Singular Solutions to the Monge-Ampère Equation

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ABSTRACT

Singular Solutions to the Monge-Ampère Equation

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This thesis contains the author's results on singular solutions to the Monge-Ampère equation

$$\det D^2u = 1.$$

We first prove that solutions are smooth away from a small closed singular set of Hausdorff $n-1$ dimensional measure zero. We also construct solutions with a singular set of Hausdorff dimension $n-1$, showing that this result is optimal. As a consequence we obtain unique continuation for the Monge-Ampère equation. Finally, we prove an interior $W^{2,1}$ estimate for singular solutions, and we construct an example to show that this estimate is optimal.

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Acknowledgments

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to my parents

Chapter 1

Introduction

The Monge-Ampère equation for a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\det D^2 u = f(\nabla u, u, x). \tag{1.1}$$

Equations of the form (1.1) arise in several contexts. For example, the Gauss curvature K of the graph of u is given by

$$K(x) = \frac{\det D^2 u}{(1 + |\nabla u|^2)^{\frac{n+2}{2}}},$$

so the equation for prescribed Gauss curvature has the form (1.1).

The Monge-Ampère equation also arises in optimal transportation. For quadratic cost, the cost-minimizing map transporting a probability density g on $\mathbb{R}^n$ to another probability density h is the gradient map of a convex function u solving the Monge-Ampère equation

$$\det D^2 u = \frac{g}{h(\nabla u)}$$

in a certain weak sense ([B]).

Many interesting questions involving applications of the Monge-Ampère equation are related to the regularity of solutions. The regularity theory for strictly convex solutions to (1.1) is well-developed. In particular, if f is a positive constant then strictly convex solutions are smooth. However, there are examples of (convex) singular solutions to $\det D^2 u = 1$ which are not strictly convex. In this thesis, we obtain estimates on:

1. the Hausdorff dimension of the singular set where a convex solution to $\det D^2 u = 1$ is not strictly convex, and

2. the integrability of second derivatives for singular solutions.

We also present some interesting consequences of these estimates, including unique continuation.

In this introductory chapter we first discuss some important structural properties of equation (1.1). We then describe previous results for strictly convex solutions and we introduce the notion of singular solution. Finally, we state and discuss our main theorems.

1.1 Structure of the equation

We first make some remarks on the structure of equation (1.1). For u and v smooth one computes

$$\det\left(D^2u + \epsilon D^2 v\right) = \det D^2 u + \epsilon \det D^2 u\, u^{ij} v_{ij} + O(\epsilon^2),$$

where (u^{ij}) is the inverse of the Hessian matrix. Thus, the equation is degenerate elliptic on convex functions, and uniformly elliptic if the (pure) second derivatives are bounded above and below by positive constants. In this paper we only consider convex solutions to (1.1), so we are in the elliptic setting.

Observe that if f is bounded above, then the equation is uniformly elliptic where u is uniformly convex. Ellipticity degenerates where u is flat in some direction.

The next observation, which plays a crucial role in the regularity theory, is that the Monge-Ampère equation is affine invariant. If u solves $\det D^2u = f(x)$ and A is an invertible linear map, then the rescaling

$$\tilde{u}(x) = |\det A|^{-\frac{2}{n}} u(Ax)$$

solves the equation $\det D^2\tilde{u} = f(Ax)$. Heuristically, squeezing a solution in one direction and stretching it in another gives a new solution. This observation, combined with the geometric fact that bounded convex sets are equivalent to ellipsoids (see John's Lemma, lemma 3.2.1 in chapter 3), allows us to rescale level sets so that they look like B_1.

Affine invariance also explains why we don't expect convex solutions to be smooth, even if $f = 1$. By performing anisotropic rescalings that squeeze in some directions but not others, one can build solutions that lose strict convexity in the limit. We discuss this idea further in chapter 2.

Finally, we remark that the level sets of determinant in the space of positive symmetric matrices are convex. Indeed, if M is a positive symmetric matrix and N is perpendicular to M^{-1} (the normal to the level set through M), then

$$\det(M + \epsilon N) = \det M - \epsilon^2 \det M\, M^{ik} M^{jl} N_{ij} N_{kl} + O(\epsilon^3).$$

Since M is positive the second term is negative. Thus, we can apply Evans-Krylov techniques to the Monge-Ampère equation.

1.2 Previous results

Now we describe previous results on the regularity of solutions to $\det D^2 u = 1$.

One consequence of the above observations is that if $f = 1$ and $D^2 u > \gamma I$ for some $\gamma > 0$, then we have interior apriori estimates (in terms of γ) on all the derivatives. Indeed, using the equation we see that the second derivatives of u are bounded above and below, so the equation is uniformly elliptic. The Evans-Krylov theorem for concave uniformly elliptic equations (see e.g. [CC]) says that the second derivatives are in fact continuous. (Actually, the first result in this direction is due to Calabi ([Cal]), see remark 1.2.1). By Schauder theory we have second derivative estimates for the linearized equation, and we can bootstrap to get estimates for all derivatives.

Remark 1.2.1. Using the equation more carefully, Calabi ([Cal]) obtained an estimate on the third derivatives at a point in terms of the maximum of the second derivates in a neighborhood of this point. Thus, once the second derivatives are bounded, they are Lipschitz. Interestingly, in dimensions five or smaller, Calabi's techniques control the third derivatives at a point in terms of the second derivatives at that point. This gives a Liouville theorem (global solutions to $\det D^2 u = 1$ are paraboloids) in dimensions five or smaller which doesn't rely on a C^2 estimate.

Thus, the regularity problem for $f = 1$ is reduced to the problem of bounding the second derivatives. The key result in this direction, due to Pogorelov ([P]), bounds $D^2 u$ above on a level set of u in terms of the geometry of the approximating ellipsoid of this level set, and the maximum of $|\nabla u|$ on this set. The idea is that by the concavity of the equation, the pure second derivatives u_{ee} are subsolutions to the linearized equation and thus cannot have local maxima. Pogorelov applies the maximum principle to the quantity

$$\log(u_{ee}) + \log|u| + \frac{1}{2} u_e^2$$

on the zero level set. The second term acts as a cutoff function.

Remark 1.2.2. Pogorelov's estimate roughly says that we can control all the derivatives of a solution to $\det D^2u = 1$ on the interior of a domain $\Omega \subset \mathbb{R}^n$ in terms of a modulus of *strict* convexity for the solution. Indeed, if u is strictly convex then any supporting plane that touches u on the interior lies strictly below the boundary data, so we can lift it a little to carve out a level set (up to subtracting a linear function) whose geometry depends only on the modulus of convexity of u. However, Pogorelov's estimate requires the solution to be C^4 apriori since we differentiate the equation twice.

This statement was not made rigorous until Caffarelli, Nirenberg and Spruck ([CNS]) solved the Dirichlet problem on uniformly convex domains with C^3 boundary and C^3 boundary data. They accomplish this by obtaining apriori $C^{2,\alpha}$ estimates up to the boundary and using the method of continuity. Interior regularity for any strictly convex solution follows by solving the Dirichlet problem on smooth approximations to level sets and using the interior derivative estimates.

Remark 1.2.3. One actually needs C^3 boundary and boundary data to get second derivative estimates up to the boundary. The idea is to use the cubic expansion to show that the tangential second derivatives at a boundary point are strictly positive. Combining this with the equation, one obtains bounds on all the second derivatives up to the boundary.

To show necessity, look for a solution $u(x, y)$ on $\{y > x^2\}$ with the homogeneity

$$u(x, y) = \frac{1}{\lambda^3}u(\lambda x, \lambda^2 y),$$

which has $C^{2,1}$ boundary data growing cubically in the tangential directions at 0. Letting

$$u(t, 1) = f(t)$$

we have $u(x, y) = y^{3/2}f\left(xy^{-1/2}\right)$, and we get the example if f solves the ODE

$$(3f + tf')f'' - f'^2 = 4.$$

1.3 Singular solutions

We now discuss the notion of singular solution. Pogorelov showed by example that solutions to $\det D^2u = 1$ are not always strictly convex ([P]). Write $x = (x', x_n)$. The Pogorelov solution has the form

$$u(x', x_n) = |x'|^{2-2/n}f(x_n), \quad n \geq 3,$$

and it solves $\det D^2u = 1$ in $\{|x_n| < \rho\}$ for some small ρ. We will discuss the Pogorelov example and more complicated constructions in detail in chapter 2. For now we highlight the important features.

First, one way to motivate this example comes from affine invariance. It is invariant under cylindrical rescalings:

$$u(x', x_n) = \frac{1}{\lambda^{2-2/n}} u(\lambda x', x_n).$$

One can think of u as the rescaling limit of some classical solution in the slab $\{|x_n| < \rho\}$. There are similar solutions, due to Caffarelli ([C3]), which are invariant under transformations preserving subspaces of any dimension strictly less than $\frac{n}{2}$. We discuss these examples in chapter 2.

