The Monge-Ampère Equation

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1 Introduction

The Monge-Ampere equation

$$\det D^2u = f$$

on a domain $\Omega \subset \mathbb{R}^n$ arises in many applications, including optimal transport and prescribed Gauss curvature. These notes sketch the important regularity theorems of Pogorelov, Caffarelli and Savin. I try to give the most direct and geometric approach to all results.

I owe much of this material to conversations with my advisor Prof. Ovidiu Savin, and with my friends Yu Wang and Daniel Rubin.

2 Basics

Structure of the Equation: The Monge-Ampere equation is

$$\det D^2 u = f.$$

In order to get a maximum principle, we restrict ourselves to u convex and $f \geq 0$; if u and v are convex and v touches u by above at x, then $\det D^2 v(x) \geq \det D^2 u(x)$, since det is increasing in the space of positive matrices. To see this rigorously, differentiate

$$\log \det D^2 u = \log f = \tilde{f}$$

once to get

$$u^{ij}(u_e)_{ij} = \tilde{f}_e.$$

Differentiating once more we obtain

$$u^{ij}(u_{ee})_{ij} = u^{ip}u^{jq}(u_e)_{ij}(u_e)_{pq} + \tilde{f}_{ee}.$$

This demonstrates that the Monge-Ampere equation is degenerate elliptic on convex functions, uniformly elliptic if we control $D^2 u$ from above and below (which in the nondegenerate case $f > 1$ means all we need is by above, using the equation) and concave. These observations allow us to apply much of the theory of fully nonlinear equations (e.g. Evans-Krylov) to the Monge-Ampere equation.

Invariance Group: An important observation is that the equation is invariant under affine transformations. If

$$\det D^2 u(x) = f(x)$$

then we can "stretch" u one way and "contract" in the other to get another solution; if A is an affine transformation then

$$\tilde{u}(x) = (\det A)^{-2/n} u(Ax)$$

solves

$$\det D^2 \tilde{u}(x) = \tilde{f}(x) = f(Ax).$$

This is extremely important in the regularity theory; we must understand the invariance group of the equation in order to do rescaling. This observation coupled with John's lemma (below), the observation that all convex sets are equivalent to ellipsoids, will allow us to focus on the case when our domain is roughly B_1 and our function has height 1.

John's Lemma:

Theorem 1. *Let K be a bounded nonempty convex set in $\mathbb{R}^n$. Then there exists a point $x \in k$ and an ellipsoid E such that*

$$\frac{1}{n}E \subset K - x \subset E.$$

Proof. Take the smallest-volume ellipsoid containing K by compactness. Assume without loss of generality that it is B_1. Take the largest η such that $B_\eta \subset K$. Then by rotating we get that $K \subset B_1 \cap \{x_n \geq -\delta\}$.

Squeeze B_1 in the x_n direction a little so that the center is ϵx_n, and stretch it by $(1-\epsilon)^{1-n}$ to preserve volume. We compute the x_n coordinate δ of the intersection. The ellipsoid equation tells us that

$$\frac{(\delta+\epsilon)^2}{(1-\epsilon)^2} + (1-\delta^2)(1-\epsilon)^{2/(n-1)} = 1.$$

Expanding in ϵ we obtain

$$1 + 2\epsilon\left(\frac{n}{n-1}\delta^2 + \delta - \frac{1}{n-1}\right) + O(\epsilon^2) = 1.$$

The second term factors as $\frac{2n}{n-1}\epsilon\left(\delta - \frac{1}{n}\right)(\delta+1)$. We are assuming $\delta > 0$ so this gives

$$\delta - \frac{1}{n} = O(\epsilon).$$

Finally, we conclude that if $\eta < \frac{1}{n}$ a volume-preserving deformation of B_1 contains K with some extra room, a contradiction.

□

Sections and Rescaling A section $S_h(x)$ of a convex function u is

$$\{y : u(y) < u(x) + \nabla u(x) \cdot (y-x) + h\}.$$

Geometrically, take a supporting hyperplane at x and lift it by h, carving out some convex set S_h. These are the "balls of radius $\sqrt{h}$" in the Monge-Ampere geometry. Using affine invariance and John's lemma, we will be able to "rescale to B_1" as follows: The approximating ellipsoid to $S_h(x)$ is given by $E = A_h(B_1)$. Setting $\tilde{u}(x) = (\det A_h)^{-2/n} u(A_h x)$ we obtain a new solution on a normalized region

$$B_1 \subset \tilde{\Omega} \subset B_n.$$

Certain assumptions on f will allow us to estimate $|S_h|$ and the location of x within S_h.

Alexandrov Maximum Principle The Alexandrov maximum principle states roughly that if the Monge-Ampere measure is bounded, we must step in a bit from the boundary in order to drop down:

Theorem 2. *Assume* $\det D^2 u \leq \Lambda$ *in* Ω *with* $u|_{\partial\Omega} = 0$. *Then*

$$|u(x)| \leq C(n, \Lambda, diam(\Omega), |\Omega|) dist(x, \partial\Omega)^{1/n}.$$

Proof. Let K be the cone generated by $\partial\Omega$ and $(x, u(x))$. By convexity, $\nabla K(x) \subset \nabla u(\Omega)$. Furthermore, it is clear that every plane of slope less than $\frac{|u(x)|}{diam(\Omega)}$ is contained in $\nabla K(x)$, and there is also a subgradient of K a distance $\frac{|u(x)|}{d}$ from the origin. Since $\nabla K(x)$ is convex, this region has volume like $\frac{|u(x)|^n}{diam(\Omega)^{n-1} d}$. Thus, we have

$$\frac{|u(x)|^n}{diam(\Omega)^{n-1} d} \leq C(n)|\nabla K(x)| \leq C\Lambda|\Omega|,$$

completing the proof. □

Compactness Let the space of normalized solutions be defined by

$$NS_\Lambda = \{\det D^2 u \leq \Lambda, u|_{\partial\Omega} = 0, B_1 \subset \Omega \subset B_{C(n)}\}.$$

We claim that NS_Λ is compact. Indeed, by Alexandrov and convexity, we have an interior gradient estimate and therefore uniform convergence on compact subsets. Since viscosity solutions are closed under local uniform convergence, we have compactness.

3 Solving the Dirichlet Problem

The following apriori estimate allows us to solve the Dirichlet problem for the Monge-Ampere equation with sufficiently smooth boundary and boundary data. It is interesting that the proof makes full use of the C^3 assumption on the boundary data; this is in fact sharp, as there are examples of solutions with $C^{2,1}$ boundary data with normal second derivative blowing up towards the boundary (see the section on $C^{1,\alpha}$ regularity).

Theorem 3. *Let $u \in C^{2,\alpha}(\bar{\Omega})$ solve $\det D^2u = f$ with $f \in C^2(\bar{\Omega})$, $\lambda \leq f \leq \Lambda$. Assume the boundary condition $u|_{\partial\Omega} = g$ with $g \in C^3(\bar{\Omega})$. Assume further that $\partial\Omega$ is C^3 and is uniformly convex. Then we have*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(n, \lambda, \|f\|_{C^2}, \|g\|_{C^3}, \|\Omega\|_{C^3}).$$

Proof. Assume for simplicity that $\Omega = B_1(e_n)$.

Step 1: The L^∞ estimate. We have the easy lower barrier $\Lambda(|x - e_n|^2 - 1) - \|g\|_{L^\infty}$ to estimate u from below. By convexity, u is bounded above by $\|g\|_{L^\infty}$.

Step 2: The $\|\nabla u\|_{L^\infty}$ estimate. By convexity, $|\nabla u|$ achieves its maximum on the boundary, say at 0. The tangential derivatives u_i for $i < n$ are exactly g_i, so they are controlled. Subtract $g(0) + \nabla g(0) \cdot x$ from u so that the tangential derivatives at 0 are 0. Then the boundary data is trapped by $\pm Cx_n$ where C depends on $\|g\|_{C^2}$ and the modulus of convexity of $\partial\Omega$. It follows that u is bounded by Cx_n and $\Lambda(|x - e_n|^2 - 1) - Cx_n$, which bounds the normal derivative.

Step 3: The $\|D^2u\|_{L^\infty}$ estimate. We focus on the boundary first. Write $\partial\Omega$ as a C^3 graph $\phi(x')$ and define

$$w(x') = u(x', \phi(x')).$$

We compute the derivatives, which are tangential and denoted by T_i:

$$w_i = u_{T_i} = u_i + u_n\phi_i,$$

$$w_{ij} = u_{T_iT_j} = u_{ij} + u_{in}\phi_j + u_{jn}\phi_i + u_{nn}\phi_i\phi_j + u_n\phi_{ij}.$$

We assume that $\phi = \frac{1}{2}|x'|^2 + O(|x'|^3)$ (using that $\phi \in C^3$) to obtain

$$u_{ij}(0) = u_{T_iT_j}(0) - u_n(0)\delta_{ij},$$

for $i, j < n$, taking care of this case at the boundary.

