Singularities in the Calculus of Variations

Connor Mooney

2017

Contents

1 Introduction

In these notes we discuss regularity results for minimizers in the calculus of variations, with a focus on the vectorial case. We then discuss some important singular examples.

The notes follow a mini-course given by the author for the INdAM intensive period "Contemporary research in elliptic PDEs and related topics" in April 2017. I am very grateful for the generous hospitality of Serena Dipierro and her family, Enrico Valdinoci, and l'Università degli Studi di Bari during this time. I am also grateful to the anonymous referee for helpful suggestions that improved the exposition. This work was partially supported by National Science Foundation fellowship DMS-1501152 and by the ERC grant "Regularity and Stability in Partial Differential Equations (RSPDE)."

2 Preliminaries

In this section we introduce the main question of the course.

Let $F : M^{m\times n} \to \mathbb{R}$ be a smooth, convex function satisfying

$$\lambda I < D^2 F < \lambda^{-1} I$$

for some positive constant $\lambda \leq 1$. Let $\mathbf{u} = (u^1, ..., u^m) \in H^1(B_1 \subset \mathbb{R}^n; \mathbb{R}^m)$ be the unique minimizer of the functional

$$E(\mathbf{u}) = \int_{B_1} F(D\mathbf{u})\, dx, \tag{1}$$

subject to its own boundary data. A classical example is $F(p) = |p|^2$ (the Dirichlet energy), whose minimizers are harmonic maps.

Exercise: Show the existence and uniqueness of minimizers in $H^1(B_1)$ of (1), subject to the boundary condition $\mathbf{u}|_{\partial B_1} = \psi \in H^1(B_1)$. Use the direct method (take a minimizing sequence).

Hints: Use the bounds on D^2F to find a subsequence that converges weakly in H^1. Use the convexity of F to show that the limit is a minimizer, and the strict convexity to show it is unique.

For classical examples like $F(p) = |p|^2$, minimizers are smooth. The main question of the course is:

Are minimizers always smooth?

Our approach to the regularity problem is to study the PDE that minimizers and their derivatives solve. By minimality we have

$$0 \leq \int_{B_1} (F(D\mathbf{u} + \epsilon D\varphi) - F(D\mathbf{u}))\, dx = \epsilon \int_{B_1} \nabla F(D\mathbf{u}) \cdot D\varphi\, dx + O(\epsilon^2)$$

for all ϵ and all $\varphi \in C_0^\infty(B_1; \mathbb{R}^m)$. In particular, $\mathbf{u}$ solves the Euler-Lagrange system

$$\mathrm{div}(\nabla F(D\mathbf{u})) = \partial_i(F_{p_i^\alpha}(D\mathbf{u})) = 0 \tag{2}$$

in the distributional sense.

Exercise: Show that if $\mathbf{u} \in H^1(B_1)$ solves the Euler-Lagrange system (2), then it is the unique minimizer of (1).

Remark 1. *An interesting question is the uniqueness for (2) in weaker Sobolev spaces. Examples of Šverák-Yan [13] show non-uniqueness in $W^{1,p}$ for $p < 2$. We discuss these examples in Section 4.*

Equation (2) is invariant under translations, and under the Lipschitz rescalings

$$\mathbf{u} \to \mathbf{u}_r = \frac{1}{r}\mathbf{u}(rx).$$

This scaling invariance plays an important role in regularity results. The classical approach to regularity is to differentiate the Euler-Lagrange system. Formally, we have

$$\text{div}(D^2F(D\mathbf{u})D^2\mathbf{u}) = \partial_i(F_{p_i^\alpha p_j^\beta}(D\mathbf{u})u_{kj}^\beta) = 0. \tag{3}$$

We then treat the problem as a linear system for $D\mathbf{u}$ with coefficients $D^2F(D\mathbf{u})$.

Remark 2. *For justification that $D\mathbf{u} \in H^1_{loc}(B_1)$ and solves (3), see the exercises in the next section.*

If $D\mathbf{u}$ is continuous, then the coefficients $D^2F(D\mathbf{u})$ are continuous. By perturbation theory from the constant coefficient case (see e.g. [3]), we obtain that $\mathbf{u}$ is smooth. However, we have no a priori regularity for $D\mathbf{u}$, so we can only assume the coefficients are bounded and measurable. As a result, below we will consider the linear system

$$\text{div}(AD\mathbf{v}) = \partial_i(A^{ij}_{\alpha\beta}(x)v_j^\beta) = 0 \tag{4}$$

in B_1, where $A^{ij}_{\alpha\beta}|^{\alpha,\beta=1,\dots,m}_{i,j=1,\dots,n}$ are bounded measurable coefficients satisfying the ellipticity condition

$$\lambda|p|^2 \leq A(x)(p,\, p) < \lambda^{-1}|p|^2$$

for all $x \in B_1$ and $p \in M^{m\times n}$, and $\mathbf{v} = (v^1,\, ...,\, v^m) \in H^1(B_1;\mathbb{R}^m)$ solves the system in the distribution sense.

Exercise: Show that if $\mathbf{v} \in H^1(B_1)$ solves (4), then $\mathbf{v}$ is a minimizer of the functional

$$J(\mathbf{v}) = \int_{B_1} A(x)(D\mathbf{v},\, D\mathbf{v})\, dx. \tag{5}$$

If one can show that solutions to (4) are continuous, then minimizers of (1) are smooth.

This course consists of two main parts. In the first part (Section 3) we discuss estimates for the linear system (4), and consequences for minimizers of (1). In the second part (Section 4) we discuss some examples that show the optimality of the linear results, and also the optimality of their consequences for minimizers.

In Section 5 we discuss the parabolic case (which was not covered in the lectures). We emphasize some striking differences with the elliptic case.

3 Linear Estimates and Consequences

In this section we discuss the key estimates for solutions to the linear system (4), and their consequences for minimizers of (1).

3.1 Energy Estimate

Recall that solutions to the linear system (4) minimize the energy (5). Thus, the natural quantity controlled by the linear system (4) is the H^1 norm of $\mathbf{v}$. By using minimality or the equation, we can get more precise information.

Exercise: Let φ be a cutoff function that is 1 in $B_{1/2}$ and 0 outside B_1. Use $\mathbf{v}\varphi^2$ as a test function in (4) to derive the Caccioppoli inequality

$$\int_{B_{1/2}} |D\mathbf{v}|^2 \, dx < C(\lambda) \int_{B_1} |\mathbf{v}|^2 |\nabla \varphi|^2. \tag{6}$$

Exercise: Derive the Caccioppoli inequality by using $\mathbf{v}(1 - \epsilon\varphi^2)$ as a competitor for $\mathbf{v}$ in the energy (5). This gives a perhaps more illuminating way to understand the inequality: the energy density of $\mathbf{v}$ cannot concentrate near the center of B_1, since then the energy lost by dilating $\mathbf{v}$ by a factor less than 1 in $B_{1/2}$ is more than the energy paid to reconnect to the same boundary data.

One consequence of the Caccioppoli inequality is the following energy loss estimate

$$\int_{B_{r/2}} |D\mathbf{v}|^2 \, dx < \gamma(n, \lambda) \int_{B_r} |D\mathbf{v}|^2 \, dx, \tag{7}$$

for some $\gamma < 1$ and all $r < 1$. This inequality says that the energy density must "spread evenly at all scales."

Exercise: Prove Inequality (7).

Hints: Reduce to the case $r = 1$ by scaling. Since the system (4) is invariant under adding constant vectors, we can replace $\mathbf{v}$ by $\mathbf{v} - \text{avg.}_{\{B_1 \backslash B_{1/2}\}}\mathbf{v}$ in Inequality (6). (By $\text{avg.}_\Omega \mathbf{v}$ we mean the average of $\mathbf{v}$ in Ω). Finally, note that $\nabla\varphi$ is supported in $B_1 \backslash B_{1/2}$. The result follows by applying the Poincarè inequality in the annulus $B_1 \backslash B_{1/2}$.