Remark 1.3.1. Caffarelli also shows in [C3] that the agreement set between a solution to $\det D^2u = 1$ and a tangent plane has dimension strictly less than $\frac{n}{2}$, so his examples are optimal. We give a short new proof of this result (see lemma 3.4.4) in chapter 3. This implies in particular that in two dimensions, solutions are strictly convex, hence smooth. The strict convexity of solutions in two dimensions is a classical result (see e.g. [Al]).

Second, this example agrees with its tangent plane on the x_n axis. If a supporting plane agrees with a solution at more than one point, we call the agreement set a "singularity," and we say that the solution is a "singular solution." The Pogorelov solution has a singularity along the x_n axis. It is smooth away from the singularity and $C^{1,1-2/n}$ in $\{|x_n| < \rho\}$, and as we approach the x_n axis the radial (x') second derivatives blow up while the second derivatives in the e_n direction go to zero.

Singular solutions cannot be classical because at any point on a singularity, one of the pure second derivatives would be zero. However, singular solutions are weak solutions in for example the viscosity and Alexandrov senses (see chapter 3 for a precise definition).

Finally, an important observation is that this example is not global (it is only defined in $\{|x_n| < \rho\}$) and the singularity extends to the boundary. Caffarelli ([C1]) showed that the (convex) agreement set between a solution to $0 < \lambda \leq \det D^2u \leq \frac{1}{\lambda}$ and a tangent plane has no interior extremal points; if it is not a single point, then it must extend to the boundary, and it has only "flat edges" in the interior. This observation, along with affine invariance, allows one to control the geometry of level sets. It facilitated the proof of a $C^{1,\alpha}$ estimate for strictly convex solutions to $\lambda \leq \det D^2u \leq \frac{1}{\lambda}$, and gave a modulus of strict convexity when the boundary data are $C^{1,\beta}$ for any

$\beta > 1 - 2/n$. In light of the Pogorelov example, this gave a more or less optimal characterization of when solutions to $\det D^2u = 1$ are strictly convex, hence smooth.

Remark 1.3.2. Caffarelli's results also allowed the extension of perturbation results and covering arguments from uniformly elliptic theory to strictly convex solutions to the Monge-Ampère equation. In particular, he obtained interior $C^{2,\alpha}$ estimates when the right side is C^α, and $W^{2,p}$ estimates when the right side has small oscillation depending on p ([C2]).

1.4 Main results

The above results give context to the work in this thesis, which gives estimates for singular solutions to $\det D^2u = 1$ which are not strictly convex.

Since there are many examples of singular solutions (see chapter 2), a natural problem is to find the largest possible Hausdorff dimension of the singular set. Our first theorem addresses this problem. We say a convex function u is strictly convex at x_0 if there exists a supporting plane L such that

$$\{u = L\} = \{x_0\}.$$

Our first theorem is:

Theorem 1.4.1. *Assume u is an Alexandrov solution to*

$$\det D^2u \geq 1$$

in $B_1 \subset \mathbb{R}^n$. Then u is strictly convex away from a singular set Σ with

$$\mathcal{H}^{n-1}(\Sigma) = 0.$$

We also construct examples of solutions to $\det D^2u = 1$ with a singular set of Hausdorff dimension as close as we like to $n - 1$, which shows that this estimate is optimal.

By the previous discussion, this theorem implies that solutions to $\det D^2u = 1$ are smooth away from a closed singular set of Hausdorff $n - 1$ dimensional measure zero.

The key new idea is to show that u separates from its tangent plane much faster than quadratically in at least two directions perpendicular to a singular line. We do this by examining the geometry of level sets of $u + |x|^2$ near points in Σ.

Theorem 1.4.1 also implies that the singular set cannot disconnect the domain. We obtain as a result the following unique continuation theorem:

Theorem 1.4.2. *Assume that u and v are Alexandrov solutions to*

$$\det D^2 u = \det D^2 v = f$$

in an open connected set $\Omega \subset \mathbb{R}^n$, with $f \in C^{1,\alpha}(\Omega)$ strictly positive. Then if $\{u = v\}$ has nonempty interior, then u agrees with v in all of Ω.

The idea is that $u - v$ solves a linear equation with Lipschitz coefficients where both are strictly convex. Since solutions are strictly convex on an open connected dense set by theorem 1.4.1, theorem 1.4.2 follows from a classical unique continuation theorem for linear equations.

Finally, our last theorem concerns the integrability of second derivatives for singular solutions. Convex functions are almost $W^{2,1}_{loc}$ because they are locally Lipschitz and the second derivatives are dominated by the Laplace. However, the second derivatives can concentrate on sets of measure zero (take for example the function $|x_n|$, whose second derivatives concentrate on the hyperplane $\{x_n = 0\}$). An interesting question is whether this can happen when the product of second derivatives is between positive constants. De Philippis and Figalli ([DF]) recently proved a $W^{2,1}$ estimate for strictly convex solutions to $0 < \lambda < \det D^2 u < \frac{1}{\lambda}$. A modification of their original argument (see [DFS]) shows that the second derivatives are in fact in $L^{1+\epsilon}$ for some small ϵ depending only on dimension and λ.

It is natural to ask if the second derivatives of singular solutions still have some integrability properties. It is not hard to show (see chapter 6) that the singular examples we construct to show optimality of theorem 1.4.1 are not in $W^{2,1+\epsilon}$ for ϵ as small as we like, so we cannot get a $W^{2,1+\epsilon}$ estimate. On the other hand, theorem 1.4.1 says roughly that the singular set is too small for the second derivatives to concentrate there, giving $W^{2,1}$ regularity. Our last theorem quantifies this result, showing that the second derivatives are in fact in $L \log^{\epsilon} L$:

Theorem 1.4.3. *Assume that*

$$0 < \lambda \leq \det D^2 u \leq \Lambda \quad \text{in } B_1 \subset \mathbb{R}^n, \qquad \|u\|_{L^\infty(B_1)} < K.$$

Then for some $\epsilon(n)$ and $C(n, \lambda, \Lambda, K)$ we have $\Delta u \in L \log^{\epsilon} L$ and

$$\int_{B_{1/2}} \Delta u \, (\log(1 + \Delta u))^{\epsilon} \, dx \leq C.$$

The key idea is to refine our techniques from theorem 1.4.1 to show that u separates from its tangent plane logarithmically faster than quadratic in at least two directions perpendicular to a singular line. We also construct a singular solution to $\det D^2u = 1$ whose second derivatives are not in $L\log^M L$ for some M, showing that theorem 1.4.3 is almost optimal.

Remark 1.4.4. All of our three main theorems are new even in the case $f = 1$. The difficulties of the proofs come from the degeneracy of level sets near singularities. The hypotheses on f in theorems 1.4.2 and 1.4.3 are simply those required to apply the appropriate regularity results for strictly convex solutions away from the singular set.

The paper is organized as follows. In chapter 2 we present some important examples of singular solutions to $\det D^2u = 1$. These include the well-known Pogorelov example and some generalizations. In chapter 3 we prove some results from convex analysis and use them to prove useful localization properties of level sets. We also prove some previous results on the geometry of singularities. In chapter 4 we use the results from chapter 3 to prove theorem 1.4.1, and we construct solutions to $\det D^2u = 1$ with singular sets of Hausdorff dimension as close as we like to $n-1$, showing that theorem 1.4.1 is optimal. In chapter 5 we prove theorem 1.4.2 by applying a classical unique continuation theorem in the set of strict convexity. We also discuss the linear theory. Finally, in chapter 6 we prove theorem 1.4.3 and we construct a singular example showing it is optimal.

Chapter 2

Examples

In this chapter we present some instructive examples of singular solutions to $\det D^2u = 1$. We begin by closely examining the well-known example of Pogorelov. We then discuss more complicated examples, which are constructed in a similar way.

2.1 The Pogorelov example

Harmonic functions on some domain are smooth in the interior even if the boundary data are irregular. In contrast, we do not expect a purely local regularity theory for the Monge-Ampère equation due to affine invariance. Indeed, if we have a smooth solution to

$$\det D^2u = 1$$

in $\{|x_n| < 1\}$ then the anisotropic rescaling

$$u_\lambda(x', x_n) = \frac{1}{\lambda^{2-2/n}}u(\lambda x', x_n)$$

is also a solution. As λ gets large, the second derivatives in the x' directions blow up on the x_n-axis, and we lose strict convexity there.