To control u_{in} we want to imitate the ∇u estimate above, but we don't have control on u_i in C^2 along the boundary. We instead know the rotational derivative u_{T_i} along the boundary. Taking the rotational derivative of the equation we get

$$U^{ij}(u_{T_k})_{ij} = f_{T_k}$$

and we can trap u_{T_i} (up to subtracting the tangential linear part) on the boundary by $\pm Cx_n$, where C now depends on $\|g\|_{C^3}$. Furthermore, note that the trace of U^{ij} is $ftr((D^2u)^{-1}) \geq \lambda\Lambda^{-1/n}$. Thus, for sufficiently large K depending on $\lambda, \Lambda, \|f\|_{C^1}$ we have that

$$\pm(K(|x - e_n|^2 - 1) - Cx_n)$$

are upper and lower barriers for u_{T_i}, allowing control of $u_{in}(0)$ since

$$u_{T_i,n}(0) = u_{in}(0) - u_i(0).$$

To control u_{nn} we cannot apriori appeal to uniform ellipticity since we don't have the C^2 estimate yet. Using the equation, all we need is that $U^{nn} \geq \delta$ for some small universal δ. Suppose by way of contradiction that it can be arbitrarily small (say it's 0). Subtract the linear part of u at 0 so that $u(0) = 0$ and $\nabla u(0) = 0$, and $u \geq 0$ by convexity. Choose coordinates so that

$$u(x', \phi(x')) = \sum_{i=1}^{n-1} \alpha_i x_i^2 + Q(x') + o(|x'|^3),$$

where Q is a cubic. Suppose that α_1 is 0. The goal is to show that the section $S_h(0)$ is much too large for small h. To see this, note that since $u \geq 0$ and $\alpha_1 = 0$ we need the coefficient of x_1^3 in Q to be 0. It follows that S_h goes out by $M(h)h^{1/3}$ in the x_1 direction along the boundary, where $M(h) \to \infty$ as $h \to 0$. In the other directions, we go out by at least $ch^{1/2}$ since α_i are bounded above. By convexity of S_h and quadratic separation of the boundary, we must go at least $M(h)^2 h^{2/3}$ in the x_n direction. This gives the volume estimate

$$|S_h| \geq C h^{n/2} M(h)^3.$$

But it is clear using a barrier with constant boundary data on an interior John ellipse that u must drop by $c|S_h|^{2/n} >> h$, contradiction of $u \geq 0$.

Step 4: The global C^2 estimate gives uniform ellipticity. The general strategy from here is to note that the tangential second derivatives are regular by the regularity of boundary data, apply the boundary Harnack inequality to get $C^{1,\alpha}$ at the boundary for the tangential derivatives (which gives the C^α estimate for the mixed tangential-normal second derivatives), and apply the equation to conclude $C^{2,\alpha}$ regularity for the normal derivatives. Rescaled Evans-Krylov then gives the global $C^{2,\alpha}$ estimate. □

3.1 Boundary Harnack Inequality

The Harnack inequality for linear equations says that if we have two solutions, one on top of the other, then if they are close at a point they are close in a neighborhood. One can view this in several ways; as a quantitative version of the strong maximum principle, or as an inequality for the gradients for translation-invariant equations.

In general the Harnack inequality cannot be extended to the boundary; take for example $|x|^{2-n}$ in $B_1(e_n)$, where we have power growth of the fundamental solution near the boundary. This is in a sense optimal. If we have a regular boundary, and a positive solution u which is 1 a distance 1 away from the boundary, then we can iterate the Harnack inequality to get

$$u < C^k$$

for $d(x, \partial\Omega) < 2^{-k}$, i.e. polynomial growth.

However, given certain assumptions on the equation, the boundary and boundary data we can extend the Harnack inequality to the boundary.

Theorem 4 (Krylov Boundary Harnack). *Assume that $a^{ij}(x)u_{ij} = 0$ for bounded measurable a^{ij} with ellipticity constants λ, Λ in B_1^+, that $|u| \leq 1$ and that*

$$u|_{x_n=0} = 0.$$

Then for some universal $a, \alpha \in \mathbb{R}$ we have

$$|u - ax_n|_{B_r} \leq Cr^{1+\alpha}.$$

In particular,

$$\frac{u}{x_n} \in C^\alpha(\{x_n = 0\}).$$

This theorem says that even if the coefficients are terrible, the boundary and boundary data are so nice that they have some influence on (the gradients of) solutions to elliptic equations when we step in.

Proof. **Step 1:** We show that the gradient is bounded on $\{x_n = 0\} \cap B_{1/2}$. Take a barrier of the form $v = |x|^{-\beta}$ for β large universal and cut it off in $B_{1/2}$. By translating it and multiplying by a constant we can make sure $v \leq 0$ in B_1^+, touches 0 along $\{x_n = 0\} \cap B_{1/2}$ and $v < -1$ on ∂B_1. Thus, $v \leq u$ and at the touching point we have $u_n \geq -C$. The bound by above is similar. In particular, we can reduce to the case that

$$0 \leq u \leq x_n$$

in B_1^+.

Step 2: We show that we can improve the trapping angle: there exist $0 \leq a \leq b \leq 1$ such that

$$ax_n \leq u \leq bx_n$$

in $B_{1/2}^+$ with $b - a \leq \gamma$ for some universal $\gamma < 1$. To show this we use the Harnack inequality and barriers. We have $u(e_n/2)$ is closer to 0 or $1/2$; say $1/2$. Then $u > c$ universal in a region around $e_n/2$. We can use the same v as before (translated and rescaled if necessary) so that v lies below u in this region, $v \leq 0$ on ∂B_1 and v touch 0 at a point in $\{x_n = 0\}$. These functions intersect horizontal planes at a positive angle, giving that

$$u \geq cx_n$$

in $B_{1/2}^+$ (take v centered at points in a whole strip).

Step 3: Iterate. We obtain that there exist an increasing sequence $0 \leq a_k$ and a decreasing sequence $b_k \leq 1$ such that

$$0 \leq b_k - a_k \leq \gamma^k,$$

and

$$a_k x_n \leq u \leq b_k x_n$$

in $B_{2^{-k}}^+$. The sequences converge to some number a, and it is easy to verify that

$$|u - ax_n|_{B_{2^{-k}}} \leq C2^{-k}\gamma^k \leq C2^{-(1+\alpha)k}$$

where $2^{-\alpha} = \gamma$. □

4 Pogorelov's Estimate

In this section we prove the Pogorelov interior estimate:

Theorem 5. *Suppose u is a normalized solution to* $\det D^2u = 1$ *in* Ω *with* $u|_{\partial\Omega} = 0$. *Assume further that* $u \in C^4(\bar{\Omega})$. *Then at the maximum of the function*

$$w = \log u_{11} + \log u + \frac{1}{2}u_1^2$$

we have

$$|u|u_{11} \leq C(n, |\nabla u|).$$

Observe that w attains its maximum on the interior because u is 0 at the boundary. The Pogorelov apriori estimate tells us that with sufficiently regular right hand side and linear boundary data (like for a section) we have an interior bound on $\|D^2u\|$. This implies, in particular, that we are in the uniformly elliptic setting, so by applying Evans-Krylov and Schauder theory we have an interior estimate on all derivatives of u.

The philosophy of the proof is to look at the quantities u_{ee} which are subsolutions to a linear equation (since $\log\det$ is concave), and thus cannot attain interior maxima. We obtain the estimate by cooking up a modification (w) of u_{ee} with lower-order terms that has an interior maximum and examining u_{ee} at this point.

Proof. Suppose u achieves its maximum at x_0. Then at x_0, by doing a "sliding" transformation in the $x_2, ..., x_n$ variables we preserve all x_1 derivatives and make D^2u diagonal for computational convenience. To see what is happening, take the quadratic approximation to u at x_0 and look at the section of height 1. This is some ellipsoid with axes $\frac{1}{\sqrt{\lambda_i}}$ where λ_i are the eigenvalues the Hessian. The aforementioned slide makes the ellipsoid axes agree with the coordinate axes without changing the x_1 direction.

The following identites hold by differentiating the equation:

$$u^{ii}u_{1ii} = 0,$$

$$u^{ii}u_{11ii} = u^{ii}u^{jj}u_{1ij}^2.$$

Since we are at a maximum of w, the gradient vanishes:

$$w_i = \frac{u_{11i}}{u_{11}} + \frac{u_i}{|u|} + u_1u_{1i} = 0.$$

Finally, we observe that at a local maximum w_{ii} are all nonpositive, giving $u^{ii}w_{ii} \leq 0$. Writing it out, this becomes

$$\frac{u^{ii}u_{11ii}}{u_{11}} - \frac{u^{ii}u_{11i}^2}{u_{11}^2} + \frac{n}{|u|} - \frac{u^{ii}u_i^2}{u^2} + u_{11}^2 + u^{ii}u_1u_{1ii} \leq 0.$$

The last term vanishes by the first identity. Using the second identity on the first term and the third identity on the fourth term, we obtain

$$u_{11}^2 + \frac{n}{|u|} - \frac{u_1^2}{u_{11}u^2} + \left(\frac{u^{ii}u^{jj}u_{1ij}^2}{u_{11}} - \frac{u^{ii}u_{11i}^2}{u_{11}^2} - \Sigma_{i=2}^n\frac{u^{ii}u_{11i}^2}{u_{11}^2}\right) \leq 0.$$

Clearly, the term in parentheses is positive, so we may discard it. Multiplying by u^2u_{11} we have

$$u^2u_{11}^2 + n|u|u_{11} \leq u_1^2,$$

completing the proof.