As a consequence of the energy loss estimate, we have that the mass of the energy in B_r decays like a power of r:

$$\int_{B_r} |D\mathbf{v}|^2 \, dx < C(n, \lambda) \left(\int_{B_1} |\mathbf{v}|^2 \, dx \right) r^{2\alpha}, \tag{8}$$

for all $r < 1/2$ and some $\alpha > 0$. Inequality (8) is our main result for the linear system (4).

Exercise: Prove Inequality (8) by iterating Inequality (7) on dyadic scales.

Remark 3. *The energy decay estimate (8) says that $D\mathbf{v}$ behaves as if it were in L^q for q slightly larger than 2. It is in fact true that $\mathbf{v} \in W^{1,\,2+\delta}$ for some $\delta > 0$. This result is part of the "reverse-Hölder theory" (see e.g. [3]). This stronger result will not be required for our purposes.*

The energy decay estimate is particularly powerful in the case $n = 2$, due to the invariance of the H^1 norm under the rescaling $\mathbf{v} \to \mathbf{v}(rx)$. More specifically, by standard embeddings for Morrey-Campanato spaces, if

$$\frac{r^2}{|B_r|} \int_{B_r(x)} |D\mathbf{v}|^2 \, dx < Cr^{2\alpha}$$

for all $r < 1/4$ and all $x \in B_{1/2}$, then $\mathbf{v} \in C^\alpha(B_{1/2})$. In particular, in the case $n = 2$, we conclude from the energy decay (8) that $\mathbf{v} \in C^\alpha$.

We conclude by noting that (8) also holds for inhomogeneous systems when the right side is sufficiently integrable.

Exercise: Consider the inhomogeneous system

$$\text{div}(A(x)D\mathbf{v}) = \text{div}(\mathbf{g}),$$

and assume that

$$\int_{B_r} |\mathbf{g}|^2 \, dx < r^{2\beta}$$

for some $\beta > 0$ and all $r < 1$. Repeat the above line of reasoning to show that

$$\int_{B_r} |D\mathbf{v}|^2 \, dx < Cr^{2\gamma}$$

for some $\gamma(n,\, \lambda,\, \beta) > 0$ and $C\left(\int_{B_1} |\mathbf{v}|^2 \, dx,\, n,\, \lambda\right)$.

Hint: Note that the system solved by the rescaling $\mathbf{v}(rx)$ has right side $\text{div}(r\mathbf{g}(rx))$.

Remark 4. *The required condition for $\mathbf{g}$ is satisfied e.g. when $\mathbf{g} \in L^q$ for some $q > 2$. We will use this result when we discuss the parabolic case in Section 5.*

3.2 Consequences for Minimizers

Now we investigate the consequences (8) for minimizers of (1). Below we assume that $\mathbf{u}$ is a minimizer of the regular functional (1).

Exercise: Let $\mathbf{u},\, \mathbf{w} \in H^1(B_1)$ solve the Euler-Lagrange equation (2). Show using the fundamental theorem of calculus that

$$\partial_i \left(\left(\int_0^1 F_{p_i^\alpha p_j^\beta}(D\mathbf{u} + s(D\mathbf{w} - D\mathbf{u})) \, dx \right) (w_j^\beta - u_j^\beta) \right) = 0,$$

i.e. that the difference $\mathbf{w} - \mathbf{u}$ solves a linear system of the type (4).

Exercise: Using the previous exercise for difference quotients $h^{-1}(\mathbf{u}(x+he)-\mathbf{u}(x))$ and the Caccioppoli inequality, justify that $\mathbf{u} \in W^{2,2}_{loc}(B_1)$ and that $D\mathbf{u}$ solves the differentiated Euler-Lagrange equation (3).

As a consequence of the the estimate (8) for linear systems, we have

$$\int_{B_r} |D^2\mathbf{u}|^2 \, dx < Cr^{2\alpha} \tag{9}$$

for some $\alpha > 0$ and all $r < 1/2$.

Exercise: Using embedding theorems from Sobolev and Campanato-Morrey spaces, conclude from Inequality (9) the following results:

- In the case $n = 2$, $D\mathbf{u} \in C^\alpha$, hence $\mathbf{u}$ is smooth.
- In the cases $n = 3$ and $n = 4$, $\mathbf{u} \in C^\beta$ for some $\beta > 0$.
- In the case $n \geq 5$, unbounded minimizers are not ruled out.

Hint: In the case $n = 4$, $W^{2,2}$ embeds into $W^{1,4}$, which nearly embeds to continuous. Using the decay estimate one can improve. Apply the Sobolev-Poincaré inequality to obtain $\int_{B_r} |D\mathbf{u} - (D\mathbf{u})_{B_r}|^4 \, dx < Cr^{4\alpha}$. (Here $(D\mathbf{u})_{B_r}$ is the average in B_r). Then use the Cauchy-Schwarz inequality to reduce to a Morrey-Campanato embedding.

We will show in the next section that when $m > 1$, both the decay estimate (8) and the above consequences for minimizers are optimal. We discuss examples of De Giorgi [2], Giusti-Miranda [4], and Šverák-Yan [12], [13].

Remark 5. *The energy estimate (8) and its consequences for minimizers are due to Morrey, in the 1930s (see e.g. [7]).*

3.3 Scalar Case

The energy decay estimate (8) came from comparison with a simple competitor obtained by slightly deforming $\mathbf{v}$. It is natural to ask whether one can improve upon this result.

As the examples in the next section show, the answer is in general no. However, in the scalar case $m = 1$, one can improve to $\mathbf{v} \in C^\alpha$. The key property of solutions to (4) in the scalar case is the maximum principle: $\mathbf{v}$ never goes beyond its maximum or minimum values on the boundary. Indeed, we get competitors with smaller energy by truncating $\mathbf{v}$ where it goes beyond its boundary data (e.g. if $v \geq 0$ on ∂B_1, then consider $\max\{v, 0\}$). In the vectorial case, making truncations of certain components doesn't send the full gradient to 0, so truncations are not always energetically favorable.

Remark 6. *It is instructive to consider a simple example. In dimensions $n = m = 2$ let F be the quadratic $|p|^2 - 2\epsilon(p_1^1 p_2^2 + p_2^1 p_1^2)$. It is clear that F is uniformly convex for ϵ small. Direct computation shows that $\mathbf{v} = \left(x_1x_2, \, \frac{\epsilon}{2}(|x|^2 - 1)\right)$ is a minimizer of $\int_{B_1} F(D\mathbf{u})\, dx$. However, the second component of $\mathbf{v}$ vanishes on ∂B_1; in particular, the "truncation" $(x_1x_2, \, 0)$ has larger energy. One also checks that $|\mathbf{v}|$ has a local maximum at 0.*

As a consequence of the maximum principle, solutions exhibit oscillation decay in L^∞ when we decrease scale. By quantifying the maximum principle, one can obtain C^α regularity. This breakthrough result is due to De Giorgi [1], and at the same time Nash [8], in the late 1950s.

To illustrate the role of the maximum principle, it is instructive to consider the two dimensional case. Assume that $v \in H^1(B_1;\mathbb{R})$, with $B_1 \subset \mathbb{R}^2$ and

$$\int_{B_1} |\nabla v|^2\,dx \leq 1.$$

Assume further that the maximum and minimum of v on B_r occur on ∂B_r, for all $r < 1$. (For convenience, assume that v is continuous so that we can make sense of these values, and derive a priori estimates). Such v share the key properties of solutions to (4) in the scalar case. We indicate how to use the maximum principle to find a modulus of continuity for v at 0. Let

$$osc_{B_r} v = \max_{B_r} v - \min_{B_r} v = \max_{\partial B_r} v - \min_{\partial_{B_r}} v.$$

Exercise: Show using the fundamental theorem of calculus that

$$\frac{1}{2\pi r}(osc_{B_r} v)^2 \leq \int_{\partial B_r} |\nabla v|^2\,ds.$$

(This is the only place where we use that $n = 2$). Using the maximum principle, show that $osc_{B_r} v$ is increasing with r. Combine with the above inequality to obtain

$$(osc_{B_\delta} v)^2 \frac{|\log(\delta)|}{2\pi} \leq \int_{B_1 \backslash B_\delta} |\nabla v|^2\,dx.$$

Conclude that

$$osc_{B_\delta} v < \left(\frac{2\pi}{|\log \delta|}\right)^{1/2},$$

for all $\delta < 1/2$.