To construct the well-known Pogorelov solution one searches for functions that are invariant under this rescaling:

$$u(x', x_n) = |x'|^{2-2/n}f(x_n), \quad n \geq 3.$$

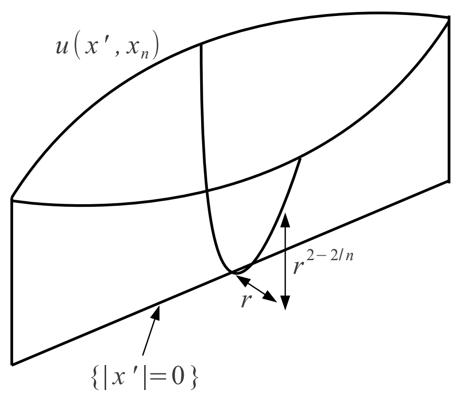

Figure 2.1: Graph of the Pogorelov solution restricted to a plane through the x_n axis.

For appropriate f, this solves $\det D^2u = 1$ in $\{|x_n| < \rho\}$ for some $\rho > 0$. This solution vanishes on the x_n axis, and grows in a $C^{1,1-2/n}$ fashion from this axis. The condition $n \geq 3$ is important, since in two dimensions solutions are in fact strictly convex (see chapter 3).

We now compute the ODE for f and show why we can only solve it for $|x_n| < \rho$. In cylindrical coordinates (r, x_n), the Hessian of $u = r^\alpha f(x_n)$ is

$$D^2u = r^{\alpha-2}\left[\mathrm{diag}(\alpha(\alpha-1)f, \alpha f, ..., \alpha f, r^2 f'') + \alpha r f'(e_r \otimes e_n + e_n \otimes e_r)\right].$$

Taking $\alpha = 2 - 2/n$ and using the equation $\det D^2u = 1$ we find that

$$f^{n-2}\left(ff'' - \frac{\alpha}{\alpha-1}f'^2\right) = c > 0.$$

If we search for solutions with $f(0) = 1$ and $f'(0) = 0$ then by the ODE, f is convex and even, and $f''(0) > 0$. Furthermore,

$$|f'| > cf^{\frac{\alpha}{\alpha-1}} \tag{2.1}$$

where $|f'| > 0$. Since $\frac{\alpha}{\alpha-1} > 1$ it follows from differential inequality 2.1 that f blows up in finite time. The graph of u resembles a half-pipe that closes up at the values of x_n where f blows up (see figure 2.1).

In the following remarks we make some qualitative observations about u and show why they hold more generally for singular solutions.

Remark 2.1.1. Observe that u is a solution in $\{|x_n| < \rho\}$, but not globally. It is in fact true that there are no global singular solutions. If there were, then there are two possibilities. Suppose that 0 is a tangent plane to u and that $\{u = 0\}$ contains more than one point.

The first possibility is that $\{u = 0\}$ has no extremal points. Then it contains a whole line. However, if 0 is a tangent plane to any convex function v and $v = 0$ on a line l in the direction e, then v is constant on all lines in this direction. In particular, ∇v maps to a hyperplane, so $\det D^2 v = 0$.

To see this let L be a supporting plane for v with slope p. If $|p \cdot e| > 0$ then L restricted to l is a nontrivial linear function. In particular, it is positive somewhere on l, contradicting that supporting planes lie below v.

The second possibility is that $\{u = 0\}$ has extremal points. This is ruled out by an important result of Caffarelli ([C1]) which says that singularities have no extremal points. We present this result chapter 3.

Remark 2.1.2. The Pogorelov solution is not a classical solution, but it is a weak solution in for example the Alexandrov sense (see chapter 3). This means that the gradient map preserves volume. This heuristic gives another useful way to predict the exponent $\alpha = 2 - 2/n$: the gradient map of $r^\alpha f(x_n)$ (roughly) takes the cylinder of radius ϵ around the x_n axis to the shape

$$\{|x_n| < r^{\frac{\alpha}{\alpha-1}}\}, \quad r < \epsilon^{\alpha-1}$$

(see figure 2.2). The image has volume like $\epsilon^{(n-1)(\alpha-1)+\alpha}$. Imposing volume preservation gives the expected value of α.

This perspective also gives a heuristic reason that in two dimensions solutions are strictly convex. If not, assume by rotation and scaling that $u \geq 0$ and $u = 0$ on the x_2 axis, and by subtracting a linear function of the form ax_1 assume that $u(0, x_1) = o(x_1)$. By monotonicity of the gradient map for convex functions, the gradient map of u takes a thin triangle with vertices $(0, \pm 1)$ and $(\epsilon, 0)$ into the thin wedge $\{|x_2| < O(\epsilon)x_1\}$, and because the tangent plane at 0 is "tilted as much as possible" the gradient map is also in $\{x_1 < o(1)\}$. Thus, the gradient map shrinks volume near a singularity in two dimensions.

We rigorously prove the strict convexity of solutions in two dimensions in the next chapter, using a technique based on this principle.

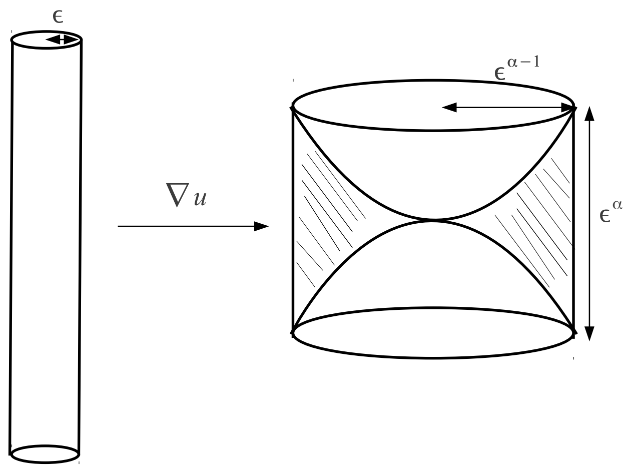

Figure 2.2: The image of the gradient map for solutions of the form $u = r^{\alpha} f(x_n)$.

Remark 2.1.3. We remark that $C^{1,1-2/n}$ is the best possible regularity for a singular solution. One can show using the tools from the next chapter that solutions with $C^{1,\beta}$ boundary data are strictly convex for $\beta > 1-2/n$. Also, since the second derivatives blow up like $r^{-2/n}$ the Pogorelov example is $W^{2,p}$ for $p < \frac{n(n-1)}{2}$.

2.2 Generalizations of the Pogorelov example

The Pogorelov example is cylindrically symmetric, degenerates along a line segment, and is differentiable but not C^2. In this section we construct cylindrically symmetric analogues to the Pogorelov example that degenerate on larger subspaces. We also construct similar examples that are merely Lipschitz. Finally, we present examples that have different homogeneity, which will be useful for later constructions.

Examples that degenerate on larger subspaces:

By looking for solutions which are invariant under rescalings that preserve larger-dimensional subspaces, we find generalizations of the Pogorelov solution that degenerate on subspaces of any dimension strictly less than $\frac{n}{2}$. These are due to Caffarelli ([C3]). We denote a point in $\mathbb{R}^n$ by

(x, y) with $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$, and we write $r = |x|$, $t = |y|$. These solutions have the form

$$u(x, y) = r^{2-\frac{2k}{n}} f(t), \quad n \geq 3, \quad k < \frac{n}{2}.$$

Let $\alpha = 2 - \frac{2k}{n}$. In these coordinates the Hessian is

$$D^2u = r^{\alpha-2} \left(\begin{array}{cccc|cccc} \alpha(\alpha-1)f & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha r f' \\ 0 & \alpha f & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \alpha f & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & \frac{r^2}{t} f' & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & \cdots & \frac{r^2}{t} f' & \cdots \\ \alpha r f' & 0 & \cdots & 0 & 0 & \cdots & 0 & r^2 f'' \end{array} \right).$$

One computes

$$\det D^2u = c \frac{f^{n-k-1} f'^{k-1}}{t^{k-1}} \left(f f'' - \frac{\alpha}{\alpha - 1} f'^2 \right).$$

Taking for example $f(t) = 1 + t^2$ we can produce a solution with smooth right hand side bounded between two positive constants for t bounded.

Remark 2.2.1. These solutions vanish on subspaces of dimension k for any $k < \frac{n}{2}$. They are $C^{1,1-\frac{2k}{n}}$, and $W^{2,p}$ for $p < \frac{n(n-k)}{2k}$. In [C3] Caffarelli shows this dimension is optimal. We give a new proof (see also [M1]) in the next chapter.