□

5 Singular Solutions

A natural question is whether we can find singular solutions, i.e. weak solutions that don't have 2 derivatives everywhere. Pogorelov constructed several examples which degenerate along lines, which we describe here from several perspectives. Caffarelli later observed that we can always find singular solutions that are 0 on subspaces of dimension $< \frac{n}{2}$, and that this is sharp; it is impossible to find solutions degenerating on subspaces of dimension $\frac{n}{2}$ or higher.

5.1 Scaling to a Singularity

For harmonic functions we have derivative estimates independent of the regularity of the boundary data.

For Monge-Ampere, this situation is ruled out by invariance under anisotropic rescaling. Indeed, if we have a smooth solution to

$$\det D^2 u = 1$$

in B_1 then the "squeezing" in the x' directions given by

$$u_\lambda(x', x_n) = \frac{1}{\lambda^{2-2/n}} u(\lambda x', x_n)$$

is also a solution. As λ gets large, note that

$$|D^2_{x'} u_\lambda(0, x_n)| = \lambda^{2/n} |D^2_{x'} u(0, x_n)| \to \infty$$

so that in the limit there is the possibility of obtaining a singularity along $\{|x'| = 0\}$.

Indeed, Pogorelov first wrote down an example of a singular solution with this rescaling invariance, i.e. a solution of the form

$$|x'|^{2-2/n} f(x_n).$$

5.2 Explicit Computation of Solutions

We seek a solution to $\lambda \leq \det D^2 u \leq \Lambda$ of the form

$$u(x_1, ..., x_k, y_1, ..., y_s) = |x|^\alpha f(|y|),$$

which vanishes along a subspace of dimension s, and for $\alpha > 1$ and certain f we can make convex. The rigorous approach to finding α and f is to explicitly compute the determinant of the Hessian. Choosing coordinates cylindrical in x and y, the Hessian is

$$\left(\begin{array}{cccc|cccc}
\alpha(\alpha-1)|x|^{\alpha-2}f & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha|x|^{\alpha-1}f' \\
0 & \alpha|x|^{\alpha-2}f & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \alpha|x|^{\alpha-2}f & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & |x|^\alpha|y|^{-1}f' & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & |x|^\alpha|y|^{-1}f' & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
\alpha|x|^{\alpha-1}f' & 0 & \cdots & 0 & 0 & \cdots & 0 & |x|^\alpha f''
\end{array}\right).$$

One easily computes the determinant:

$$\det D^2 u = \alpha^k |x|^{n\alpha - 2k} |y|^{1-s} f^{k-1} f'^{s-1} \left((\alpha - 1) f f'' - \alpha f'^2 \right).$$

For right side bounded away from 0 we thus need $\alpha = \frac{2k}{n}$, and for a convex positive function f to give a positive bounded right side we require $\alpha > 1$ and (for example) $f(|y|) = 1 + |y|^2$, $1 < |y| < 2$. Thus, we can always construct a singular solution on a subspace of dimension $s < \frac{n}{2}$ of the form

$$u(x, y) = |x|^{2k/n} f(|y|).$$

Note that for $k = n-1$, $s = 1$ we can solve the explicit ODE $f^{n-2}((\alpha-1)ff'' - \alpha f'^2) = 1$ to find an f with $f(0) = f'(0) = 0$, f convex and smooth in a neighborhood of 0 (blows up to infinity in finite time), giving a solution to $\det D^2 u = 1$ of the form

$$u(x', x_n) = |x'|^{2-2/n} f(x_n),$$

Pogorelov's original example. Observe that these examples have deneracy that extends to the boundary, and that the regularity of these functions is at best $C^{1,1-2/n}$.

Exercise: Do a similar computation to the one above to show that there are merely Lipschitz solutions of the form

$$|x'| + |x'|^{(k+1)/(s+1)} f(|y|).$$

As a quick technical remark, we know these are viscosity solutions because the only places a C^2 function can touch by above is away from the singularity, and if we touch by below with a convex C^2 function on the singularity the determinant is 0 because it must vanish on a line. Alternatively, these are Aleksandrov solutions because the Monge-Ampere measure cannot concentrate on subspaces of dimension ≥ 1 using monotonicity of Du. We will discuss this more geometric approach to singular solutions next.

5.3 Geometric Perspective

Using the requirement that Du is roughly measure-preserving, we have a nice quick way to compute the exponents α in the above examples and a proof that the above examples are sharp. More precisely, any convex function vanishing on a subspace of dimension $\geq n/2$ cannot solve the Monge-Ampere equation with strictly positive bounded right hand side.

In the above examples, take a cylinder of length 1 in the y direction and length ϵ in the x direction, so that it has volume like ϵ^k. Taking Du of this cylinder and denoting $r = |x|$ we see it is a shape growing in the y direction like $r^{\alpha/(\alpha-1)}$, from 0 to $\epsilon^{\alpha-1}$. Integrating to find the volume we get

$$\int_0^{\epsilon^{\alpha-1}} r^{\alpha s/(\alpha-1)+k-1} dr = \epsilon^{s\alpha + k(\alpha-1)}.$$

Equating exponents, we see that $\alpha = 2k/n$ as expected.

Exercise: Do the same for the Lipschitz solutions above. Note that the only difference is that the image Du of the cylinder has a flat disk in dimension k of radius 1 before we start growing, and so the growth of volume needs to be slower (don't have the r^{k-1} near 0 helping us).

The deeper reason that all of the above works is that for convex functions, Du is monotonic:

$$\langle Du(x) - Du(y), x - y \rangle \geq 0.$$

Suppose for instance that we have a convex solution u to $\lambda \leq \det D^2 u \leq \Lambda$ vanishing on the y axis in $\mathbb{R}^2$. Take the largest C so that Cx is a supporting hyperplane at 0, and tilt u so that without loss of generality $C = 0$. Then look at the the thin triangle with base length 1 on the y-axis and height ϵ on the x axis. This has area of order ϵ. By monotonicity, Du takes this triangle into cone of vertex 0 and opening ϵ in the x direction, but since 0 is the best supporting hyperplane in this direction, we know the image goes out at most $O(\epsilon)$, so that the image has area of order $\epsilon O(\epsilon)$, a contradiction! Note that in 3 dimensions we would have volume ϵ^2 to start, so there is no contradiction, as expected given the examples. This is a sketch of the proof that we can't have a solution vanishing on a subspace of dimension $\geq n/2$.

5.4 Boundary Data and Hausdorff Dimension of Singular Set

A natural question to ask is whether we can have a singular solution with better than $C^{1,1-2/n}$ regularity. The answer is in fact no. We will later show that the singular set has no interior extremal points, i.e. every singular point lies on a line extending to the boundary. Suppose that this line is the x_n axis and that the boundary data grow slower that $C|x'|^{1+\beta}$. Then by convexity $u \leq C|x'|^{1+\beta}$ in the cylinder. We construct the barrier w in the cylinder $C_r = \{|x'| < r, |x_n| < 1\}$ satisfying $\det D^2 w = \lambda$, $w = 0$ on ∂C_r. We know that w can drop no lower than $-Cr^{1+\beta}$. However, from a later section we have an exact estimate on the height of w given the volume of its section:

$$|\min w| = (vol)^{2/n} = r^{(n-1)2/n} = r^{2-2/n}.$$

Thus, we have $r^{2-2/n} \leq Cr^{1+\beta}$ for all r small, giving $\beta \leq 1 - 2/n$. (We have strict convexity otherwise, i.e. w forces u to drop below 0.)

One conclusion is that away from a singular point, the maximum of u grows at slowest like $r^{2-2/n}$, which along with convexity allows us to estimate the Hausdorff dimension of the singular set! Suppose we have a convex function u with $u(0) = 0$, $\nabla u(0) = 0$ and $u \geq 0$ and let 1 be the maximum of u on $B_{1/2}$. Then $u \geq 1$ on a universal portion of ∂B_1 by convexity, and on this part $u_\nu \geq 1$. Thus, we conclude

$$\int_{B_1} \Delta u = \int_{\partial B_1} u_\nu \geq C.$$

By rescaling around a singular point for a solution, i.e. letting $\tilde{u}(x) = \frac{1}{r^{2-2/n}} u(rx)$, we are in this situation. This gives us the estimate

$$\int_{B_r} \Delta u = r^{-2/n} \int_{B_r} \Delta \tilde{u}(x/r) dx = r^{n-2/n} \int_{B_1} \Delta \tilde{u} \geq Cr^{n-2/n}.$$

Hence, the singular set has Hausdorff dimension at most $n - 2/n$.

Note that for the mass of Δ to be large for a convex function all we need is a large second derivative in one direction. However, the second derivatives perpendicular to the singularity can blow up in many directions. By considering the Monge-Ampère mass of u restricted to lower dimensional slices, one can easily improve this estimate of the Hausdorff dimension to $n-1$, and with some harder work we have shown that

$$\mathcal{H}^{n-1}(\Sigma) = 0.$$

Examples show that this estimate is optimal.