It is instructive to investigate why this argument doesn't work in higher dimensions. Scaling provides a useful explanation. Roughly, if a function v oscillates order 1 on S^{n-1}, then we expect that $\int_{S^{n-1}} |\nabla v|^2\,ds$ is order 1 (see the remark below). If v oscillates order 1 on ∂B_r for all $r > 0$, then applying the the unit-scale estimate to $v_r = v(rx)$ we obtain that the Dirichlet energy on ∂B_r is order r^{n-3}. In the case $n \geq 3$ this is not enough to contradict H^1 boundedness. De Giorgi's argument overcomes this difficulty by using the Caccioppoli inequality for a sequence of truncations of v.

Remark 7. *Even the "expectation" that if v oscillates order 1 on S^{n-1} then $\int_{S^{n-1}} |\nabla v|^2\,dx$ has order at least 1 is not quite true when $n \geq 3$ (unlike the case $n = 2$). Consider for example the functions on $B_1 \subset \mathbb{R}^2$ (rather than S^2, for simplicity) equal to $-\log r/\log R$ on $B_1 \backslash B_{1/R}$ and equal to 1 in $B_{1/R}$. These have small Dirichlet energy going like $(\log R)^{-1}$.*

To conclude the section, we remark that for systems with special structure, we can sometimes find a quantity that solves a scalar equation or inequality. In these cases we have stronger regularity results. Here is an important example due to Uhlenbeck [14].

Assume (like above) that F is a smooth, uniformly convex function on $M^{m\times n}$ with bounded second derivatives. Assume further that F has radial symmetry, i.e. $F(p) = f(|p|)$, with $0 < \lambda \leq f'' \leq \lambda^{-1}$. Let $\mathbf{u}$ be a minimizer to the corresponding functional.

Exercise: Show that $\nabla F(p) = \frac{f'}{|p|}p$. Conclude that the Euler-Lagrange equation is

$$\partial_i \left(\frac{f'}{|D\mathbf{u}|} u_i^\alpha \right) = 0,$$

i.e. that the components of $\mathbf{u}$ solve elliptic equations. Give a variational explanation that each component satisfies the maximum principle.

Hint: If we truncate a component, then $|D\mathbf{u}|$ (hence F) decreases.

As a consequence, minimizers of rotationally symmetric functionals are continuous. We can in fact show that $|D\mathbf{u}|^2$ is a subsolution to a scalar equation (it takes its maxima on the boundary):

Exercise: Show that

$$D^2F(p) = \frac{f'}{|p|}I + \left(f'' - \frac{f'}{|p|}\right)\frac{p\otimes p}{|p|^2}.$$

Conclude that

$$\partial_i \left(\frac{f'}{|D\mathbf{u}|} u_{ik}^\alpha + \left(f'' - \frac{f'}{|D\mathbf{u}|}\right)\frac{u_i^\alpha u_j^\beta}{|D\mathbf{u}|^2} u_{jk}^\beta \right) = 0.$$

Multiply this equation by u_k^α and sum over α and k to conclude that

$$\mathrm{div}(A(x)\nabla|D\mathbf{u}|^2) \geq \lambda|D^2\mathbf{u}|^2, \tag{10}$$

where $A(x)$ are uniformly elliptic coefficients.

By using De Giorgi's results for the inequality (10), one can show that $\mathbf{u}$ is smooth (see e.g. [3], Chapter 7). Radial symmetry for F is one of the few structure conditions known to ensure full regularity of minimizers.

4 Singular Examples

We discuss some examples of singular minimizers. The examples show optimality of the linear estimates, and of their consequences for minimizers of (1)

4.1 Linear Elliptic Examples

Here we describe examples of discontinuous homogeneous solutions to (4), that show the optimality of the energy decay estimate (8) in the vectorial case. The examples are due to De Giorgi [2] and Giusti-Miranda [4] in 1968, about 10 years after the De Giorgi proved continuity of solutions in the scalar case.

We first establish some notation. For $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$ we let $\mathbf{a} \otimes \mathbf{b} \in M^{m\times n}$ act on $\mathbb{R}^n$ by $(\mathbf{a}\otimes\mathbf{b})(x) = (\mathbf{b}\cdot x)\mathbf{a}$. In particular, $(\mathbf{a}\otimes\mathbf{b})^\alpha_i = a^\alpha b_i$. Likewise, if A, $B \in M^{m\times n}$ we let $A\otimes B$ be the linear map on $M^{m\times n}$ defined by $(A\otimes B)(p) = (B\cdot p)A$, where the dot product on matrices is defined by $B\cdot p = \mathrm{tr}(B^T p) = B^\alpha_i p^\alpha_i$. In particular, $A\otimes B$ is a four-index tensor with components $(A\otimes B)^{ij}_{\alpha\beta} = A^\alpha_i B^\beta_j$.

It is natural to start the search for singular examples by considering 0-homogeneous maps, which have a bounded discontinuity at the origin. Let $|x| = r$ and let $\mathbf{v} = \nu := r^{-1}x$ be the radial unit vector. The De Giorgi construction is based on the observation that

$$D\nu = r^{-1}(I - \nu\otimes\nu),$$

the matrix that projects tangent to the sphere, is non-vanishing and is perpendicular to $B := \nu\otimes\nu$ in $M^{n\times n}$. In particular, ν clearly minimizes the functional $\int_{B_1} A(x)(D\mathbf{v},\, D\mathbf{v})\,dx$ for $A = B\otimes B$. (Note that A is zero-homogeneous, with a discontinuity at the origin). Since this functional is degenerate convex (indeed, $A^{ij}_{\alpha\beta}(x)p^\alpha_i p^\beta_j = 0$ when p is perpendicular to $\nu\otimes\nu$), we need to make a small perturbation.

We first do some simple calculations. We compute

$$\Delta\nu = \nabla(\Delta r) = -\frac{n-1}{r^2}\nu, \quad \mathrm{div}\left(\nu\otimes\frac{\nu}{r}\right) = \nu\Delta(\log(r)) = \frac{n-2}{r^2}\nu. \tag{11}$$

Remark 8. *Note that the last expression vanishes in the plane.*

Now we take coefficients

$$A = \delta I_{n^2} + (B + \gamma(I_n - B))\otimes(B + \gamma(I_n - B)).$$

It is useful to think that δ and γ are small, so that A is a perturbation of $B\otimes B$. We compute

$$AD\nu = \delta D\nu + \gamma(n-1)(\gamma D\nu + \nu\otimes\nu/r).$$

Taking the divergence and using (11), we obtain

$$\mathrm{div}(AD\nu) = [-\delta(n-1) + \gamma(n-1)(n-2-\gamma(n-1))]r^{-2}\nu.$$

The example follows provided

$$\delta = (n-2)\gamma - (n-1)\gamma^2 > 0,$$

which is true when $n \geq 3$ and $\gamma < \frac{n-2}{n-1}$.

Thus, ν solves a system of the type (4) in $\mathbb{R}^n \backslash \{0\}$, for $n \geq 3$ and zero-homogeneous coefficients that are analytic away from the origin. It remains to verify that ν solves the equation globally.

Exercise: Show that $\nu \in H^1_{loc}(\mathbb{R}^n)$ when $n \geq 3$, but not $n = 2$. Show that ν solves (4) in B_1 in the sense of distributions when $n \geq 3$.

