alpha F)'(u,r) &= \frac{1}{r}(E-\alpha F)'(u_r,1) \\ &\geq \frac{1}{r}\int_{\partial B_1}\left(\frac{ru_\nu(rx)}{r^\alpha} - \alpha\frac{u(rx)}{r^\alpha}\right)^2 d\mathcal{H}^{n-1} \\ &= \frac{1}{r^{n+2\alpha}}\int_{\partial B_r}(ru_\nu - \alpha u)^2 d\mathcal{H}^{n-1},\end{aligned}$$

and in particular the quantity $E - \alpha F$ is constant if and only if

$$u_\nu = \frac{\alpha}{r}u,$$

i.e. u is homogeneous of degree α.

The theorem is then:

Theorem 20 (**Almgren-Related Monotonicity Formulae**). *If u is harmonic in B_1 then*

$$\frac{1}{r^{n-1+2\alpha}}\left(r\int_{B_r}|\nabla u|^2 - \alpha\int_{\partial B_r} u^2\right)$$

is monotone increasing in r for each α, and is constant (and vanishes) if and only if u is homogeneous of degree α.

To prove this theorem rigorously (exercise), one may imitate the proof of the monotonicity formula for minimal surfaces by differentiating the given quantity, and control the $\int_{\partial B_r}|\nabla_T u|^2$ term by comparing with the radially homogeneous function

$$v(x) = r^\alpha u\left(\frac{x}{r}\right).$$

In particular, use the observation that

$$\int_{B_1} |\nabla u|^2 \leq \int_{B_1} |\nabla v|^2 = \frac{1}{n-2+2\alpha} \int_{\partial B_1} (|\nabla_T u|^2 + \alpha^2 u^2).$$

An important observation is that if we choose

$$\alpha = \frac{r_0 \int_{B_{r_0}} |\nabla u|^2}{\int_{\partial B_{r_0}} u^2}$$

then we have

$$(E - \alpha F)(u, r_0) = 0, \quad (E - \alpha F)'(u, r_0) \geq 0$$

which gives

$$\left(\frac{E}{F}\right)'(r_0) \geq 0$$

and we have equality if and only if u is homogeneous. In particular, we have derived the important monotonicity formula of Almgren:

Theorem 21 (**Almgren's Monotonicity Formula**). *If u is harmonic in B_1 then the quanitity*

$$J(r) = \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} \nearrow \text{ in } r,$$

and it is constant if and only if u is homogeneous.

Note that if u is homogeneous of degree α then

$$u_\nu = \frac{\alpha}{r} u,$$

so the Almgren quantity picks out the frequency/growth rate of u.

Alternatively, one may apply the divergence theorem to the vector field

$$x|\nabla u|^2 - 2(x \cdot \nabla u)\nabla u$$

to get the useful identity

$$\int_{\partial B_1} |\nabla u|^2 = \int_{\partial B_1} (2u_\nu^2 + (n-2)uu_\nu).$$

Differentiating and using this formula (exercise), one can give another proof the Almgren monotonicity.

An interesting and immediate application is an estimate by below on the growth rate of a harmonic function from a 0. Let $\alpha = J(1)$ and assume $u(0) = 0$. Then notice that quantity

$$f(r) = \frac{1}{r^{n-1}} \int_{\partial B_r} u^2$$

satisfies

$$f'(r) = \frac{2}{r^{n-1}} \int_{\partial B_r} u u_\nu = \frac{2}{r} \left(\frac{1}{r^{n-2}} \int_{B_r} |\nabla u|^2 \right)$$

which gives

$$\log(f)'(r) = \frac{2}{r} F(r) \leq \frac{2\alpha}{r}$$

by Almgren monotonicity. Integrating we obtain that

$$f(r) \geq f(1) r^{2\alpha},$$

so heuristically u grows at least as fast as the homogeneity detected by J at $r = 1$. In particular, if u is nonzero at any other point, then u cannot vanish near 0.

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