6 Geometric Properties of Sections

The geometry of the sections of solutions to the Monge-Ampere equation gives essential information about the second derivatives of solutions.

Definition 1. *Let u be a convex function and let $\nabla u(x)$ be a subgradient of u at x. A section of u at x of height h, denoted by $S_h(x)$, is defined by*

$$S_h(x) = \{y : u(y) < u(x) + \nabla u(x) \cdot (y - x) + h\}.$$

As an example consider the sections of $u(x) = x^T \cdot M \cdot x$ for a symmetric positive matrix M. Then S_h are ellipsoids with axes of length $\sqrt{h/\lambda_i}$ where λ_i are the eigenvalues of M. Thus, the eccentricity of the sections squared geometrically reflects $\|D^2u\|$. Note also that $|S_h| = \frac{h^{n/2}}{\sqrt{\det D^2 u}}$, so that the volume growth doesn't depend on the eccentricity but only the determinant.

If $u \in C^2$ then since u is approximated by such a polynomial to order better than 2, the sections $S_h(x)$ look like those of $y^T \cdot D^2u(x) \cdot y$ for h small.

The first key property we wish to prove, first observed by Caffarelli, is the sections are "balanced" around the point from which we lift, and we know precisely how the volume of the sections grow.

Theorem 6. *Suppose u is a convex solution to the Monge-Ampere equation $\lambda \leq \det D^2u \leq \Lambda$ in Ω and that $S_h(x) \subset\subset \Omega$. Then there is an ellipsoid E of volume $h^{n/2}$ centered at 0 such that*

$$x + cE \subset S_h(x) \subset x + CE$$

where c, C are constants depending only on λ, Λ, n.

Proof. By subtracting the tangent plane at x shifted up by h, we may assume $u|_{\partial S_h(x)} = 0$ and u achieves its minimum at x. By John's lemma, we can normalize $S_h(x)$ by an affine transformation A, i.e. let $\tilde{S} = A^{-1}(S)$ so that $B_1 \subset \tilde{S} \subset B_n$. Let $\tilde{u}(y) = (\det A)^{-2/n}u(Ay)$. Then $\lambda \leq \det D^2\tilde{u} \leq \Lambda$. Using the lower and upper barriers $\frac{\Lambda^{1/n}}{2}(|y|^2 - n^2)$, $\frac{\lambda^{1/n}}{2}(|y|^2 - 1)$ we have that $|\min \tilde{u}|$ is of order 1. Since $|\min u| = h$, we conclude that $|\det A|$ is $h^{n/2}$ up to multiplication by universal constants. Observing $|\det A|$ is the volume of $S_h(x)$ we obtain the volume growth of sections.

For the balancing, recall the Alexandrov estimate $|\tilde{u}(y)| \leq C dist(y, \partial\tilde{S})^{1/n}$ (using the boundedness of $\det D^2\tilde{u}$ and that $\tilde{S}$ has size of order 1). Since $\tilde{u}$ has a minimum of order 1 at at $\tilde{x} = A^{-1}(x)$, we see that

$$B_c(\tilde{x}) \subset \tilde{S} \subset B_{2n}(\tilde{x}).$$

Scaling back by A, the proof is complete. □

As a corollary of the "balancing of sections," solutions cannot degenerate on sets with interior extremal points.

Theorem 7. *Suppose $\lambda \leq \det D^2u \leq \Lambda$ in Ω. Then the set $\{u = 0\}$, if not a single point, cannot have extremal points.*

Proof. Suppose by way of contradiction that 0 is an interior extremal point, and that $\{u = 0\} \subset \{x_n \leq 0\}$. Then $\{x_n = -\delta\} \cap \{u = 0\}$ is some segment strictly contained in Ω for some small δ. Slice u with the plane P given by

$$x_{n+1} = \epsilon(x_n + \delta).$$

Observe that the maximum of $P - u$ must occur for $x_n \geq 0$, and that as $\epsilon \to 0$ the set $\{P - u > 0\}$ has largest x_n component going to 0. It follows that we can find sections with center point squeezed arbitrarily close to $\{x_n = 0\}$ but with length larger than δ in the x_n direction, contradicting balancing. □

7 $C^{1,\alpha}$ Regularity

As a consequence of section balancing and the "no line segments" results of the previous section, we obtain $C^{1,\alpha}$ regularity. This follows from the following proposition:

Proposition 1. *Assume* $\lambda \leq \det D^2u \leq \Lambda$ *in* $S_1(0)$*, with* $u(0) = 0$ *and* $\nabla u(0) = 0$*. Then*

$$(\frac{1}{2} + \delta)S_1(0) \subset S_{1/2}(0) \subset (1 - \delta)S_1(0)$$

for some small δ *universal.*

Note that without the δ the above inclusions hold for any convex function. The requirement that $\det D^2u$ is controlled as a measure gives us this more regular behavior.

Proof. The Alexandrov maximum principle tells us that to drop $\frac{1}{2}$, we must step in some universal amount from the boundary, giving the right inclusion. For the left inclusion, argue by contradiction and use the compactness of normalized solution. The limiting solution is linear on some ray from 0, contradicting the no segments result. □

Now suppose we have a normalized solution. By normalizing the interior sections, we see that

$$cB_{h^{\frac{1}{1+\alpha}}}(x) \subset S_h(x) \subset CB_{h^\mu}(x)$$

for some $0 < \alpha < 1$, μ small, c, C universal. This is precisely $C^{1,\alpha}$ regularity and strict convexity.

The Shape of the Sections: The previous results give a precise estimate on the size of the sections $S_h(x)$ and the location of x, but tell us little about the ellipticity and direction of the sections. We briefly discuss that here.

Note that if $\det D^2u = 1$ and $u|_{\partial\Omega} = 0$ then by Pogorelov we have a C^3 estimate which guarantees that as we take smaller and smaller sections, they look more and more like the sections of the approximating paraboloid, i.e. they have the same shape (stretched in directions e where u_{ee} is small and squeezed in directions where u_{ee} is large). However, if we merely have $\lambda \leq \det D^2u \leq \Lambda$ the sections may become very eccentric as we lower h. To see this, we seek solutions with different homogeneity in x and y (for 1 rhs the both are roughly homogeneous of order 2):

$$u(x, y) = \frac{1}{\lambda}u(\lambda^{1/\alpha}x, \lambda^{1/\beta}y).$$

If $\alpha > \beta$ this tells us the the sections get extremely long and narrow in the x-direction. Computing $\det D^2u$, we see that the only solutions that scale this way and have bounded right hand side must have

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Furthermore, this rescaling preserves the curves $y = x^{\beta/\alpha}$, along which u grows like x^α. With this information we can write down such functions, having reduced the problem to an ODE. Let $f(t) = u(t, 1)$. Then by the scaling we desire we have $u(x, y) = y^\beta(u(y^{-\beta/\alpha}x, 1)) = y^\beta f(y^{1-\beta}x)$. The equation gives an ODE for f:

$$\det D^2u = (\beta - 1)f''(\beta f - (\beta - 2)tf') - f'^2.$$

For example, take $\alpha = 3$ and $\beta = 3/2$, and $f(t) = 1 + t^2/2$. This gives the solution

$$u(x, y) = y^{3/2} + \frac{1}{2}x^2 y^{1/2},$$

which has bounded $\det D^2 u$ in $\{y > x^2\}$, and grows $C^{2,1}$ (cubically) along the boundary. This actually shows that the C^3 assumption we made when solving the Dirichlet problem was sharp; the second normal derivative blows up in this example at 0.

Engulfing Property Another important observation is that nearby sections must have the same shape. The way we see this is by showing that if two sections overlap, by lifing one by a universal factor we engulf the other:

Proposition 2. *Suppose $\lambda \leq \det D^2 u \leq \Lambda$ and u is normalized with $\Omega = S_1(0)$. Then there is a universal δ such that if $y \in S_\delta(0)$ then $S_\delta(0) \subset S_{1/2}(y)$.*

Proof. From above we know that $S_\delta(0) \subset CB_{\delta^\mu}$. By the $C^{1,\alpha}$ estimate, taking δ small we guarantee that the slope in S_δ, like $\delta^{\mu\alpha}$. It is clear that if we lift a plane of small slope from a small distance away by $1/2$, we must engulf S_δ. □

By rescaling we see that for any $x, y \in S_{1/2}(0)$, there is a universal δ such that if $S_{\delta h_1}(x) \cap S_{\delta h_2}(y) \neq \emptyset$ and $h_1 \geq h_2$ then

$$S_{\delta h_2}(y) \subset S_{h_1}(x).$$

Thus, if two sections of similar size intersect, a universal multiple of one engulfs the other, indicating that they must have roughly the same shape.

8 $C^{2,\alpha}$ Regularity

We have regularity theorems for regular right side; for $f \in C^2$ we have C^k interior estimates by Pogorelov. However, it is interesting to consider situations where f depends on ∇u for instance. In this case, we on have C^α right side. The following theorem treats this case:

Theorem 8. *Suppose u is a normalized solution to* $\det D^2 u = f$ *in Ω, with $f \in C^\alpha(\Omega)$ and $\lambda \leq f \leq \Lambda$. Then*

$$\|u\|_{C^{2,\alpha}(\{u<\delta\})} \leq C(\|f\|_{C^\alpha(\Omega)}, n, \lambda, \Lambda, \delta).$$

Proof. This is a perturbation argument. The idea is that by rescaling we can assume f has very small oscillation and compare u to a solution with constant right side. Using Pogorelov, we get that u is in fact very close to a quadratic polynomial, which puts us into the small perturbation setting.