Remark 2.2.2. It is not hard to build solutions which vanish on subspaces of dimension k with $k < \frac{n}{2}$ and right hand side exactly 1. Let w be the solution constructed above, multiplied by a large constant so that $\det D^2w > 1$ in the cylinder $Q = \{r < 1, t < 1\}$. Then solve the Dirichlet problem (see e.g. [Gut])

$$\det D^2\bar{u} = 1, \quad \bar{u}|_{\partial Q} = w.$$

Since w is a subsolution we have $w \leq \bar{u}$. It follows that $\bar{u} \geq 0$ when $r = 0$. On the other hand, $\bar{u} = 0$ on $\partial Q \cap \{r = 0\}$, so by convexity $\bar{u}$ vanishes when $r = 0$. This shows explicitly that solutions to the Monge-Ampère equation can have singularities that "propagate from boundary irregularities," unlike harmonic functions which become smooth as soon as we step away from the boundary.

Lipschitz examples:

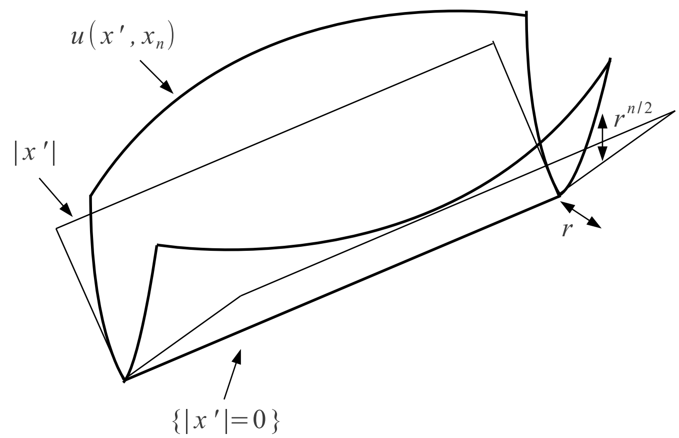

Figure 2.3: Graph of a Lipschitz singular solution.

The examples presented so far are differentiable. We now present a singular solution which is merely Lipschitz, the least possible regularity for a convex function. For $x = (x', x_n)$ with $|x'| = r$ and $x_n = t$ it has the form

$$u(x', x_n) = r + r^{\frac{n}{2}} g(t), \quad n \geq 3.$$

Let $\alpha = \frac{n}{2}$. The Hessian is

$$D^2 u = \frac{1}{r} \text{diag}\left(\alpha(\alpha-1) r^{\alpha-1} g, 1 + \alpha r^{\alpha-1} g, ..., 1 + \alpha r^{\alpha-1} g, r^{\alpha+1} g''\right) + \alpha r^{\alpha-1} g' (e_r \otimes e_n + e_n \otimes e_r).$$

One computes

$$\det D^2 u = c(1 + \alpha r^{\alpha-1} g)^{n-2} \left(g g'' - \frac{\alpha}{\alpha-1} g'^2 \right),$$

so we get solutions in B_1 to an equation with right hand side bounded between two positive constants. The graphs of these solutions are tangent to the graph of $|x'|$ on the x_n axis, and separate from $|x'|$ like $r^{n/2}$ (see figure 2.3).

Remark 2.2.3. Again, when we solve the Dirichlet problem with constant right side and the same boundary data as u, the solution “inherits” the Lipschitz singularity of u.

Remark 2.2.4. The exponent $\frac{n}{2}$ can be explained by studying the gradient map of $u = r + r^{\alpha} f(t)$. The gradient map sends the cylinder of radius ϵ around the x_n axis to the “ring”

$$\{|x_n| < ((r-1)^+)^{\frac{\alpha}{\alpha-1}}\}, \quad r < 1 + \epsilon^{\alpha-1}$$

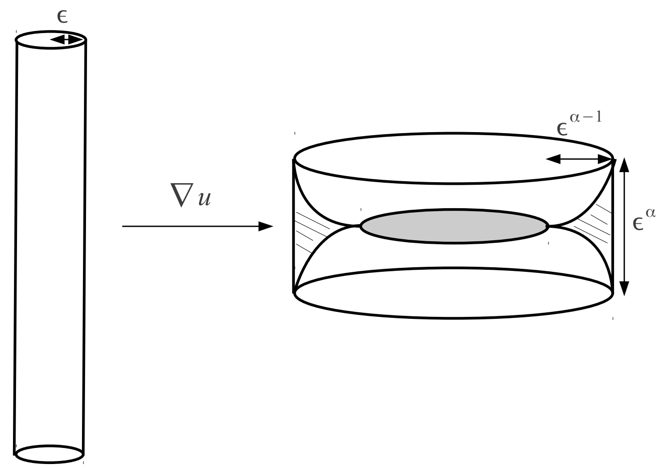

Figure 2.4: The image of the gradient map for solutions of the form $u = r + r^\alpha f(x_n)$.

(see figure 2.4). The map is discontinuous, and sends the x_n axis to a disk in $\{x_n = 0\}$. The volume of this region is like $\epsilon^{2\alpha-1}$. Imposing volume preservation gives $\alpha = \frac{n}{2}$.

Remark 2.2.5. Analagously to the above discussion, there are solutions that grow Lipschitz away from higher-dimensional subspaces. Again using the notation $(x, y) \in \mathbb{R}^n$ with $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$, these solutions can be written

$$u(x, y) = |x| + |x|^{\frac{n-k+1}{k+1}} f(|y|), \quad n \geq 3, \quad k < \frac{n}{2}.$$

Examples with different homogeneity:

The previous examples were all cylindrically symmetric. In future constructions we will use subsolutions that grow at different rates in different directions from a singularity as building blocks for more sophisticated examples. We construct these building blocks here, because they are similar in spirit to the Pogorelov examples.

For simplicity we restrict the discussion to three dimensions. It is easy to generalize the following examples to higher dimensions and to find analogues that degenerate on larger-dimensional subspaces as in the discussion above.

To that end we look for a convex function $w(x_1, x_2, x_3)$ with the homogeneity

$$w(x_1, x_2, x_3) = \frac{1}{\lambda} w(\lambda^{\frac{1}{\alpha}} x_1, \lambda^{\frac{1}{\beta}} x_2, x_3),$$

where α and β satisfy $1 < \alpha,\ \beta < 2$ and

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{2}.$$

We chose this constant so that the homogeneity respects the scaling of the Monge-Ampère equation:

$$\det D^2 w(x_1, x_2, x_3) = \lambda^{2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)-3} \det D^2 w(\lambda^{\frac{1}{\alpha}} x_1, \lambda^{\frac{1}{\beta}} x_2, x_3).$$

Note that the rescaling $(x_1, x_2) \to (\lambda^{\frac{1}{\alpha}} x_1, \lambda^{\frac{1}{\beta}} x_2)$ preserves the curves $x_2 = m x_1^{\alpha/\beta}$.

Let $f(x)$ denote $1 + x^2$. An obvious candidate for w is

$$w(x_1, x_2, x_3) = (x_1^{\alpha} + x_2^{\beta}) f(x_3).$$

One checks that

$$\begin{aligned} \det D^2 w = {} & |x_1|^{2\alpha-2}|x_2|^{\beta-2}\left(2\alpha\beta(\alpha-1)(\beta-1)f^2 - 4\alpha^2\beta(\beta-1)f x_3^2\right) \\ & + |x_1|^{\alpha-2}|x_2|^{2\beta-2}\left(2\alpha\beta(\alpha-1)(\beta-1)f^2 - 4\alpha\beta^2(\alpha-1)f x_3^2\right). \end{aligned}$$

Then for x_3 small depending on α, β we have

$$\det D^2 w \geq c(\alpha, \beta)(|x_1|^{2\alpha-2}|x_2|^{\beta-2} + |x_1|^{\alpha-2}|x_2|^{2\beta-2}).$$

Along the curves $x_2 = m x_1^{\alpha/\beta}$, we compute

$$\det D^2 w \geq c(|m|^{\beta-2} + |m|^{2\beta-2}) \geq c(\alpha, \beta),$$

since $1 < \beta < 2$.

Thus, up to rescaling the x_3-axis and multiplying by a constant, we have

$$\det D^2 w \geq 1 \quad \text{in } \Omega = \{|x'| < 1\} \times (-1, 1).$$

In chapter 4 we construct a function g on $\partial\Omega$ such that translations and "tiltings" of w touch g from below at many pairs of points on the top and bottom of Ω. The solution to the Dirichlet problem $\det D^2 u = 1$, $u|_{\partial\Omega} = g$ inherits the singularity of w on the lines connecting such pairs. By taking α very close to 2, we can produce singular sets of any Hausdorff dimension less than $n-1$. Theorem 1.4.1 shows that this is optimal.

In addition, for any $\epsilon > 0$, by taking α close enough to 2 we produce solutions that are not in $W^{2,1+\epsilon}$, which shows that $W^{2,1}$ regularity is the best we can expect (see chapter 6). See also [M1] for proofs of these results.