Hint: use that

$$\int_{B_1} A(D\mathbf{v},\, D\varphi)\, dx = \lim_{r \to 0} \int_{B_1 \backslash B_r} A(D\mathbf{v},\, D\varphi)\, dx,$$

and integrate by parts in the expression on the right.

Remark 9. *There are no discontinuous H^1_{loc} solutions to (4) in $\mathbb{R}^2$, by the energy decay estimate (8). It is interesting that the above approach doesn't even give a nontrivial zero-homogeneous solution to a uniformly elliptic system in $\mathbb{R}^2 \backslash \{0\}$. It is natural to ask whether such solutions exist. In the next section we prove a rigidity result showing that zero-homogeneous solutions to (4) in $\mathbb{R}^2 \backslash \{0\}$ are constant. We also prove a higher-dimensional analogue.*

The above example shows that De Giorgi's results for the scalar case don't extend to the vectorial case. However, observe that in the differentiated Euler-Lagrange equation (3), the coefficients $D^2F(D\mathbf{u})$ depend smoothly on the solution $D\mathbf{u}$. It is natural to ask whether this structure improves regularity.

The above example answers this question in the negative. If we choose in particular $\gamma = \frac{n-2}{n}$ and divide the coefficients by γ^2 we obtain

$$A = I_{n^2} + \left(I_n + \frac{2}{n-2}B\right) \otimes \left(I_n + \frac{2}{n-2}B\right) = I_{n^2} + C(\mathbf{v}) \otimes C(\mathbf{v}),$$

where

$$C = I_n + \frac{4}{n-2}\frac{\mathbf{v} \otimes \mathbf{v}}{1 + |\mathbf{v}|^2}$$

is bounded and depends analytically on $\mathbf{v}$. This example shows that that $\mathbf{v} = \nu$ solves a uniformly elliptic system of the form

$$\text{div}(A(\mathbf{v})D\mathbf{v}) = 0,$$

where A depend analytically on $\mathbf{v} \in \mathbb{R}^n$. The example is due to Giusti and Miranda [4].

We now modify the construction to get unbounded examples. (De Giorgi's original example was actually of this type). Let

$$\mathbf{v} = r^{-\epsilon}\nu.$$

Exercise: Take A of the same form as above, and compute in a similar way that the equation $\text{div}(AD\mathbf{v}) = 0$ in $\mathbb{R}^n \backslash \{0\}$ gives

$$-\delta(n-1-\epsilon)(1+\epsilon) + ((n-1)\gamma - \epsilon)(n-2-(n-1)\gamma - \epsilon) = 0. \tag{12}$$

Hint: For the first term note that $\Delta \mathbf{v} = \nabla(\Delta(r^{1-\epsilon}))/(1-\epsilon)$ (at least when $\epsilon \neq 1$). For the second term, divide $r^{-\epsilon-1}$ into the pieces $r^{-\epsilon}$ and r^{-1}, and use the computation in the example above.

Exercise: Using condition (12), show that $r^{-\epsilon}\nu$ solves a uniformly elliptic system in $\mathbb{R}^n\backslash\{0\}$ for any $\epsilon \neq \frac{n-2}{2}$, and any $n \geq 2$.

Exercise: Show that $r^{-\epsilon}\nu \in H^1$ for $\epsilon < \frac{n-2}{2}$, and in this case $\mathbf{v}$ solves $\text{div}(AD\mathbf{v}) = 0$ in B_1 (in particular, across the origin). Finally, show by taking ϵ arbitrarily close to $\frac{n-2}{2}$ that the decay estimate (8) is optimal in the vectorial case, in any dimension n.

Remark 10. *Observe in particular that for $n = 3, 4$ the examples are the gradients of bounded non-Lipschitz functions, and for $n \geq 5$ the examples are the gradients of unbounded functions.*

Finally, to appreciate fully the vectorial nature of the above examples, it is instructive to make similar constructions in the scalar case.

Exercise: Let $v(x) = r^{-\epsilon}g(\nu) : \mathbb{R}^n \to \mathbb{R}$ be a $-\epsilon$-homogeneous function. Let

$$A = a\nu \otimes \nu + (I - \nu \otimes \nu)$$

for some constant $a > 0$. Show that

$$A\nabla v = r^{-\epsilon-1}(-a\epsilon g\nu + \nabla_{S^{n-1}}g),$$

$$\text{div}(A\nabla v) = r^{-\epsilon-2}(\Delta_{S^{n-1}}g - a\epsilon(n-2-\epsilon)g).$$

(Here $\nabla_{S^{n-1}}$ and $\Delta_{S^{n-1}}$ denote the gradient and Laplace-Beltrami operators on S^{n-1}).

Thus, the equation $\text{div}(A\nabla v) = 0$ becomes the eigenvalue problem

$$\Delta_{S^{n-1}}g = a\epsilon(n-2-\epsilon)\, g$$

on the sphere. The maximum principle enters the picture when we consider the solvability of this problem: we need $a\epsilon(n-2-\epsilon) \leq 0$ to find nonzero solutions.

Provided that either $\epsilon < 0$ or $\epsilon > n-2$, we can find many solutions g on the sphere by choosing $a > 0$ appropriately. In the borderline cases $\epsilon = 0$ or $\epsilon = n-2$ we see that v is radial, hence v is constant resp. the fundamental solution to Δ.

Finally, observe that when $\epsilon < 0$ we have $v \in H^1(B_1)$ and the equation $\text{div}(A\nabla u) = 0$ holds across the origin. These examples show the optimality of the De Giorgi result in the scalar case.

4.2 Rigidity Result for Homogeneity $-\frac{n-2}{2}$.

Above we constructed solutions to (4) in B_1 in the case $n = m \geq 2$ that are homogeneous of degree $-\epsilon$, for any $\epsilon < \frac{n-2}{2}$. This showed the optimality of the energy decay estimate (8). We also found $-\epsilon$-homogeneous maps that solve the system in $\mathbb{R}^n\backslash\{0\}$ for all $\epsilon \neq \frac{n-2}{2}$. It is natural to ask whether there is some rigidity result for this special homogeneity. In this section we verify that this is the case. The main result is:

Theorem 1. *Assume that* $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^m$ *is* $-\frac{n-2}{2}$*-homogeneous and that* A *are bounded, uniformly elliptic coefficients. If*

$$div(AD\mathbf{v}) = 0$$

in $B_1 \backslash \{0\}$*, then* $\mathbf{v}$ *is constant. (In particular,* $\mathbf{v} = 0$ *when* $n \geq 3$*.)*

Proof. Take the dot product of the equation with $\mathbf{v}$ and integrate by parts in $B_1 \backslash B_\epsilon$ to obtain

$$\int_{B_1 \backslash B_\epsilon} A(D\mathbf{v},\, D\mathbf{v})\, dx = \int_{\partial(B_1 \backslash B_\epsilon)} A(D\mathbf{v}, \mathbf{v} \otimes \nu)\, ds.$$

Since $D\mathbf{v} \cdot \mathbf{v}$ is homogeneous of degree $-(n-1)$, the flux of the vector field $AD\mathbf{v} \cdot \mathbf{v}$ through ∂B_r is bounded independently of r. Thus, the right side of the above identity is bounded independently of ϵ. Using the ellipticity of the coefficients, we conclude that

$$\int_{B_1 \backslash B_\epsilon} |D\mathbf{v}|^2\, dx \leq C.$$

However, by the homogeneity of $\mathbf{v}$ we have

$$\int_{B_1 \backslash B_\epsilon} |D\mathbf{v}|^2\, dx \geq |\log \epsilon| \int_{\partial B_1} |D\mathbf{v}|^2\, ds.$$

Taking $\epsilon \to 0$ we conclude from the previous inequalities that $D\mathbf{v} \equiv 0$. □

Exercise: Prove Theorem (1) assuming that A are zero-homogeneous, by working only on the sphere, as follows. Write

$$\mathbf{G} = AD\mathbf{v} \cdot \mathbf{v} = r^{-(n-1)}(f(\nu)\nu + \tau(\nu)),$$

where f is a zero-homogeneous function and τ is tangential to S^{n-1}. Show that the first term is divergence-free, and that the divergence of the second term is $r^{-n}\text{div}_{S^{n-1}}\tau$. Integrate the inequality $\lambda|D\mathbf{v}|^2 \leq \text{div}_{S^{n-1}}\tau$ on the sphere to complete the proof.