Step 1: For small h take $S_h(x)$ and the normalized solution

$$\tilde{u}(y) = \frac{1}{h}(u - u(x) - \nabla u(x) \cdot (y - x))(Ay + x)$$

where $\det A = h^{n/2}$. We also know by iterating Alexandrov (end of last section) that $\|A\| \leq Ch^\mu$ for some small μ. Assume that $f(x) = 1$.

This tells us that

$$\det D^2 \tilde{u} = \tilde{f}$$

in $S_1(0)$ normalized, and that $\tilde{f}$ has small oscillation:

$$\|\tilde{f} - 1\|_{C^\alpha} \leq Ch^{\mu\alpha}\|f\|_{C^\alpha}.$$

We have thus reduced to the case where u is normalized and f is δ close to 1 in C^α for δ small to be chosen later.

Step 2: Compare with a solution of $\det D^2 w = 1$ in S_1 with $w = u = 0$ on the boundary. Then by multiplying w by $1 \pm \delta$ we get barriers:

$$|u - w| \leq C\delta.$$

Furthermore, w is close to its quadratic part Q at 0:

$$|w - Q| \leq Cr^3$$

where $\det D^2 Q = 1$ and $\|Q\|_{C^2}$ is controlled. Quadratically rescaling we get

$$\|u - Q\|_{L^\infty(B_1)} \leq C\delta r^{-2} + Cr.$$

Thus, without loss by taking r then δ small and taking an affine transformation u is δ_1 close to $\frac{1}{2}|x|^2$. We can now just apply Savin's small perturbation result, or do the next step explicitly.

Step 3: Iteration. We are in the setting $\det D^2 u = f$ with $\|f - 1\|_{C^\alpha(B_1)} \leq \delta_1$ and $|u - \frac{1}{2}|x|^2| \leq \delta_1$. The idea is that $u - P$ almost satisfies a linear equation with constant coefficients. Another way to see this is to take the solution w with 1 rhs and the same boundary data as u and note that $w-\frac{1}{2}|x|^2$ satisfies a linear equation with smooth coefficients (by Pogorelov), and thus

$$|D^3w| = |D^3w - D^3(\frac{1}{2}|x|^2)| \leq C\delta_1.$$

Hence w grows from its quadratic part P like $C\delta_1 r^3$. Putting this together we get

$$\|u - P\|_{L^\infty(B_r)} \leq C\delta_1 r^3 + C\delta_1 \leq r^{2+\alpha}$$

for r then δ chosen small enough. We then iterate, rescaling u quadratically so that now $|f-1| \leq \delta_1 r^\alpha$ and u is $\delta_1 r^\alpha$-close to another quadratic polynomial. Because we improve δ_1 geometrically at each stage, the approximating quadratic polynomials with determinant of Hessian 1 converge uniformly. $\square$

9 $W^{2,p}$ Regularity

So far, we have seen that if u is a normalized solution to $\det D^2u = f$, then $\lambda \leq f \leq \Lambda$ gives $C^{1,\alpha}$ regularity and if in addition f is C^α we get $C^{2,\alpha}$ regularity of u. Geometrically, this means that for bounded f the sections $S_h(x)$ will go to a point, with $B_{ch^{1/(1+\alpha)}} \subset S_h(x) \subset B_{Ch^\mu}$ for some small μ, but the eccentricity can get arbitrarily bad. For $f \in C^\alpha$, the oscillation of f improves with scaling so that the eccentricity remains bounded (sections all look like balls with eccentricity controlled by $\|D^2u\|$). In this section we investigate the intermediate case:

Theorem 9. *Assume u is a normalized to* $\det D^2u = f$ *with $\lambda \leq f \leq \Lambda$. Fix $1 \leq p < \infty$. Then there exists some $\epsilon(p,n)$ such that if $osc_{B_{\rho_0}} f \leq \epsilon$ for some ρ_0 and all $B_{\rho_0} \subset\subset \Omega$, then $u \in W^{2,p}(\Omega)$ and we have*

$$\int_{\{u \leq -\delta\}} \|D^2u\|^p \leq C(n, \lambda, \Lambda, p, \rho_0, \delta).$$

In particular, iteration will not necessarily improve the approximating quadratic polynomial (allowing for high eccentricity of sections, i.e. blowup of second derivatives) but in measure the second derivatives will be small. Observe that for continuous f we have $u \in W^{2,p}$ for all p except for ∞, giving $C^{1,\alpha}$ regularity for any α.

Proof. Reduce to the case $1 - \epsilon \leq \det D^2u \leq 1 + \epsilon$ in $S_1(0)$ with $B_{1-C\epsilon} \subset \Omega \subset B_{1+C\epsilon}$ and $|u - \frac{1}{2}|x|^2| \leq \epsilon$, as in the first part of the $C^{2,\alpha}$ estimates.

Step 2: By convexity, $\nabla u(x)$ is $\sqrt{\epsilon}$ close to x in $S_{1/2}(0)$, so by Green's theorem

$$\int_{S_{1/2}(0)} \left(\frac{\Delta u}{n} - 1\right) = O(\sqrt{\epsilon}).$$

Furthermore, note that

$$\Delta u/n - 1 \geq (\det D^2u)^{1/n} - 1 \geq -C\epsilon.$$

Thus, in measure, Δu must be small:

$$|\{\|D^2u\| \geq 3/2\} \cap S_{1/2}(0)| = O(\sqrt{\epsilon}).$$

Step 3: We use a covering argument and the previous step to show decay of the sets where $\|D^2u\|$ is large. Choose a Vitali cover of $A_1 = \{\|D^2u\| \geq 2\} \cap S_{1/2}$ with sections $S_{h_i}(x_i)$ with eccentricity $3/2$ such that $S_{\delta h_i}(x_i)$ are disjoint. We would like to show these sections have small total measure. By the previous lemma, $\|D^2u\|$ is close to the eccentricity (say $> 4/3$) in a fixed fraction (say $> \frac{1}{2}$) of the sections, and thus by the second lemma again we have

$$|A_1| \leq C \sum \frac{1}{2}|S_{h_i}(x_i)| \leq C \sum |S_{\delta h_i}(x_i)| \leq C\sqrt{\epsilon}.$$

Iterating we obtain

$$|\{\|D^2u\| > 2^k\} \cap S_{1/2}(0)| \leq (C\sqrt{\epsilon})^k,$$

proving the theorem. □

10 Liouville Theorem

The Pogorelov estimate puts us into the uniformly elliptic setting, and Evans-Krylov gives derivative estimates of all orders. As a consequence, when we quadratically blow up a solution to $\det D^2u = 1$ at a point we must end up with a quadratic polynomial. In this section we show that global solutions are quadratic polynomials, which one can think says that at each point, a solution looks like

$$P(x) + o(|x|^2)$$

for some quadratic polynomial P with $\det D^2P = 1$.

Theorem 10. *Assume that* $\det D^2u = 1$ *in* $\mathbb{R}^n$*. Then* $u = \frac{1}{2}|x|^2$ *up to an affine transformation.*

Proof. If the graph of u contained a line l, then by convexity (easy to show) $|\nabla u|$ would map into the subspace of $\mathbb{R}^n$ perpendicular to l, contradicting that $\det D^2u = 1$. We conclude that if L is a tangent plane to u then $\{u = L\}$ has strictly extremal points, and thus we can assume after subtracting a tangent plane that $u(0) = 0$, $\nabla u(0) = 0$ and that $u \to \infty$ as $|x| \to \infty$.

For h large, we thus have that S_h is bounded and normalized by $h^{1/2}A_h$ where A_h has determinant 1. We look at

$$\tilde{u}(x) = \frac{1}{h}u(h^{1/2}A_hx).$$

Then by Pogorelov $|D^2\tilde{u}(0)|$ are uniformly bounded with h, which gives a lower and upper bound on the eigenvalues of A_h. Geometrically, the sections are equivalent to balls of radius $h^{1/2}$.

Now, note that by Evans-Krylov and higher order derivative estimates we have for $x \in B_c$ that

$$h^{1/2}\|A_h\|^3|D^3u(h^{1/2}A_hx)| = |D^3\tilde{u}(x)| \leq C$$

for all h, so we conclude that $D^3u \equiv 0$. □

It is interesting to ask whether analagous results hold in half-spaces. The first step is the Pogorelov estimate in a half-space, due to Savin.

Theorem 11. *If* $\det D^2u = 1$ *in* $S_1 = \{u \leq 1\}$ *with* $B_1^+ \subset S_1 \subset B_C(n)^+$, $u|_{x_n=0} = \frac{1}{2}|x'|^2$ *and* $\nabla u(0) = 0$ *then*

$$|D^ku|_{B_{1/2}^+} \leq C(n,k).$$

Proof. By Evans-Krylov, the boundary Harnack inequality and Schauder all we need is a C^2 bound in $B_{1/2}^+$.