Remark 2.2.6. One can also construct subsolutions that grow just logarithmically faster than quadratic away from a singularity in some direction. These subsolutions are

$$w(x', x_3) = g(x')\left(1 + \frac{x_3^2}{|\log g(x')|}\right),$$

where

$$g(x_1, x_2) = x_1^2|\log x_1|^\alpha + \frac{|x_2|}{|\log x_2|^\beta}.$$

In chapter 6 we use these to construct singular solutions to $\det D^2u = 1$ with a singular set with Hausdorff dimension exactly $n-1$, and which have second derivatives in $L\log^c L$ for c small but not $L\log^{20} L$, which shows that Theorem 1.4.3 is optimal. See also [M2] for these results.

Chapter 3

Preliminaries

In this chapter we combine the affine invariance of the Monge-Ampère equation with convexity to obtain some useful localization properties of level sets.

In the first section of this chapter we recall the definition of Alexandrov solutions. In the second section we prove some results from convex analysis including John's lemma on the equivalence of convex sets to ellipsoids, and the Alexandrov maximum principle. We combine these results with the affine invariance of the Monge-Ampère equation in the third section to control the geometry of level sets for solutions with certain bounds on the right hand side. Finally, in the last section we use these observations to prove some previous results on the geometry of singularities.

3.1 Alexandrov solutions

We first recall the notions of Alexandrov solution and sections of convex functions.

Let v be a convex function on $\Omega \subset \mathbb{R}^n$. We say $p \in \mathbb{R}^n$ is a subgradient of v at x if it is the slope of some supporting hyperplane to the graph of v at x. The function v has an associated Borel measure Mv, called the Monge-Ampère measure, defined by

$$Mv(A) = |\partial v(A)|$$

where $|\partial v(A)|$ represents the Lebesgue measure of the set of subgradients of v in A (see [Gut] for details). If $v \in C^2$, then

$$|\partial v(A)| = \int_A \det D^2 v \, dx.$$

Given a Borel measure μ, we say that v is an Alexandrov solution to $\det D^2 v = \mu$ if $Mv = \mu$.

It is easy to check that if $\mu = f\,dx$, then for an invertible linear map L the function $v = |\det(L)|^{-2/n} u(Lx)$ solves

$$\det D^2 v = f(Lx),$$

i.e. the Monge-Ampère equation is affine invariant.

We define the sections $S^v_{h,p}(x)$ of v by

$$S^v_{h,p}(x) = \{y \in \Omega : v(y) < v(x) + p \cdot (y - x) + h\}$$

for some subgradient p at x. Geometrically, for this is the set carved out by taking the tangent plane of slope p at x and lifting it by height h. If $S^v_{h,p}(x)$ is compactly contained in Ω then it is convex, and furthermore if Ω is convex, then any section is convex.

3.2 General results on convex sets and functions

We now discuss some useful results from convex analysis.

John's lemma says that convex subsets of $\mathbb{R}^n$ are equivalent to ellipsoids. This observation allows us to exploit the affine invariance of the Monge-Ampère equation.

Lemma 3.2.1 (John's lemma). *Let K be a bounded convex subset of $\mathbb{R}^n$ with nonempty interior. Then there exists a point $x \in K$ and an ellipsoid E centered at 0 such that*

$$\frac{1}{n} E \subset K - x \subset E.$$

Proof. Let E be the ellipsoid of smallest volume containing K and translate so that 0 is the center of E. We claim that K contains the $\frac{1}{n}$-times dilation of E.

First, the conclusion of the theorem is invariant under affine transformations so we may assume that $E = B_1$. Let B_d be the largest ball contained in K. Using the convexity of K and rotating we may assume that $K \subset \{x_n \geq -d\}$. The strategy is to compute the volume of the "squeezed" ellipsoid E_ϵ with center ϵe_n that goes through e_n and $\partial B_1 \cap \{x_n = -d\}$. This ellipsoid has boundary given algebraically by

$$\partial E_\epsilon = \left\{ \frac{|x'|^2}{a^2} + \frac{(x_n - \epsilon)^2}{(1-\epsilon)^2} = 1 \right\}$$

where a is the horizontal axis length.

Inserting $|x'|^2 = 1 - d^2$ and $x_n = -d$ and doing a short computation we obtain

$$\frac{1}{a^2} = 1 - 2\epsilon\frac{d}{1-d} + O(\epsilon^2).$$

We thus have

$$\frac{|E_\epsilon|}{|B_1|} = (1-\epsilon)\left(\frac{1}{a^2}\right)^{-\frac{n-1}{2}} = 1 - \epsilon\left(\frac{(n-1)d}{1-d} - 1\right) + O(\epsilon^2).$$

The coefficient of ϵ is strictly negative if $d < \frac{1}{n}$, contradicting the definition of E and proving the lemma. $\square$

We call $x + \frac{1}{n}E$ the John ellipsoid of K. There is some linear transformation A such that $A(B_1) = E$, and we say that A normalizes K.

Remark 3.2.2. The dilation factor $\frac{1}{n}$ appearing in John's lemma is optimal. For example, in $\mathbb{R}^2$ equality is attained for triangles. More generally, in $\mathbb{R}^n$ equality is attained for convex polyhedra with $n+1$ faces.

Remark 3.2.3. The approximating ellipsoid can also be centered at the center of mass of K, at the expense of dilating by $\frac{1}{n^{3/2}}$. To see this, take the ellipsoid of smallest volume centered at the center of mass of K and containing K, normalize it to be B_1, and rotate so that $K \subset \{x_n \geq -d\}$. The point is that since 0 is the center of mass, K is squeezed from above as well. Indeed, one computes that $K \subset \{|x_n| < nd\}$. Using a similar computation to the one above, one shows that the squeezed ellipsoid centered at 0 with vertical axis length $1-\epsilon$ and going through $\partial B_1 \cap \{|x_n| = nd\}$ has volume smaller than $|B_1|$ if $d < \frac{1}{n^{3/2}}$ and ϵ is small, a contradiction.

The Alexandrov maximum principle gives a modulus of continuity for a convex function at the boundary of a sublevel set in terms of the Monge-Ampère mass and geometry of this set.

Lemma 3.2.4. *Let v be a convex function on a bounded domain $\Omega \subset \mathbb{R}^n$ with $v|_{\partial\Omega} = 0$. Let D be the diameter of Ω and for $x \in \Omega$, let $d(x)$ denote the distance from x to $\partial\Omega$. Then*

$$|v(x)|^n \leq C(n)d(x)D^{n-1}Mv(\Omega).$$

Proof. Let $C(x)$ be the function whose graph is the cone generated by $(x, v(x))$ and $\partial\Omega$. It is easy to see that $\partial C(x)$ is a convex set containing a ball of radius $\frac{|v(x)|}{D}$ centered at 0 and a point a distance $\frac{|v(x)|}{d(x)}$ from 0, giving

$$|\partial C(x)| \geq c(n)\frac{|v(x)|^n}{D^{n-1}d(x)}.$$

Furthermore, by convexity

$$\partial C(x) \subset \partial v(\Omega).$$

The estimate follows. □

Finally, we give a variant of the Alexandrov maximum principle which controls the minimum value of a convex function on a sublevel set in terms of the Monge-Ampère mass and volume of the set.

Lemma 3.2.5. *Let v be a convex function on a bounded domain $\Omega \subset \mathbb{R}^n$ with $v|_{\partial\Omega} = 0$. Then*

$$Mv(\Omega)\,|\Omega| \geq c(n)|\min_{\Omega} v|^n.$$

Proof. The estimate is affine invariant, so by John's lemma we may assume that $B_1 \subset \Omega \subset B_n$. Let x_0 be the point where $|v|$ achieves its maximum and let $K(x)$ be the function whose graph is the cone generated by $(x_0, v(x_0))$ and ∂B_n. Then $\partial K(x_0)$ contains a ball of radius at least $c(n)|v(x_0)|$ so we have

$$|\partial K(x_0)| \geq c(n)|v(x_0)|^n.$$

Finally, by convexity $\partial K(x_0) \subset \partial v(\Omega)$ and the conclusion follows. □

3.3 Localization properties of sections

When we have control on the Monge-Ampère measure of a convex function, we can combine it with the results above to obtain control on the geometry of level sets.

The first result says that if the right hand side is bounded below by a positive constant then we have an upper bound on the volume, even if the section is not compactly contained in our domain.