Observe that in the case $n = 2$, if $\mathbf{v}$ is zero-homogeneous then $D\mathbf{v}$ has rank one.

Exercise: Show in the case $n = 2$ that Theorem (1) holds when we replace uniform ellipticity with the condition that $A(x)(p,\, p) > \lambda|p|^2$ for rank-one matrices p.

4.3 Null Lagrangian Approach of Šverák-Yan

In this section we discuss an approach to constructing singular minimizers due to Šverák-Yan ([12], [13]). This approach is based on the concept of null Lagrangian. We will discuss the idea in a simple situation.

A null Lagrangian L is a function on $M^{m\times n}$ such that

$$\int_\Omega L(D\mathbf{u})\, dx = \int_\Omega L(D\mathbf{u} + D\varphi)\, dx$$

for all domains Ω and smooth deformations φ supported in Ω. In particular, every map solves the Euler-Lagrange system

$$\text{div}(\nabla L(D\mathbf{u})) = 0.$$

Any linear function is a null Lagrangian. The most important nontrivial example is the determinant.

Exercise: Let $\mathbf{u} = (u^1, u^2)$ be a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ and let Ω be a smooth bounded domain. Verify using integration by parts that

$$\int_\Omega \det D\mathbf{u}\, dx = \int_{\partial\Omega} u^1 \nabla_T u^2\, ds,$$

where ∇_T denote derivative tangential to $\partial\Omega$. Conclude that det is a null Lagrangian. Then compute directly that

$$\text{div}((\nabla \det)(D\mathbf{u})) = \partial_j(\det D\mathbf{u}\,(D\mathbf{u})^{-1}_{ji}) = 0.$$

More generally, sub-determinants are null Lagrangians. Some of the simplest non-trivial null Lagrangians are the quadratic ones. There is a useful characterization of quadratic null Lagrangians:

Exercise: Show that a quadratic form A on $M^{m\times n}$ is a null Lagrangian if and only if $A(p,\, p) = 0$ for all rank-one matrices p. (Recall that p is rank-one if and only if $p = \mathbf{a}\otimes\mathbf{b}$ for some $\mathbf{a}\in\mathbb{R}^m$ and $\mathbf{b}\in\mathbb{R}^n$.)

Hints: To show the "if" direction, use the Fourier transform, and use that $A(b\otimes a,\, c\otimes a) = 0$ for $a \in \mathbb{R}^n$ and $b,\, c \in \mathbb{R}^m$. To show the "only if" direction, use Lipschitz rescalings of a simple periodic test function whose gradients lie on a rank-one convex line. More explicitly, take $\Omega = B_1$, take $\mathbf{u} = 0$, and take $\varphi_\lambda(x) = (bf(\lambda a\cdot x)/\lambda)\eta(x)$, where $a\in\mathbb{R}^n$, $b\in\mathbb{R}^m$, f is periodic, and η is a compactly supported function in B_1 equal to 1 on $B_{1-\epsilon}$ with $|\nabla\eta| < 2/\epsilon$. In the definition of null Lagrangian, take $\lambda\to\infty$, then ϵ to 0. (The idea is to build a map whose gradient has size of order 1 and lies on a rank-one convex line, such that the map is very small in L^∞. We accomplish this by making many oscillations.Then we can cut off without changing the integral much.)

Remark 11. *The quadratic forms A on $M^{m\times n}$ that are null Lagrangians are in fact linear combinations of 2×2 sub-determinants. Indeed, by the previous exercise, they vanish on rank-one matrices. In particular, they vanish on the subspaces $\mathbb{R}^m\otimes e_i$ and $f^\alpha\otimes\mathbb{R}^n$ (where $\{e_i\}$ are the coordinate directions in $\mathbb{R}^n$ and $\{f^\alpha\}$ are the coordinate directions in $\mathbb{R}^m$), giving $A^{ii}_{\alpha\beta} = A^{ij}_{\alpha\alpha} = 0$. They also vanish on $(f^\alpha + f^\beta)\otimes(e_i+e_j)$, where $\alpha\neq\beta$ and $i\neq j$, giving $A^{ij}_{\alpha\beta} + A^{ji}_{\alpha\beta} = A^{ij}_{\alpha\beta} + A^{ij}_{\beta\alpha} = 0$. It follows that $A(p,\, p)$ is a sum of terms of the form $c(p^\alpha_i p^\beta_j - p^\beta_i p^\alpha_j)$.*

The idea of Šverák-Yan is to find a homogeneous map $\mathbf{u}$ such that the image $K := D\mathbf{u}(B_1)$ lies on or close to a subspace of $M^{m\times n}$ on which a null Lagrangian L is convex. Then one can hope to construct a smooth, convex F with the same first-order Taylor expansion as L on $D\mathbf{u}(B_1)$. The Euler-Lagrange equation $\text{div}(\nabla F(D\mathbf{u})) = \text{div}(\nabla L(D\mathbf{u})) = 0$ is then automatically satisfied.

A trivial example illustrating the idea is $\mathbf{u} = r^{-1}\nu : \mathbb{R}^2 \to \mathbb{R}^2$. Then $D\mathbf{u}(B_1)$ lies in the space of traceless symmetric matrices, and the null Lagrangian $L = -\det$ is uniformly convex when restricted to this subspace. An extension of L with the same values and gradients on the symmetric traceless matrices is $F(p) = |p|^2$. Thus, $\mathbf{u}$ is harmonic away from the origin. (Of course, this was clear from the outset since $\mathbf{u} = \nabla \log$). However, $\text{div}(\mathbf{u})$ has a Dirac mass at the origin.

Motivated by this example, consider singular candidates of the De Giorgi type

$$\mathbf{u} = r^{1-\alpha}\nu$$

from $\mathbb{R}^2 \to \mathbb{R}^2$ with $1 < \alpha < 2$. We investigate the geometry of $D\mathbf{u}(B_1)$ in matrix space.

Exercise: Show that the identification

$$(x,\, y,\, z) = \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix}$$

of the symmetric 2×2 matrices with $\mathbb{R}^3$ is an isometry. Show that the surfaces of constant trace are the horizontal planes, and the surfaces of constant determinant are hyperboloid sheets asymptotic to the cone

$$z^2 = x^2 + y^2.$$

Then show that $D\mathbf{u}(S^1)$ is a circle of constant positive trace and constant negative determinant matrices centered on the z-axis, and $D\mathbf{u}(B_1)$ is the cone centered at the origin going through this circle, minus the ball centered at the origin whose boundary passes through the circle.

The above visualization of $D\mathbf{u}(B_1)$ shows that for $\alpha > 1$, the conical surface $D\mathbf{u}(B_1)$ lies "close to" the subspace of traceless symmetric matrices, where $L = -\det$ is convex. We can in fact construct a uniformly convex, smooth function F on $Sym_{2\times 2}$ such that $F = L$ and $\nabla F = \nabla L$ on $D\mathbf{u}(B_1)$.

By homogeneity, it suffices to construct a 2-homogeneous, uniformly convex F such that $F = L$ and $\nabla F = \nabla L$ on the circle $K := D\mathbf{u}(S^1)$. Thus, it suffices to find a smooth, bounded, uniformly convex set $\Sigma \subset Sym_{2\times 2}$ containing the origin, such that $\partial\Sigma$ contains K, and the outer normal to $\partial\Sigma$ is in the direction of ∇L on K. Indeed, then we can let $F = L = \alpha - 1$ on $\partial\Sigma$, and then take the 2-homogeneous extension.