It is not hard to verify that $|\nabla u| < C$. The usual Pogorelov test function then gives an interior second derivative bound. By the boundary data $|u_{ii}| < C$ along the flat boundary for $i < n$. By the equation it suffices to bound u_{in} from above and below, say at 0.

Since $u^{ij}(u_k)_{ij} = 0$ for any direction k this is a simple barrier argument for u_k. Along the flat boundary, $u_k = x_k$ and we know u_k is locally bounded by a previous step. To construct a lower barrier we start with

$$P = x_k + |x'|^2 + C_1 x_n^2 - C_2 x_n$$

with C_1 chosen large enough that the Monge-Ampere mass is large. The problem is that along the flat boundary this is larger than u. To correct for this we use the fact that $u^{ij}u_{ij} = n$, and by taking

$$Q = P - C_3 u$$

with C_3 large enough that we are well below u on the flat boundary, and then taking C_1 large enough that $\det D^2 u > 1$, then C_2 large enough that we lie below u on the whole boundary we obtain a lower barrier and conclude $u_{kn} < C$. The other direction is similar. □

With this we can prove a similar Liouville theorem in a half-space, with some interesting features:

Theorem 12. *Assume that* $\det D^2 u = 1$ *in* $\mathbb{R}^n_+$ *with* $u|_{x_n=0} = \frac{1}{2}|x'|^2$. *Assume further that* u $|x|^2$. *Then* u *is a quadratic polynomial obtained by sliding* $|x|^2$ *along the* x' *directions.*

Of course we need to impose that u behaves like a quadratic polynomial along the boundary to get such a result. The interesting feature is that we must assume quadratic growth, due to the example

$$u(x,y) = \frac{x^2}{2(1+y)} + \frac{y^2}{2} + \frac{y^3}{6}$$

in $\mathbb{R}^2$, which may be obtained by taking the partial Legendre transform of the harmonic polynomial

$$\frac{1+y}{2}x^2 - \frac{y^2}{2} - \frac{y^3}{6}.$$

The problem with the previous proof is that we only get the second derivative bound above and below at 0 assuming nice boundary data, which may get messed up by rescaling a fat section (like the previous example).

Proof. Assuming quadratic growth means that the sections S_h are equivalent to balls of radius $h^{1/2}$. The rest goes as before. □

We remark that in 2 dimensions it is easy to classify solutions to

$$\det D^2 u = 1 \text{ in } \mathbb{R}^2_+, \quad u|_{y=0} = \frac{x^2}{2}.$$

Indeed, the partial Legendre transform u^* satisfies

$$\Delta u^* = 0 \text{ in } \mathbb{R}^2_+, \quad u^*|_{y=0} = \frac{x^2}{2}.$$

We then have $\Delta u^*_{xx} = 0, u^*_{xx}|_{y=0} = 1$ and $u^*_{xx} \geq 0$ (partial Legendre preserves convexity in the x directions). Using log as a barrier one sees that $u^*_{xx} - 1$ is in fact nonnegative. By adding a linear function assume that $u^*_{xx} - 1 = 1$ at $(0, 1)$. We apply the boundary Harnack inequality at all scales: Letting $v = u^*_{xx} - 1$ and $v_r(x) = \frac{1}{r}v(rx)$ we have that

$$c \leq v_r(0, 1) \leq C$$

and thus v has linear growth. By reflecting we obtain a global harmonic function with linear growth, so $u^*_{xx} = Cy + 1$ for some $C \geq 0$. We conclude that u^* has the form

$$\frac{1}{2}x^2(y+1) - \frac{1}{2}y^2 - \frac{1}{6}y^3$$

up to adding multiples of y and modifying the coefficients.

11 Savin's Boundary Localization Property

In this note we discuss Savin's important theorem on the geometry of boundary sections for the Monge-Ampere equation.

11.1 Statement of the Boundary Localization Theorem

Theorem 13. *Assume the following:* $\lambda \leq \det D^2 u \leq \Lambda$ *in* B_1^+, $\frac{1}{2}|x|^2 \leq u \leq 2|x|^2$ *in* $\{x_n < \mu\} \cap \partial B_1^+$, $u \geq 0$ *and* $x_{n+1} = 0$ *is the tangent plane to the graph of* u *at* 0. *Let* $S_h = \{u < h\}$. *Then for all* $h < c(\mu)$ *there is an ellipsoid* E_h *of volume* $h^{n/2}$ *such that*

$$cE_h \cap B_1^+ \subset S_h \subset CE_h \cap B_1^+,$$

and E_h *is obtained by a "slide"* $A_h B_{\sqrt{h}}$ *where* $A_h(x) = x - \nu_h x_n$, $\nu_h = (\nu_1, ..., \nu_{n-1}, 0)$, *and we have slid by at most*

$$|\nu_h| \leq C|\log h|$$

where all constants depend on λ, Λ, n.

The statement 0 is the tangent plane means that ϵx_n does not support u for any $\epsilon > 0$.

Heuristically, the theorem says that the tangential second derivatives are bounded and the mixed second derivatives u_{in} are like $|\log h|$ in S_h. The quadratic separation ensures that we have control of the tangential second derivatives by above. Quadratic separation also is a natural condition coming, for example, from constant boundary data and uniformly convex boundary (just tip the boundary to get quadratic growth) and other situations that I'll talk about when we do $C^{2,\alpha}$ estimates.

Note also that slides and adding αx_n preserve the boundary problem, so requiring 0 to be the tangent plane and allowing for slides of sections are natural conditions that remove these degrees of freedom.

11.2 Proof of Theorem

Proof. **Step 0: Rough Localization**. The reason we need to assume that $u \geq \frac{1}{2}$ on the small segments at the boundary of B_1^+ is because solutions might not see the data on a flat boundary, and could drop to 0 so fast that the sections S_h of a sequence of solutions could exit the domain in $\{x_n < \delta\}$ for h and δ going to 0. The function

$$\frac{1}{2}|x'|^2 + 2^n \Lambda x_n^2 - C(\mu) x_n$$

is an easy lower barrier for u in $\{x_n < \mu\}$ for large $C(\mu)$ (i.e. so that the n-direction is sufficiently negative in this region), which tells us that the sections can't exit the domain close to $\{x_n = 0\}$ for small $h < h_0(\mu)$.

Step 1: Volume Estimate. Normalize the solution by

$$\tilde{u}(x) = \frac{1}{h} u(Ax)$$

where $|\det A| = |S_h|$. If $|S_h|$ were much larger than $h^{n/2}$, then $\det D^2\tilde{u} >> 1$, but then the paraboloid $C|x|^2$ in a ball of radius like 1 contained in $A^{-1}S_h$ would be an upper barrier for $\tilde{u}$ and go below 0, a contradiction.

If on the other hand $|S_h| << h^{n/2}$ then $\det D^2\tilde{u} << 1$. Assume that our normalization involves a sliding to make the center of mass on the x_n-axis. Then the plane ϵx_n lies well below $\tilde{u}$ on the boundary of $A^{-1}S_h$ except for near 0, but by quadratic separation and convexity we can add $c|x|^2$ and still keep it below u. Then this is a lower barrier with a positive x_n-slope at 0, a contradiction.

More Explicitly: For the other direction, we want a quadratic polynomial lower barrier. To make things easier, let's slide S_h to make it more symmetric; if x^* is the center of mass of S_h, take the slide $A_h(x) = x - \frac{x^{*\prime}}{x^*\cdot e_n}x_n$ and define $\tilde{u}(x) = u(A_h^{-1}x)$. Then $\tilde{S_h}$ has center of mass $\tilde{x^*}$ with no x' component. Forget the tildes for simplicity of notation. Then after this slide we have $S_h - x^*$ equivalent to

$$diag(d_1, ..., d_n)(B_1),$$

where by quadratic separation $d_i \geq c\sqrt{h}$ for $i < n$. We can now write an easy lower barrier candidate:

$$v = ch\sum_{i=1}^{n}\left(\frac{x_i}{d_i}\right)^2 + \epsilon x_n.$$

Since v has positive x_n slope, it cannot be a lower barrier, but if $|S_h|$ is too small, i.e. $d_1...d_n << h^{n/2}$ then

$$\det D^2 v = \frac{h^n}{(d_1 d_2...d_n)^2} \geq \Lambda,$$

giving the desired contradiction.

We would be done if $d_n \geq c\sqrt{h}$, i.e. if the section is not too short and fat. Indeed, the equation then tells us that d_i all go like $\sqrt{h}$ so that $cE_h \cap B_1^+ \subset S_h \subset CE_h \cap B_1^+$, and E_h are slides. To get the estimate on sliding distance, observe that

$$cE_{h/2} \subset CE_h$$

. Since the sliding distance for these are of size $\sqrt{h}|\nu_h|$ and $\sqrt{h}\nu_{h/2}$, we must have

$$|\nu_h - \nu_{h/2}| \leq C,$$

which would finish the proof. We reduce the statement $d_n \geq c\sqrt{h}$ a new statement in the following steps.