Lemma 3.3.1. *Assume that $\det D^2u \geq \lambda$ in a bounded convex domain $\Omega \subset \mathbb{R}^n$. Then if $S^u_{h,p}(x)$ is any section of u, we have*

$$|S^u_{h,p}(x)| \leq C(n, \lambda)h^{n/2}.$$

Proof. Assume by translation that 0 is the center of the John ellipsoid for $S^u_{h,p}(x)$. By subtracting a linear function we can assume that

$$p = 0, \quad u|_{\partial S^u_{h,0}(x)} \leq 0, \text{ and } \quad |\min_{S^u_{h,0}(x)} u| = h.$$

By John's Lemma, there is a linear transformation A that normalizes $S^u_{h,0}(x)$. Let

$$\tilde{u}(x) = |\det A|^{-2/n} u(Ax).$$

It is easy to check that

$$\det D^2 \tilde{u} \geq \lambda, \quad \tilde{u}|_{\partial\tilde{\Omega}} \leq 0$$

and $B_1 \subset \tilde{\Omega} \subset B_{C(n)}$. Then $\frac{\lambda^{1/n}}{2}(|x|^2 - 1)$ is an upper barrier for $\tilde{u}$, so

$$|\min_{\tilde{\Omega}} \tilde{u}| \geq c.$$

Since $|\det A| \geq c(n)|S^u_{h,0}(x)|$, the conclusion follows. □

The next result, due to Caffarelli ([C1]), says that if in addition the right hand side is bounded above, then compactly contained sections $S_h(x)$ are "balanced" around x and we control the volume from both sides.

Lemma 3.3.2. *Assume that $\lambda \leq \det D^2 u \leq \Lambda$ in $\Omega \subset \mathbb{R}^n$. Then there exist $c, C(n, \lambda, \Lambda)$ such that for all $S^u_{h,p}(x) \subset\subset \Omega$, there is an ellipsoid E centered at 0 of volume $h^{n/2}$ with*

$$cE \subset S^u_{h,p}(x) - x \subset CE.$$

Proof. Normalize exactly as in the beginning of the proof of lemma 3.3.1, and let $\tilde{x} = A^{-1}x$. Then $\frac{\lambda^{1/n}}{2}(|x|^2 - 1)$ and $\frac{\Lambda^{1/n}}{2}\left(\frac{|x|^2}{n^2} - 1\right)$ are upper and lower barriers, respectively, for $\tilde{u}$. This gives

$$c \leq |\tilde{u}(\tilde{x})| \leq C,$$

hence

$$ch^{n/2} \leq |\det A| \leq Ch^{n/2}.$$

Finally, by the Alexandrov maximum principle we have

$$d(\tilde{x}, \partial\tilde{\Omega}) > c.$$

We conclude that

$$B_c(\tilde{x}) \subset \tilde{\Omega} \subset B_C(\tilde{x}),$$

and scaling back proves the lemma. □

Finally, using similar techniques and the results from the next section one can establish the following engulfing and covering properties of compactly contained sections (see [CG] and [DFS]). In the following $\alpha S^u_{h,p}(x)$ denotes the α-dilation of $S^u_{h,p}(x)$ around x.

Lemma 3.3.3. *Assume that $\lambda \leq \det D^2 u \leq \Lambda$ in Ω. Then there exists $\delta > 0$ universal such that:*

1. *If $S^u_{h,p}(x) \subset\subset \Omega$ then*
$$S^u_{\delta h,p}(x) \subset \frac{1}{2} S^u_{h,p}(x).$$
2. *Suppose that for some compact $D \subset \Omega$, we associate to each $x \in D$ some $S^u_{h,p}(x) \subset\subset \Omega$. Then we can find a finite subcollection $\{S^u_{h_i,p_i}(x_i)\}_{i=1}^M$ such that $S^u_{\delta h_i,p_i}(x_i)$ are disjoint and*
$$D \subset \cup_{i=1}^M S^u_{h_i,p_i}(x_i).$$

3.4 Previous results on singularities

In this section we describe two important results of Caffarelli on the geometry of a set where a solution to $\lambda < \det D^2 u \leq \Lambda$ agrees with a tangent plane.

The first says that a singularity cannot have any interior extremal points. It extends to the boundary of the domain, possibly with some flat edges in the interior.

Proposition 3.4.1. *Assume that $\lambda \leq \det D^2 u \leq \Lambda$ in a domain $\Omega \subset \mathbb{R}^n$ and let l be a linear function whose graph supports the graph of u. Then the set $\{u = l\}$ has no interior extremal points.*

Proof. By subtracting a linear function assume that $l = 0$. Assume by way of contradiction that $\{u = 0\}$ has an extremal point in Ω. By translating and scaling we may assume that 0 is an extremal point and $\Omega = B_1$. Rotating, we may assume that

$$\{u = 0\} \cap \{x_n \geq 0\} = \{0\}$$

and that for some $d, d' > 0$, the solution u is strictly positive in $(B_1 - B_{1-d'}) \cap \{x_n > -d\}$. Let $l_\epsilon = \epsilon(x_n + d)$. Then the sets $S_\epsilon = \{u < l_\epsilon\}$ are compactly contained in B_1 for ϵ sufficiently small. Write $S_\epsilon = S^u_{h,p}(y)$ for some h, p, y depending on ϵ.

We first argue that $y_n \geq 0$. Indeed, since $u \geq 0$, the maximum of $l_\epsilon - u$ in $\{x_n \leq 0\}$ occurs on $\{x_n = 0\}$. By lemma 3.3.2, $S^u_{h,p}(y)$ has points in $\{x_n \geq cd\}$.

However, since $u > 0$ in $\{x_n > 0\}$, we have

$$S_\epsilon \subset \{x_n < o(1)\}$$

which contradicts the balancing of sections. □

Remark 3.4.2. We saw in previous chapters that the key obstruction to regularity is the loss of strict convexity. The importance of Proposition 3.4.1 is that it implies that solutions with linear boundary data are strictly convex and $C^{1,\alpha}$ with a quantitative interior estimate. The idea is that for a general convex function, $\frac{1}{2}S^u_{h,p}(x) \subset S^u_{h/2,p}(x)$, but using a compactness argument and Proposition 3.4.1 we can improve the dilation constant on the left to $\frac{1+\delta}{2}$. Rescaling and iterating gives the result.

Remark 3.4.3. One can in fact show using proposition 3.4.1 and the volume estimate lemma 3.3.1 that if the boundary data are $C^{1,\beta}$ for $\beta > 1 - \frac{2}{n}$ then solutions are strictly convex. Indeed, since a singularity doesn't have extremal points it contains a line segment that hits the boundary at both ends. If the boundary data were $C^{1,\beta}$ then the volume of the section of height h taken with the plane that agrees with u on this segment is of order $h^{\frac{n-1}{1+\beta}}$, since the section has length of order 1 for all h. The volume estimate gives the conclusion.

Finally, the last result of this chapter says that if the right side is bounded below by a positive constant, then the set where u agrees with a tangent plane has affine dimension strictly less than $\frac{n}{2}$. In light of the examples from chapter 2, this estimate is sharp. Caffarelli proved this result in [C3]. We provide a short proof using a technique related to our proof of theorem 1.4.1. The idea is that if the agreement set has large dimension, then even if u grows as fast as possible (Lipschitz) in the remaining directions, we can find a section whose volume is too large by "tilting" the tangent plane. This proof can also be found in [M1].

Lemma 3.4.4. *Assume*

$$\det D^2 u \geq \lambda > 0$$

in $B_1 \subset \mathbb{R}^n$. Then u cannot vanish on a subspace of dimension $\frac{n}{2}$ or higher.

Proof. Suppose u vanishes on

$$\{x_{k+1} = ... = x_n = 0\} \cap B_1.$$

By subtracting a linear function of the form $a_{k+1}x_{k+1} + ... + a_n x_n$ we may assume that $u(te_n) = o(t)$. Then $S^u_{h,0}(0)$ has length $R(h)h$ in the e_n direction, where $R(h) \to \infty$ as $h \to 0$. Furthermore, $S^u_{h,0}(0)$ has length exceeding $\frac{1}{C}h$ in the $e_{n-k}, ..., e_{n-1}$ directions, where C is the Lipschitz constant of u in $B_{1/2}$. Finally, $S^u_{h,0}(0)$ contains the unit ball in the subspace spanned by $\{e_1, ..., e_k\}$. We conclude that

$$|S^u_{h,0}(0)| \geq C^{-k} R(h) h^{n-k},$$

which contradicts Lemma 3.3.1 as $h \to 0$ for $k \geq \frac{n}{2}$. □

Remark 3.4.5. Lemma 3.4.4 implies in particular that every solution to $\det D^2 u \geq 1$ in two dimensions is strictly convex.

Chapter 4

Partial Regularity

Recall that we say that a convex function u is strictly convex at x if some supporting plane touches u only at x. In this chapter we prove theorem 1.4.1, which says that if $\det D^2 u$ is bounded below by a positive constant, then u is strictly convex away from a small singular set of Hausdorff $n-1$ dimensional measure zero.