As a first step to constructing Σ, consider the surface $\Gamma = \{L = \alpha - 1\}$. In the coordinates introduced above, Γ is a hyperboloid of revolution around the z axis, asymptotic to $z^2 = x^2 + y^2$. Thus, there is a circular cone centered on the negative z axis that is tangent to Γ on K. This cone divides $Sym_{2\times 2}$ into two components; let Σ_0 be the component containing the origin. Then Σ_0 is convex, and the outer normal to $\partial\Sigma_0$ on K is in the direction of ∇L. From here, it is easy to find a smooth, uniformly convex surface of revolution $\partial\Sigma$ bounding a region $\Sigma \subset \Sigma_0$, such that $\partial\Sigma$ touches $\partial\Sigma_0$ on K and $0 \in \Sigma$. This completes the construction.

Remark 12. *To make F smooth, we need to modify it near the origin. This doesn't affect the equation since $D\mathbf{u}(B_1)$ stays outside a ball around the origin. To make a uniformly convex extension G of F to all of $M^{2\times 2}$, decompose $p \in M^{2\times 2}$ into its symmetric and anti-symmetric parts S and A and let $G(p) = F(S) + |A|^2$.*

Exercises:

- Show that for $1 < \alpha < 2$, the map $\mathbf{u} \in W^{1,p}$ for $p < \frac{2}{\alpha}$. Show that $\mathbf{u}$ solves the Euler-Lagrange equation $\text{div}(\nabla F(D\mathbf{u})) = 0$ in the distribution sense.
- Show that linear maps are minimizers of (1) subject to their own boundary data. Conclude that $\mathbf{w} = x$ is the unique minimizer in $H^1(B_1)$ for the functional corresponding to F.
- Conclude that there is non-uniqueness for the Euler-Lagrange equation (2) in the spaces $W^{1,p}$ for $p < 2$.

Remark 13. *The quadratic analogue of $-\det$ in higher dimensions is $L = -\sigma_2$, where $\sigma_2(M) = \frac{1}{2}(tr(M)^2 - |M|^2)$. Thus, L is a uniformly convex, radial quadratic on the constant-trace symmetric matrices.*

We can repeat the above procedure for $\mathbf{u} = r^{1-\alpha}\nu$ in higher dimensions provided $L > 0$ on $D\mathbf{u}(B_1)$. This gives the condition $\alpha > \frac{n}{2}$. Again, these maps provide counterexamples to uniqueness for (2) in $W^{1,p}$ when $p < 2$.

The above examples are due to Šverák-Yan ([13]). To find examples of singular minimizers to smooth, uniformly convex functionals, more complicated maps are required. In [12], Šverák and Yan use the null Lagrangian technique to show that the one-homogeneous map

$$\mathbf{u}(x) = r\left(\nu \otimes \nu - \frac{1}{3}I\right),$$

viewed as a map from $\mathbb{R}^3$ to the space of symmetric traceless matrices (isomorphic to $\mathbb{R}^5$), is a Lipschitz but not C^1 singular minimizer in the case $n = 3$, $m = 5$. We describe the construction here.

To understand the geometry of $K := D\mathbf{u}(B_1)$, it is useful to use the symmetries of $\mathbf{u}$. Letting $a_{ijk} = \partial_k u^{ij}$ and $R \in SO(3)$, we have

$$\mathbf{u}(Rx) = R\mathbf{u}(x)R^T,$$

$$a_{ijk}(Rx) = R_{il}R_{jm}R_{kn}a_{lmn}(x).$$

It is not hard to check that two invariant subspaces of $\{a_{iik} = 0,\, a_{ijk} = a_{jik}\} \cong M^{5\times 3}$ are the space of traceless tensors $T^0 = \{a_{ikk} = 0\}$, and its orthogonal complement T_3. Among the traceless matrices, two invariant subspaces are the permutation-invariant subspace $T_7 = \{a_{ijk} = a_{jki}\}$, and its orthogonal complement T_5. (The subscripts represent the dimension of the subspace). The quadratic forms invariant under the above action take the simple form $\alpha|X|^2 + \beta|Y|^2 + \gamma|Z|^2$, where X, Y, Z are the projections to T_3, T_7 and T_5 respectively.

Recall that the quadratic null Lagrangians correspond to quadratic forms that vanish in the rank-one directions. By imposing the condition $L(C_{ij}\eta_k) = 0$ for all symmetric traceless C_{ij} and $\eta \in \mathbb{R}^3$ (and using explicit formulae for the projections of a_{ijk} to T_3, T_7 and T_5), Šverák-Yan compute

$$L = 3|X|^2 - 2|Y|^2 + |Z|^2,$$

up to multiplication by constants.

Ideally, we would be able to say that the Y projection of $D\mathbf{u}$ vanishes, so that K lies in a subspace where L is convex. This is not quite the case. However, a computation (see [12]) shows that

$$|D\mathbf{u}_{T_3}|^2 = \frac{64}{15}, \quad |D\mathbf{u}_{T_7}|^2 = \frac{2}{5}$$

and the remaining projection vanishes. Heuristically, the example works because $D\mathbf{u}$ is closer to the T_3 subspace where L is convex.

One can compute explicitly that $L = 12$ on K, and that L separates from its tangent planes quadratically on K:

$$L(Y) - L(X) - \nabla L(X) \cdot (Y - X) = -L(X - Y) > c|Y - X|^2$$

for $X, Y \in K$ and some $c > 0$. This is enough to construct a uniformly convex, smooth function F on $M^{5\times 3}$ with the same first-order expansion as L on K.

Remark 14. *This beautiful example was the first singular minimizer to a smooth, uniformly convex functional in dimension $n = 3$. The first singular minimizer, constructed by Nečas in 1977 [9], was also one-homogeneous and worked in high dimensions.*

It is natural to ask whether the regularity results for minimizers obtained from linear theory (Hölder continuity in dimensions 3 and 4, and possible unboundedness in dimension 5) are optimal. Šverák-Yan accomplish this in [13] using modifications of the above example. More precisely, they consider $\mathbf{u} = r^{1-\alpha}(\nu\otimes\nu - (1/n)I) : \mathbb{R}^n \to M^{n(n+1)/2-1}$. In higher dimensions one can perform the same decomposition of $M^{(n(n+1)/2-1)\times n}$. An important observation is that the coefficient of L in the higher-dimensional version of T_3 (the trace part of a_{ijk}) is n, and the other coefficients remain the same. As a result, the higher the dimension, the better the convexity in the trace subspace. Furthermore, the component of $D\mathbf{u}$ in the direction of this subspace grows roughly linearly with n, while the other component remains bounded.

This allows the construction of increasingly singular examples in higher dimensions. They show quadratic separation of L from its tangent planes on $D\mathbf{u}(B_1)$ when $0 \leq \alpha < C(n)$, where $C(n) > 0$ for $n \geq 3$ and increases with n. In the cases $n = 3$ and $n = 4$ this gives non-Lipschitz minimizers. Furthermore, a careful computation shows that $C(5) > 1$, providing examples of unbounded singular minimizers in the optimal dimension.

Remark 15. *The most recent approach to constructing singular examples is based on constructing a singular minimizer to a degenerate convex functional in the scalar case, and coupling two such minimizers together in a way that removes the degeneracy. Using this approach, Savin and the author constructed a one-homogeneous singular minimizer in the minimal dimensions $n = 3$, $m = 2$ (see [6]).*

5 Parabolic Case

In the final section we discuss the regularity problem for the parabolic case. To emphasize ideas, we assume that solutions are smooth and obtain a priori estimates.