Reduction 1: Define $b(h) = h^{-1/2}\sup_{S_h} x_n$. We want to show that $b(h) \geq \delta$ for some small δ universal. We'll work in $\mathbb{R}^2$ from hereon out. This is implied by the following:

Proposition 3. *If $b(h) < \delta$ for some universal threshold δ, then $b(\tilde{h}) > 2b(h)$ for some $ch \leq \tilde{h} < h$ of comparable size.*

Reduction 2: Rescaled version of reduction 1. Let's slide like above and let $v(x) = \frac{1}{h}u(cD_h x)$, with c chosen so that $b(1) = 1$. Then we want to show:

Proposition 4. *If $\partial S_1 \cap \{x_2 = 0\} \subset \{|x_1| < \delta\}$ then $b(h) > 2$ for some $c \leq h < 1$.*

Reduction 3: Limiting situation. In the above picture, it seems clear the the sections should get long and skinny in the x_2 direction as we would like. If reduction 2 fails for a sequence of functions u_k with $\delta_k \to 0$ one can show that the graphs of u_k converge in the Hausdorff sense to the graph of a discontinuous solution u satisfying: $\lambda \leq \det D^2 u \leq \Lambda$ in Ω, $0 \in \partial\Omega$ and Ω equivalent to the unit ball, $u(0) = 0$ and $u|_{\partial\Omega - 0} = 1$, $b(1) = 1$ and $|S_h|$ go like $h^{n/2}$. Furthermore, for this limit solution we'd have $b(h) < 2$ for all h. This is a contradiction of the last reduction:

Proposition 5. *Such a limit solution has $b(h) > 2$ for some $c \leq h < 1$.*

We finally prove this statement by showing with barriers that S_h are as narrow as h, (this is suggested by the picture of what the limit solutions look like; they have a tilted tangent cone at 0) so if they were bounded in the x_2 direction by $\sqrt{h}$ we would contradict the volume estimate. The first lower barrier is easy:

First Barrier:

$$u \geq \epsilon(|x_1| + x_1^2/2) + \frac{\Lambda}{\epsilon}x_2^2 - Mx_2 \geq \epsilon(|x_1| - Cx_2),$$

where M is chosen so large that the x_2 polynomial is $\leq 0 \leq u$. This tells us that S_h is contained in roughly a triangle of universal opening and height $2\sqrt{h}$. The idea is that we can improve this barrier using this localization of S_h:

Improvement of Barrier: Let $z = x_1 - Cx_2$. Then $-C\sqrt{h} \leq z \leq ch$ in S_h. Consider the function

$$\frac{\Lambda}{\epsilon^{n-1}}z^2 + \sqrt{h}Mz + \epsilon(|x|^2 + \sqrt{h}x_2).$$

By choosing ϵ small we can get the second two terms smaller than $h/2$ since $x_2 \leq 2\sqrt{h}$. By choosing M large we can guarantee that the first two terms are ≤ 0 when $z \leq 0$ and we are in S_h since $z \geq -C\sqrt{h}$. Finally, by choosing h_0 small we see that the first two terms are like $h_0^{3/2} < h_0/2$ where $z > 0$ since $z \leq ch$. Thus, this is a lower barrier for u. This gives

$$u \geq \sqrt{h_0}M(|x_1| - (C - \epsilon/M)x_2)$$

in S_h for $h < h_0$, so we have improved the slope of our barrier, i.e. the opening angle of the triangle trapping S_h is smaller by a universal number. Iterating finitely many times, we see that

$$u \geq c(|x_1| + \eta x_2)$$

in S_h for h small, showing that S_h has narrowness of order h but length of order $\sqrt{h}$, a contradiction.

11.3 Some Philosophical Remarks

Heuristically we expect regularity in the situation $\det D^2 u = f$ with f strictly positive and smooth near the boundary and u with quadratic separation because then:

- $|D^2 u|$ is bounded above and below on the boundary by the equation,

- u_{ee} are all subsolutions, so when we step in we expect them to decay if anything, but by the equation they remain bounded above and below, giving uniform ellipticity up to the boundary,

- Evans-Krylov gives us interior $C^{2,\alpha}$ regularity and combined with the boundary Harnack inequality gives global regularity.

Quadratic separation is a natural condition because it arises by "tilting" the picture when we have linear boundary data and uniformly convex domain, and more generally (as we saw) if we have C^3 boundary data and uniformly convex domain. (This is sharp because of the very important homogeneous examples in $2d$ we discussed above).

If on the other hand f goes to 0 near the boundary, the problem is more difficult because some second derivatives are guaranteed to go to 0 and there is no hope for uniform ellipticity (i.e., being subsolutions doesn't help us since the rhs goes to 0). Savin's recent work shows that the tangential second derivatives in fact remain positive as we step in.

A fun remark is that in 2 dimensions the second derivatives are in fact solutions to an elliptic equation with no $0th$ order terms. This gives a heuristic reason why we expect this result in 2 dimensions. To see this:

$$\det D^2 u = 1$$

$$u^{ij} u_{1ij} = 0$$

$$u^{ij} u_{11ij} = u^{ik} u^{jl} u_{1ij} u_{1kl}.$$

Any terms with $i, j, k, l = 1$ we may move to the other side as gradient terms. On the other hand, by the first derivative of the equation we have

$$u^{22} u_{122} = -b^i u_{11i}$$

so in fact the whole right side is a gradient term.

□

12 Optimal Transportation

Given a distribution of goods, how can we transport them to a desired distribution with least cost? This is the question that optimal transportation theory aims to answer.

12.1 Mathematical Formulation

We start with probability measures μ and ν, say with support in smooth bounded domains Ω_1 and Ω_2 respectively, and a cost function $c(x, y)$ giving the cost of moving x to y.

The Monge Problem: Minimize

$$\int_{\Omega_1} c(x, Tx)d\mu$$

over all measure-preserving maps T, i.e. maps such that

$$\nu(B) = \mu(T^{-1}(B))$$

for all measurable $B \subset \Omega_2$. (This just says that the mass ending up at B should be the same as the mass transported to it). The problem with this is that measure-preserving maps might not exist. For example, if we want to transport a point mass to the uniform distribution, we need to "spread" that point out. Furthermore, if T is smooth this is formally

$$d\mu = |\det DT|d\nu,$$

a nonlinear constraint that is not closed under weak notions of convergence in case we wanted to use direct methods to minimize. We take care of the problem as follows.

The Kantorovich Problem: Minimize

$$\int_{\Omega_1 \times \Omega_2} c(x, y)d\sigma(x, y)$$

over all σ with marginals μ, ν. Now point masses can be spread ($\mu \times \nu$ is an obvious candidate, the plan that spreads each point over Ω_2 with distribution ν). Furthermore, the only way a sequence of probability measures doesn't converge is if they give mass drifting to the boundary; the set Π of all candidates avoids this since μ and ν are fixed, so it is tight. By a standard theorem we have weak compactness of Π, giving a general existence theorem:

Theorem 14. *If c is bounded and continuous then the Kantorovich problem has a minimizer.*

Remark 1. *If the support of a minimizer σ is contained in the graph of a map $T : \Omega_1 \to \Omega_2$, modulo sets of μ-measure zero, then T is a solution to the Monge problem. Indeed, that T is measure-preserving follows from the fact that the marginals of σ are μ and ν, and since any competitor for T induces a competitor for σ we have that T must be a minimizer.*

The following example sheds light on the questions of geometry and uniqueness for general costs. Let $\mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$ and $\nu = \sum_{j=1}^{N} \delta_{y_j}$ for points $x_i, y_j \in \mathbb{R}^n$. A general transportation plan is represented by the $N \times N$ matrix

$$(\sigma_{ij})$$

where σ_{ij} is the mass going from x_i to y_j. Clearly, the rows and columns must sum to 1. The cost of transportation is

$$\sum_{i,j} \sigma_{ij} c(x_i, x_j),$$

so this is a linear optimization problem. It is easy to see that the collection B_N of possible σ is convex, so the minimum is achieved at an extremal point. Finally, if an entry is not 0 or 1, we can move alternately along the rows and columns selecting entries between 0 and 1 until we return to the same row or column, creating a "cycle" of points that we can self-consistently perturb (1 up, 1 down, etc.) and maintain the condition that sums of rows and columns are 1 (up to deleting the first point to keep the number in the cycle even). Thus, the only extremal points are the permutation matrices.

We conclude that at least one minimizer is a map taking x_i to y_i (up to permuting the x_i). However, if we take for example $N = 2$ and x_i diagonally opposing corners on a square, y_i the other corners, with cost $c = c(|x - y|)$, it is easy to see that all transportation plans have the same cost, so uniqueness is not true in general.