Previous results on the singularities of convex functions include those of Alberti, Ambrosio and Cannarsa (see [A], [AAC]), who show that the nondifferentiability set of a semi-convex function is $n-1$ rectifiable. Theorem 1.4.1 may be viewed as a strengthening of this result when we have positive lower *and* upper bounds on $\det D^2 u$, in which case Caffarelli's regularity theory gives differentiability at points of strict convexity (see chapter 3). In fact, if $\det D^2 u = 1$ then Σ is precisely the set where u is not smooth. However, it is important to note that points in Σ may still be points of differentiability for u (see for example the Pogorelov solutions in chapter 2), and without an upper bound on $\det D^2 u$ the points of non-differentiability for u may not be in Σ (take for example $u = |x|^2 + |x_n|$, which solves $\det D^2 u \geq 1$ and is strictly convex everywhere).

The chapter is organized as follows. In the first section we prove theorem 1.4.1 using the results on section geometry from chapter 3 and the useful technique of replacing u by $u + \frac{1}{2}|x|^2$. The key estimate shows that u grows much faster than quadratically in at least two directions perpendicular to a singular line. In the last section we construct, for any δ small, a solution to $\det D^2 u = 1$ with a singular set of Hausdorff dimension $n - 1 - \delta$, which shows that our partial regularity theorem is optimal.

4.1 Proof of Theorem 1.4.1

In this section assume that

$$\det D^2 u \geq 1$$

in $B_1 \subset \mathbb{R}^n$. Fix $x \in \Sigma$ and a subgradient p at x. By translation and subtracting a linear function assume that $x = p = 0$. Then $\{u = 0\}$ contains a line segment of some length l. By Lemma 3.3.1,

$$|S^u_{h,0}(0)| \leq C(n)h^{n/2}$$

for all $h > 0$.

Letting $v = u + \frac{1}{2}|x|^2$, it follows that

$$|S^v_{h,0}(0)| \leq \frac{C(n)}{l}h^{\frac{n+1}{2}}$$

for all h small. In fact, for any $x_0 \in \Sigma$ and subgradient p_0 to v at x_0 we have

$$|S^v_{h,p_0}(x_0)| < Ch^{\frac{n+1}{2}}$$

for some C which may depend on x_0 and p_0. Indeed, p_0 can be written as $p + x_0$ for some subgradient p of u at x_0, and one easily checks that

$$S^v_{h,p_0}(x_0) = S^{u+\frac{1}{2}|x-x_0|^2}_{h,p}(x_0),$$

so by subtracting a linear function with slope p and translating we are in the situation described above.

Theorem 1.4.1 thus follows from the following more general result:

Theorem 4.1.1. *Let v be any convex function on $B_1 \subset \mathbb{R}^n$ with sections $S^v_{h,p}$, and let Σ_v denote the set of points x such that for all subgradients p at x, there is some $C_{x,p}$ such that*

$$|S^v_{h,p}(x)| < C_{x,p}h^{\frac{n+1}{2}}$$

for all h small. Then

$$\mathcal{H}^{n-1}(\Sigma_v) = 0.$$

Proof of Theorem 1.4.1: Let $v = u + \frac{1}{2}|x|^2$. By the discussion preceding the statement of Theorem 4.1.1, $\Sigma \subset \Sigma_v$. The conclusion follows from Theorem 4.1.1. □

We briefly discuss the main ideas of the proof. Fix $x \in \Sigma_v$ and a subgradient p at x. In the following analysis c, C will denote small and large constants depending on n and $C_{x,p}$. If $S^v_{h,p}(x) \subset\subset B_1$ then the definition of Σ_v and Lemma 3.2.5 give

$$Mv(S^v_h(x)) \geq ch^{\frac{n-1}{2}} = c(h^{1/2})^{n-1} \tag{4.1}$$

for all h small.

An important technique of the proof is to replace v by $v + \frac{1}{2}|x|^2$. Since adding a quadratic can only decrease section volume, we have

$$\Sigma_v \subset \Sigma_{v+\frac{1}{2}|x|^2}$$

and it suffices to prove Theorem 4.1.1 for this case. Then all of the sections are compactly contained in B_1 for h small, and the diameter of sections is at most $h^{1/2}$. By replacing the sections $S^v_{h,p}(x)$ by $B_{\sqrt{h}}(x)$ and using a covering argument, we easily obtain that Σ_v has Hausdorff dimension at most $n - 1$.

Lemmas 4.1.2 and 4.1.3 improve this result as follows. We aim to rule out behavior like

$$|x|^2 + |x_n|,$$

which has a singular hyperplane. For this example, the sections at $\{x_n = 0\}$ have the correct growth when we take supporting slopes with no x_n-component, but the sections are too large when we take supporting slopes with x_n-component 1.

In the first lemma we use that the sections are small for *all* supporting planes at $x \in \Sigma_v$ to show that v must grow much faster than quadratically in at least two directions, unlike the example above:

Lemma 4.1.2. *Assume that $v = v_0 + \frac{1}{2}|x|^2$ for some convex function v_0. Fix $x \in \Sigma_v$. For a supporting slope p of v at x, let*

$$d_1(h) \geq d_2(h) \geq ... \geq d_n(h)$$

denote the axis lengths of the John ellipsoid of the section $S^v_{h,p}(x)$. *Then*

$$\frac{d_{n-1}(h)}{h^{1/2}} \to 0 \text{ as } h \to 0.$$

The heuristic idea is that if not, then u resembles $|x|^2 + x_n$ in some system of coordinates, and by tilting the tangent plane we get sections whose volumes are too large.

In the second lemma we use the above observation about the Monge-Ampère mass of v (inequality 4.1) in the directions where v grows much faster than quadratically from x. Since we replaced v by $v + \frac{1}{2}|x|^2$ we also know that v grows at least quadratically in the remaining directions. This allows us to cover Σ_v with balls in which the Monge-Ampère mass of v is much larger than the radius to the $n-1$, giving the desired improvement.

Lemma 4.1.3. *Assume that* $v = v_0 + \frac{1}{2}|x|^2$ *for some convex function* v_0. *Fix* $x \in \Sigma_v$. *For any* $\epsilon > 0$, *there is a sequence* $r_k \to 0$ *such that*

$$Mv(B_{r_k}(x)) > \frac{1}{\epsilon} r_k^{n-1}.$$

The proof of Theorem 4.1.1 follows easily from Lemma 4.1.3.

Proof of Theorem 4.1.1: Since $\Sigma_v \subset \Sigma_{v+\frac{1}{2}|x|^2}$, we may assume without loss of generality that v has the form $v_0 + \frac{1}{2}|x|^2$ with v_0 convex.

Fix ϵ small. By Lemma 4.1.3, for each $x \in \Sigma_v$ we can choose an arbitrarily small r such that

$$Mv(B_r(x)) > \frac{1}{\epsilon} r^{n-1}.$$

Cover $\Sigma_v \cap B_{1/2}$ with such balls, and choose a Vitali subcover $\{B_{r_i}(x_i)\}_{i=1}^N$, i.e. a disjoint subcollection such that $B_{3r_i}(x_i)$ cover $\Sigma_v \cap B_{1/2}$. Then

$$\begin{aligned}\sum_{i=1}^N (3r_i)^{n-1} &\leq C\epsilon \sum_{i=1}^N Mv(B_{r_i}(x_i)) \\ &\leq C\epsilon,\end{aligned}$$

since v is locally Lipschitz and the B_{r_i} are disjoint. This means exactly that

$$\mathcal{H}^{n-1}(\Sigma_v \cap B_{1/2}) = 0.$$

The above reasoning also gives $\mathcal{H}^{n-1}(\Sigma_v \cap B_{1-\beta}) = 0$ for any β small, but not necessarily for $\beta = 0$ since we only know v is locally Lipschitz. To get

$$\mathcal{H}^{n-1}(\Sigma_v \cap B_1) = 0,$$

use that $\Sigma_v \cap B_1 = \cup_{k=1}^{\infty}\{\Sigma_v \cap B_{1-1/k}\}$ and apply countable subadditivity. $\square$

We now prove Lemmas 4.1.2 and 4.1.3.

Proof of Lemma 4.1.2: By translating and subtracting a linear function assume that $x = p = 0$. Assume by way of contradiction that we can find $h_k \to 0$ and some $\delta > 0$ such that

$$d_{n-1}(h_k) > \delta h_k^{1/2} \tag{4.2}$$

for all k. We first show that v is trapped by two tangent planes at 0.