The gradient flow $\mathbf{u} : Q_1 = B_1 \times [-1, 0) \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ of the regular functional (1) solves

$$\partial_t \mathbf{u} - \operatorname{div}(\nabla F(D\mathbf{u})) = 0. \tag{13}$$

Differentiating (13), we see that the space and time derivatives of $\mathbf{u}$ solve a linear uniformly parabolic system of the form

$$\mathbf{v}_t - \operatorname{div}(A(x, t)D\mathbf{v}) = 0, \tag{14}$$

where $A(x, t)$ are bounded measurable, uniformly elliptic coefficients.

Remark 16. *As in the elliptic case, the coefficients of the system obtained by differentiating (13) depend smoothly on $D\mathbf{u}$. By perturbation theory, a continuity result for (14) implies smoothness for gradient flows. Around the same time that De Giorgi proved continuity of solutions to the scalar elliptic problem, Nash showed continuity of solutions to (14) in the scalar case. Again, the maximum principle plays an important role. We will focus on the vectorial case.*

We will discuss the key estimate for (14), its consequences for gradient flows, and some singular examples. While much of the theory is motivated by the elliptic case, some of the parabolic results required significant new ideas, and some have no elliptic analogue. We emphasize these differences in the discussion.

5.1 Linear Estimates

The classical energy estimate for (14) says that the L^2 norm of $\mathbf{v}$ is controlled uniformly in time, and the H^1 norm is controlled on average in time:

$$\sup_{\{t>-1/2\}} \int_{B_{1/2}} |\mathbf{v}|^2 \, dx + \int \int_{Q_{1/2}} |D\mathbf{v}|^2 \, dx \, dt < C(n, \lambda) \int \int_{Q_1} |\mathbf{v}|^2 \, dx \, dt. \tag{15}$$

(Here Q_r is the parabolic cylinder $B_r \times [-r^2, 0)$.)

Recall that in the elliptic case, linear estimates give control on the H^2 norms of minimizers, and in the case $n = 2$ the H^2 norm is invariant under the natural rescalings that preserve the Euler-Lagrange equation. In the parabolic case, the natural scaling is

$$\mathbf{u} \to \mathbf{u}_r = r^{-1}\mathbf{u}(rx, r^2 t).$$

The quantities controlled in (15) obtained by taking $\mathbf{v} = D\mathbf{u}$ are not scaling-invariant in the case $n = 2$; under the above scaling, they increase by a factor of r^{-2}. However, for $\mathbf{v} = \partial_t \mathbf{u}$, the quantities controlled are (15) are scaling-invariant in the case $n = 2$. Roughly, one time derivative plays the role of two spatial derivatives.

This observation suggests the following approach: obtain a version of "energy decay" for (14), and apply it to the "second-order" quantity $\mathbf{v} = \mathbf{u}_t$. Nečas and Šverák accomplished this in 1991 (see [10]). Precisely, they show for solutions $\mathbf{v}$ to (14) that

$$\sup_{t>-1/2} \int_{B_{1/2}} |\mathbf{v}|^\gamma \, dx < \infty, \tag{16}$$

for some $\gamma > 2$. We can then treat the parabolic system (13) as an elliptic system for each fixed time.

Remark 17. *If we could apply (16) directly to $D^2\mathbf{u}$, we recover the elliptic result uniformly in time. We cannot do this, since the second derivatives don't solve (14). The key observation is that (16) does apply to $\mathbf{u}_t$, and estimates for $\mathbf{u}_t$ are as good as estimates for $D^2\mathbf{u}$ by elliptic theory.*

We sketch the argument here.

Exercise:

- Derive the energy estimate (15) by taking the time derivative of $\int_{B_1} |\mathbf{v}|^2 \varphi^2 \, dx$, where φ is a spacetime cutoff function that is 1 in $Q_{1/2}$ and vanishes outside Q_1.
- Apply the Sobolev inequality to the second term in (15) and use the interpolation

$$\int_{B_1} w^{2+2/q} \, dx < \left(\int_{B_1} w^{2^*} \, dx \right)^{2/2^*} \left(\int_{B_1} w^2 \, dx \right)^{1/q}$$

  to conclude that $\mathbf{v} \in L^\gamma_{loc}(Q_1)$ for some $\gamma > 2$. (Here q is the Hölder conjugate of $2^*/2$).
- Apply the same procedure as in the first exercise to the integral of $|\mathbf{v}|^\gamma$ to obtain

$$\sup_{t>-1/2} \int_{B_{1/2}} |\mathbf{v}|^\gamma \, dx + \int \int_{Q_{1/2}} |\mathbf{v}|^{\gamma-2} \left(\lambda |D\mathbf{v}|^2 - (\gamma-2)\gamma A \left(\frac{\mathbf{v} \otimes \mathbf{v}}{|\mathbf{v}|^2} (D\mathbf{v}), D\mathbf{v} \right) \right) dx \, dt$$
$$< C \int \int_{Q_{3/4}} |\mathbf{v}|^\gamma \, dx \, dt.$$

  Conclude that if $\gamma - 2 = \delta > 0$ is small, then $\mathbf{v}$ is bounded in $L^{2+\delta}(B_{1/2})$ uniformly in $t > -1/4$.

Remark 18. *Observe that the improved parabolic energy estimate does not imply continuity of solutions to (14) in the case $n = 2$, unlike in the elliptic case.*

As a consequence of the Nečas-Šverák result, the regularity results for gradient flows coming from linear theory mirror those of the elliptic case. Let $\mathbf{u}$ solve (13).

Exercise:

- Using (15) for $\mathbf{v} = D\mathbf{u}$, show that $D^2\mathbf{u} \in L^2_{loc}(Q_1)$. Conclude from the equation (13) that $\mathbf{u}_t \in L^2_{loc}(Q_1)$.

- Using the previous exercise, show that $\mathbf{u}_t \in L^\gamma_{loc}(Q_1)$ for some $\gamma > 2$.

- Using the previous exercise, conclude that $\mathbf{u}_t$ is bounded in $L^{2+\delta}(B_{1/2})$, independently of $t > -1/4$.

- Conclude that $D\mathbf{u}$ solves at each time an inhomogeneous linear elliptic system with right side $D(\mathbf{g})$, where $\mathbf{g}$ is bounded in $L^{2+\delta}(B_{1/2})$ uniformly in $t > -1/4$. (Here $\mathbf{g} = \mathbf{u}_t$). Conclude from the elliptic theory that $\int_{B_r} |D^2\mathbf{u}|^2\, dx < Cr^{2\alpha}$ for some $\alpha > 0$ and all $r < 1/2$, with C independent of $t > -1/4$.

As a consequence, Nečas-Šverák show that $\mathbf{u}$ is smooth in the case $n = 2$, and continuous in the case $n \leq 4$, as in the elliptic case.

5.2 Singularities from Smooth Data

The elliptic examples of De Giorgi, Giusti-Miranda, and Šverák-Yan are of course parabolic examples, with singularities on the cylindrical set $\{x = 0\}$. It is natural to ask for examples that develop a singularity from smooth data. In addition, a difference between the elliptic and parabolic theory is that the energy estimate (15) does not imply continuity of solutions to (14) in the case $n = 2$. (However, a version of it implies that there are no examples of a singularity that persists in time in the case $n = 2$, unlike in higher dimensions.)

In this last section we discuss examples of finite time singularity in the case $n = m \geq 3$ due to Stará-John-Malý [11]. We then describe a more recent example in the case $n = m = 2$ [5].

To find examples of discontinuity from smooth data, it is natural to seek examples that are invariant under parabolic rescalings that preserve zero-homogeneous maps:

$$\mathbf{v}(x,\, t) = \mathbf{V}\left(\frac{x}{\sqrt{-t}}\right), \quad A(x,\, t) = A\left(\frac{x}{\sqrt{-t}}\right). \tag{17}$$

Exercise: Show that imposing the self-similarity (17) reduces (14) to the elliptic system

$$\mathrm{div}(AD\mathbf{V}) = \frac{1}{2} D\mathbf{V} \cdot x. \tag{18}$$

This approach reduces the problem to constructing a global, bounded solution to the elliptic system (18). In [11] Stará-John-Malý use a perturbation of the De Giorgi example of the form $\varphi(r)\nu$, with φ asymptotic to 1 near ∞. The resulting solution $\mathbf{v}$ becomes the De Giorgi example at time $t = 0$.