12.2 The Geometry of Optimal Transportation Maps

The fact that a transportation plan σ is a minimizer strongly restricts the geometry of the map. To explore this take

$$c = |x - y|^2$$

and look at the discrete case. By the above discussion, at least one minimizer comes from a map, and any other map will increase cost:

$$\sum_{i=1}^{N} |x_i - y_i|^2 \leq \sum_{i=1}^{N} |x_{\tau(i)} - y_i|^2$$

for some permutation τ. Since permutations are compositions of cycles, this is equivalent to

$$|x_1 - y_1|^2 + |x_2 - y_2|^2 + ... + |x_k - y_k|^2 \leq |x_2 - y_1|^2 + ... + |x_1 - y_k|^2$$

for any subcollection (up to relabelling) $x_1, ..., x_k, y_1, ..., y_k$. We rewrite this as

$$(x_2 - x_1) \cdot y_1 + ... + (x_1 - x_k) \cdot y_k \leq 0,$$

a geometric condition known as cyclical monotonicity. In fact, this is true of the supports of the minimizer for any distributions:

Proposition 6. *The support of a minimizer σ is cyclically monotonic.*

Heuristically, if $\{(x_i, y_i)\}$ are in the support we can cyclically permute the targets of little balls around x_i to increase cost and keep the same marginals. The following theorem of Rockafeller geometrically characterizes such collections of points:

Theorem 15. *A collection of points in $\mathbb{R}^n \times \mathbb{R}^n$ is cyclically monotonic if and only if there exists a convex function u such that if (x, y) is in the collection then $y \in \partial u(x)$.*

Proof. Fix x_1, y_1 in our cyclically monotonic set and define

$$\phi(x) = \sup\{(x_2 - x_1) \cdot y_1 + ... + (x - x_k) \cdot y_k\}$$

over all collections $x_2, ..., x_k$. By cyclic monotonicity, $\phi(x_1) \leq 0$. Think of cyclic monotonicity as a compability condition that lets us take the plane of desired slope at all x_i, and move them to appropriate heights so that their maximum has the subgradient given by the plane at x_i.

It is clear that ϕ is well-defined and convex. For any (x, y) in the cyclically monotonic set, take an $a < \phi(x)$ so that for some collection we have

$$a + y \cdot (z - x) \leq (x_2 - x_1) \cdot y_1 + ... + (x - x_n) \cdot y_n + y \cdot (z - x) \leq \phi(z)$$

where the second inequality comes from the writing $x = x_{n+1}$ and $y = y_{n+1}$ in the definition of $\phi(z)$. Thus, y is indeed in the subgradient of x. □

As a result we have the following theorem:

Theorem 16. *The support of a minimizer σ lies in the subgradient of a convex function ϕ.*

With additional information on the measures we obtain the important Brenier theorem:

Theorem 17 (Brenier's Theorem). *If $\mu = f\,dx$ with $f > 0$ then there is a convex function u such that ∇u (interpreted as an almost-everywhere map) is a solution to the Monge problem for quadratic cost. Furthermore, u is unique up to constants.*

Proof. By the previous discussion, there is a convex potential u whose subgradient contains the support of a minimizer. Since u is differentiable μ almost everywhere, the support of the minimizer is contained in the graph of ∇u as an almost-everywhere map. Finally, any other convex potential must have the same gradient at points of differentiability (since at such points x, there is only one point y such that (x, y) is in the support of the minimizer). We conclude that u is unique up to constants. □

12.3 Connection to the Monge-Ampère Equation

The convex potential satisfies

$$\int_{\Omega_2} h(y)g(y)dy = \int_{\Omega_1} h(\nabla u(x))f(x)dx$$

for all bounded continuous functions h on Ω_2. If u were smooth, this is equivalent (via change of variable) to

$$\det D^2 u(x) = \frac{f(x)}{g(\nabla u(x))}.$$

However, there are geometric conditions needed to ensure this. For example, the function $\nabla(|x|^2/2 + |x_1|)$ maps the ball to two half-balls and is an optimal map, and if we connect the target with a thin strip it remains singular.

Convexity of Ω_2, for example, is sufficient (Caffarelli) to avoid this situation (but not necessary, try mapping a half annulus to a half ball). The idea is that at a point of non-differentiability we can tilt the supporting plane until we can't any more (it's an extremal point of the set of subgradients at the points), and the resulting slope is arbitrarily close to the slopes of supporting planes at regular points nearby. This gives that the subgradients lie in the convex hull of Ω_2. (If a subgradient were outside we could find an extremal point outside and get a contradiction).

If $f > 0$ and $g > 0$ are bounded below and above we have C^α for the map by Caffarelli's affine-invariance theory outlined above, and higher regularity as well.

Interesting questions include partial regularity when the target is not convex (explored by Figalli and others), behavior when f or g goes to 0 (difficult since one of the eigenvalues of D^2u is guaranteed to go to 0 or ∞ and we are guaranteed not to have uniform ellipticity), and regularity when the cost function is not quadratic.

12.4 Application: Isoperimetric Inequality

The existence of a measure-preserving map T between domains allows us to infer global information from algebraic inequalities for local quantities. A striking example is the Isoperimetric inequality: Given a (say smooth) domain $\Omega \subset \mathbb{R}^n$, we have

$$\frac{|\partial\Omega|_{\mathcal{H}^{n-1}}}{|\Omega|^{\frac{n-1}{n}}} \geq \frac{|\partial B_1|_{\mathcal{H}^{n-1}}}{|B_1|^{\frac{n-1}{n}}}.$$

Since the left side is scaling invariant we may as well assume that $|\Omega| = |B_1|$.

Ignoring regularity issues, we have

$$\det DT = 1$$

and DT is positive (Hessian of a convex function). The AGM inequality implies that

$$\text{div}(T) = tr(DT) \geq n.$$

We tie this local inequality to a global one by the divergence theorem:

$$n|\Omega| \leq \int_\Omega \text{div}(T) = \int_{\partial\Omega} T \cdot \nu.$$

Since T maps into B_1 we have $|T| \leq 1$, so we conclude that

$$n|\Omega| \leq |\partial\Omega|.$$

Since $|\Omega| = |B_1| = \frac{1}{n}|\partial B_1|$, we have the isoperimetric inequality. Note that T is the identity map for $\Omega = B_1$ so all of the inequalities become equalities.

Remark 2. *Since the target is convex, we have that T is in fact a smooth map on the interior by Caffarelli. If $\partial\Omega$ is sufficiently regular we get continuity up to the boundary and everything works. One highlight of the optimal transport approach is that it gives the sharp constants.*

13 The Legendre Transform

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth, strictly convex function. Then the gradient map $x \to \nabla u(x)$ is smooth and invertible, with inverse $X(p)$. The Legendre transform of u is its convex dual:

$$u^*(p) = \sup\{p \cdot x - u(x)\}.$$

One way to see the Legendre transform is to take the plane of slope p tangent to u and look at the height of this plane at 0. It is clear that the above supremum is achieved for $x = X(p)$, so we see that

$$u^*(p) = p \cdot X(p) - u(X(p)).$$

As we vary p by δ, we are rotating the plane with lever $X(p)$ at first order, so we expect that the value at 0 changes by $\delta X(p)$. Indeed, differentiating the expression we get

$$\nabla u^*(p) = X(p) + p \cdot \nabla X(p) - \nabla u(X(p)) \cdot \nabla X(p) = X(p)$$

since $\nabla u(X(p)) = p$. Furthermore, since $X(p)$ is the inverse of the gradient map, we have

$$D^2 u^*(p) = (D^2 u)^{-1}(X(p)).$$

Thus, roughly taking the Legendre transform means finding a convex function with inverts the derivative and has reciprocal eigenvalues of the Hessian.

Convexity: The Legendre transform is convex, as seen by the second derivative matrix being strictly positive or checking directly.

Examples: The Legendre transform of $\frac{1}{2}|x|^2$ is $\frac{1}{2}|p|^2$ as follows: The point with slope p is p. The tangent plane goes from $u^*(p)$ at 0 to $\frac{1}{2}|p|^2$ at p, so $u^*(p) = \frac{1}{2}|p|^2$.

The Legendre transform of the cone $|x|$ is identically 0 and only defined on B_1. If we take $|x|+\frac{1}{2}|x|^2$, the Legendre transform is now defined on all of $\mathbb{R}^n$ and we have $X(p) = p(1-1/|p|)$ for $|p| > 1$, so it is 0 on B_1 then grows $C^{1,1}$ away from B_1.

Partial Legendre Transform

Suppose $\det D^2 u(x, y) = f(y)$. If we take a Legendre transform in x for each fixed y, we get the partial Legendre transform, and now X depends both on p and y.

$$u^*(p, y) = \sup\{p \cdot x - u(x, y)\} = p \cdot X(p, y) - u(X(p, y), y).$$

Roughly, this inverts the eigenvalue in the x-direction $\lambda_1 \to \frac{1}{\lambda_1}$ and flips the other eigenvalue $\lambda_2 \to -\lambda_2 = -f/\lambda_1$, so the transform solves the equation

$$f(y)u^*_{pp} + u^*_{yy} = 0,$$

i.e. if $f = 1$ then the partial Legendre transform is harmonic! To show this rigorously, we compute:

$$u^*_p = X(p, y), u^*_{pp} = \frac{1}{u_{xx}}, u^*_y = -u_y, u^*_{yy} = -u_{yy} - u_{yx}X_y.$$

Since the mixed partials agree, we must have

$$X_y = -u_{xy}/u_{xx},$$

so that

$$u^*_{yy} = -\det D^2 u/u_{xx},$$

giving that u^* solves the aforementioned equation. This is a very useful tool for studying the Monge-Ampere equation on $\mathbb{R}^2$.

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