To simplify computations they make the useful observation that

$$\left(\delta I_{n^2} + \frac{B - \delta D\mathbf{V}}{[(B - \delta D\mathbf{V}) \cdot D\mathbf{V}]^{1/2}} \otimes \frac{B - \delta D\mathbf{V}}{[(B - \delta D\mathbf{V}) \cdot D\mathbf{V}]^{1/2}}\right) \cdot D\mathbf{V} = B.$$

This reduces the problem further to constructing a matrix field $B(x)$ whose divergence is the right side of (18), such that $B \cdot D\mathbf{V} \sim |D\mathbf{V}|^2$ and $|B| \sim |D\mathbf{V}|$. (Here $\sim$ denotes equivalent up to multiplication by positive constants).

Exercise:

- Use the observation that $B := r^{-1}\left((n-2)I_{n\times n} + \nu \otimes \nu\right)$ is divergence-free and $B \cdot D\nu \sim |D\nu|^2$ when $n \geq 3$, to re-derive the De Giorgi example.

- Now take $\mathbf{V} = \varphi(r)\nu$ and take $B = r^{-1}f(r)\nu \otimes \nu + h(r)D\nu$. With the choice of coefficients above, show that the system (18) becomes
$$\frac{f'}{r} + (n-2)\frac{f}{r^2} - (n-1)\frac{h}{r^2} = \frac{1}{2}r\varphi'.$$
(Hint: The left side is just the divergence of B).

- If φ is asymptotically homogeneous of degree zero, then the left side of the system (18) scales like r^{-2}. It is thus natural to take $r\varphi' \sim r^{-2}$. Show that for the the choice $\varphi = \frac{r}{(1+r^2)^{1/2}}$ and $f = \varphi$, we have in dimension $n \geq 3$ that
$$B \cdot D\mathbf{V} \sim |D\mathbf{V}|^2.$$

- Show that f and h depend analytically on φ. Show similarly that the coefficients depend analytically on $\mathbf{V}$ in a neighborhood of the image of $\mathbf{V}$ $(= B_1)$. Conclude that $\mathbf{v} = \mathbf{V}(x/\sqrt{-t})$ solves an equation of the form $\partial_t \mathbf{v} - \mathrm{div}(A(\mathbf{v})D\mathbf{v}) = 0$, with coefficients that depend smoothly on $\mathbf{v}$.

- Taking $\mathbf{V}$ and $\mathbf{B}$ of the above form, show that there are smooth solutions to the uniformly elliptic system $\mathrm{div}(AD\mathbf{V}) = 0$ that approximate $r^{-\epsilon}\nu$ for all $\epsilon < \frac{n-2}{2}$. (Hint: Take φ linear near the origin, and smoothly connect to $r^{-\epsilon}$ near $r = 1$. Then rescale).

This gives a parabolic analogue of the De Giorgi example in dimension $n = m \geq 3$.

It is natural to ask whether a map of the form $\varphi(r)\nu$ can work in two dimensions. The following exercise reveals an important restriction on the "shape" of possible φ:

Exercise: Observe that in the above example, $|\mathbf{V}|$ is radially increasing. For any such map solving the system (18), show that
$$\mathbf{V} \cdot \mathrm{div}(AD\mathbf{V}) \geq 0.$$
Multiply by a cutoff ψ that agrees with 1 in B_1 and integrate by parts to conclude that if $\mathbf{V}$ is bounded, then
$$\int_{B_1} |D\mathbf{V}|^2\,dx < C \inf_{\psi|_{\partial B_1}=1,\,\psi|_{\partial B_R}=0} \int |\nabla\psi|^2\,dx,$$
for each $R > 2$. Show that the quantity on the right approaches 0 as $R \to \infty$ in the case $n = 2$, by taking ψ to be the harmonic function with the given boundary data.

Thus, bounded solutions to (18) with radially increasing modulus are constant in two dimensions.

In [5] we construct a solution to (18) in the case $n = m = 2$, using a different perspective. The idea is to show that for the correct choice of $\varphi(r)$, each component of $\varphi\nu$ solves the scalar version of (18) away from an annulus where the error in each equation is small. By introducing off-diagonal coefficients in this region, we cancel the errors without breaking the uniform ellipticity of the coefficients.

In view of the previous exercise, the function $\varphi(r)$ is not increasing. In fact, the equations fail to hold exactly where φ has a local maximum. The philosophy of the example is to capture in an explicit way how coupling can cancel the regularizing effect of the maximum principle.

Remark 19. *In this example, the coupling coefficients are changing near the maximum of φ. Thus, the coefficients can not be written as functions of the solution. However, by considering a pair of similar maps $(\varphi\nu, \tilde{\varphi}\nu)$ we obtain a solution to (18) in the case $n = 2$, $m = 4$ that is injective into $\mathbb{R}^4$ (see [5]). In this way we get an example with coefficients that depend smoothly on the solution.*

Remark 20. *To obtain examples of L^∞ blowup from smooth data, look for self-similar solutions that are invariant under parabolic rescalings that preserve $-\epsilon$-homogeneous maps:*

$$\mathbf{v}(x, t) = \frac{1}{(-t)^{\epsilon/2}} \mathbf{V} \left(\frac{x}{\sqrt{-t}} \right).$$

This reduces the problem to finding asymptotically $-\epsilon$-homogeneous solutions to a certain elliptic system. The methods described above adapt to this case.

References

[1] De Giorgi, E. Sulla differenziabilità e l'analicità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino cl. Sci. Fis. Fat. Nat.* **3** (1957), 25-43.

[2] De Giorgi, I. Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. *Boll. UMI* **4** (1968), 135-137.

[3] Giaquinta, M. *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Princeton University Press, Princeton, 1983.

[4] Giusti, E.; Miranda, M. Un esempio di soluzione discontinua per un problem di minimo relativo ad un integrale regolare del calcolo delle variazioni. *Boll. Un. Mat. Ital.* **2** (1968), 1-8.

[5] Mooney, C. Finite time blowup for parabolic systems in two dimensions. *Arch. Ration. Mech. Anal.* **223** (2017), 1039-1055.

[6] Mooney, C., Savin, O. Some singular minimizers in low dimensions in the calculus of variations. *Arch. Ration. Mech. Anal.* **221** (2016), 1-22.

[7] Morrey, C. B. *Multiple Integrals in the Calculus of Variations.* Springer-Verlag, Heidelberg, NY (1966).

[8] Nash, J. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* **80** (1958), 931-954.

[9] Nečas, J. Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions of regularity. *Theory of Non Linear Operators*, Abhandlungen Akad. der Wissen. der DDR (1997), Proc. of a Summer School held in Berlin (1975).

[10] Nečas, J., Šverák, V.: On regularity of solutions of nonlinear parabolic systems. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* 18, 1-11 (1991).

[11] Stará, J., John, O., Malý, J.: Counterexample to the regularity of weak solution of the quasilinear parabolic system. *Comment. Math. Univ. Carolin.* 27, 123-136 (1986).

[12] Šverák, V.; Yan, X. A singular minimizer of a smooth strongly convex functional in three dimensions. *Cal. Var. PDE* **10** (2000), no.3, 213-221.

[13] Šverák, V.; Yan, X. Non-Lipschitz minimizers of smooth uniformly convex functionals. *Proc. Natl. Acad. Sci. USA* **99** (2002), no. 24, 15269-15276.

[14] Uhlenbeck, K. Regularity for a class of nonlinear elliptic systems. *Acta Math.* **138** (1977), 219-240.