Let $x_{1,k}$ and $x_{2,k}$ be the points on $\partial S^v_{h_k,0}(0)$ where the hyperplanes perpendicular to the shortest axis of the John ellipsoid become tangent to $\partial S^v_{h_k,0}(0)$, and let $p_{1,k}$ and $p_{2,k}$ denote subgradients at these points. Since

$$d_1(h_k)d_2(h_k)...d_n(h_k) < Ch_k^{\frac{n+1}{2}},$$

we have by the inequality 4.2 that $d_n(h_k) < \frac{C}{\delta^{n-1}}h_k$ for all k. By this observation and convexity we can rotate and pass to a subsequence such that

$$p_{1,k} \to c_1(\delta)e_n, \quad p_{2,k} \to -c_2(\delta)e_n.$$

Then v is trapped by the planes $\pm c(\delta)x_n$. We conclude that

$$S^v_{h_k,0}(0) \subset \{|x_n| < C(\delta)h_k\}.$$

To complete the proof, we show that the volumes of sections obtained with tilted supporting planes are too large. Take the largest a such that $v \geq ax_n$ and consider the sections

$$S_k = S^v_{(1+aC(\delta))h_k,ae_n}(0).$$

Then S_k engulf $S^v_{h_k,0}(0)$. Furthermore,

$$\sup\{|x_n| : x \in S_k\} = R_kh_k,$$

where $R_k \to \infty$ as $k \to \infty$. Indeed, if not, then for some small ϵ and a sequence $b_i \to 0$ we would have $v(x', b_i) > (a+\epsilon)b_i$ for all x'. Convexity and $v(0) = 0$ imply that $v > (a+\epsilon)x_n$ for all $x_n > b_i$, which in turn implies that

$$v > (a+\epsilon)x_n,$$

contradicting the definition of a.

Finally, let $x_k = (x_k', R_k h_k) \in S_k$ be the point in S_k furthest in the e_n direction. Since v grows at least quadratically away from every tangent plane, we have

$$|x_k'| < C(\delta, a)h_k^{1/2}. \tag{4.3}$$

Explicitly, since ae_n is a subgradient at 0 and v is of the form $v_0 + \frac{1}{2}|x|^2$ with v_0 convex, we have that ae_n is a subgradient of v_0 at 0, giving

$$ax_n + \frac{1}{2}|x|^2 \leq v \leq C(\delta, a)h_k + ax_n$$

in S_k, giving the desired bound on $|x_k'|$.

Recall that

$$S^v_{h_k,0}(0) \subset \{|x_n| < C(\delta)h_k\} \cap B_{h_k^{1/2}}(0).$$

Take any two points y, z in $\{|x_n| < C(\delta)h_k\} \cap B_{h_k^{1/2}}(0)$ a distance $\delta h_k^{1/2}$ apart, take the lines from these points to $(x_k', R_k h_k)$ and denote the intersections of these lines with $\{x_n = C(\delta)h_k\}$ by $\tilde{y}$ and $\tilde{z}$. Since $|y_n - z_n| < Ch_k$ and $|y - z| > \delta h_k^{1/2}$, it is obvious that $|y' - z'| > \frac{\delta}{2} h_k^{1/2}$ for k large. By similar triangles and inequality 4.3, we also have

$$|y' - \tilde{y}'| = \frac{C}{R_k}|y' - x_k'| \leq \frac{C}{R_k}h_k^{1/2},$$

and we have the same bound on $|z' - \tilde{z}'|$ (see Figure 4.1). We conclude that

$$|\tilde{y} - \tilde{z}| \geq |y' - z'| - |y' - \tilde{y}'| - |z' - \tilde{z}'| \geq (\delta/2 - C/R_k)h_k^{1/2}. \tag{4.4}$$

Since $d_i(h_k) > \delta h_k^{1/2}$ for all $i \leq n-1$, inequality 4.4 (applied to the center of the John ellipsoid for $S^v_{h_k,0}(0)$ and the $n-1$ dimensional ball of radius δh_k it contains) implies that S_k contains the cone with vertex $(x_k', R_k h_k)$ and base containing a ball of radius $(\delta/2 - C(a,\delta)/R_k)h_k^{1/2}$ on the hyperplane $\{x_n = C(\delta)h_k\}$.

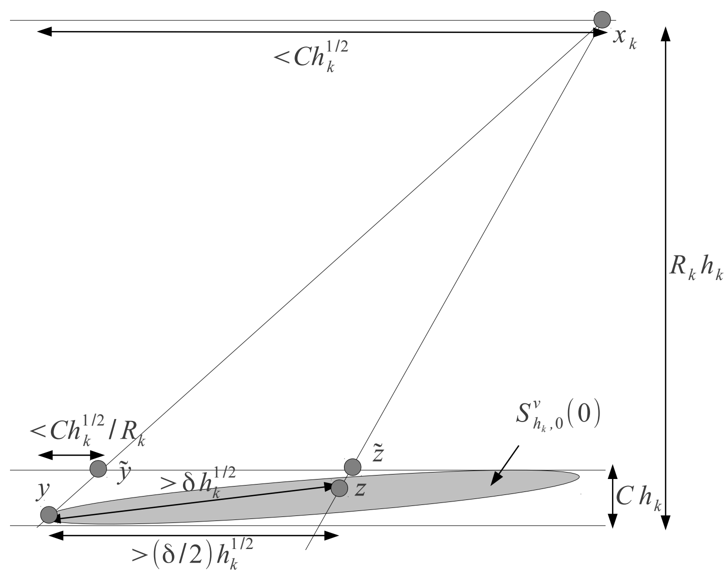

Figure 4.1: The cone above $\{x_n = Ch_k\}$ generated by x_k and the John ellipsoid of $S^v_{h_k,0}(0)$ has a base containing a ball of radius at least $(\delta/2 - C(a,\delta)/R_k)h_k^{1/2}$.

We conclude that

$$|S_k| \geq c(\delta, a) R_k h_k^{\frac{n+1}{2}},$$

contradicting our definition of Σ_v for k large. □

Proof of Lemma 4.1.3: Fix a subgradient p at x and let $d_1(h), ..., d_n(h)$ be defined as in the statement of Lemma 4.1.2. Let

$$I = \min\left\{i : \frac{d_i(h)}{h^{1/2}} \to 0 \text{ as } h \to 0\right\}.$$

Fix δ small. Then we can find a sequence $h_k \to 0$ and η depending only on p such that

$$d_I(h_k) < \delta h_k^{1/2}, \tag{4.5}$$

and

$$d_i(h_k) > \eta h_k^{1/2} \tag{4.6}$$

for all $i < I$. Rotate the axes so that the e_i are the axes for the John ellipsoid of $S^v_{h_k,p}(x)$ and assume by translation that $x = 0$.

Take the restriction of v to the subspace spanned by $e_I, ..., e_n$, and call this restriction w. Let

$$S_k^w = S_{h_k,p}^v(x) \cap \{x_1 = ... = x_{I-1} = 0\},$$

the slice of the section $S_{h_k,p}^v(x)$ in this subspace. Then since

$$d_1(h_k)d_2(h_k)...d_n(h_k) \leq Ch_k^{\frac{n+1}{2}}$$

and v grows at most quadratically in the first $I-1$ directions (inequality 4.6), we have

$$|S_k^w|_{\mathcal{H}^{n-I+1}} \leq \frac{C}{\eta^{I-1}} h_k^{\frac{n+2-I}{2}}.$$

Using this and Lemma 3.2.5,

$$Mw(S_k^w) \geq c\eta^{I-1} h_k^{\frac{n-I}{2}}. \tag{4.7}$$

Finally, let $r_k = C(n)d_I(h_k)$, with $C(n)$ taken large enough that

$$S_k^w \subset B_{r_k/2}(x).$$

By strict quadratic growth in all directions, $\partial v(B_{r_k}(x))$ contains a ball of radius $r_k/2$ around every point in $\partial v(S_k^w)$. (See figure 4.2 for the simple case $n = 3$ and $I = 2$).

It follows that

$$\begin{aligned} Mv(B_{r_k}(x)) &\geq c(n)Mw(S_k^w)r_k^{I-1} \\ &\geq ch_k^{\frac{n-I}{2}} r_k^{I-1} \text{ (inequality 4.7)} \\ &\geq \frac{c}{\delta^{n-I}} r_k^{n-1} \text{ (inequality 4.5)}. \end{aligned}$$

By Lemma 4.1.2 we have $I \leq n-1$, so the conclusion follows. □

Remark 4.1.4. Replacing Σ_v with the set Σ_v^k of points such that

$$|S_{h,p}^v| < C_{x,p}h^{\frac{n+k}{2}}$$

for all h small ($1 \leq k \leq n-1$) and replacing 1 with k in the preceding, one obtains that $\mathcal{H}^{n-k}(\Sigma_v^k) = 0$. If $\det D^2u \geq 1$, such growth happens for $v = u + \frac{1}{2}|x|^2$ at points where u agrees with a linear function on a k-dimensional subspace. This shows that the Hausdorff dimension of the k-dimensional singularities is at most $n-k$. In particular, we recover Lemma 3.4.4 since for $k \geq \frac{n}{2}$ we would have a k-dimensional singularity with Hausdorff k-dimensional measure 0.

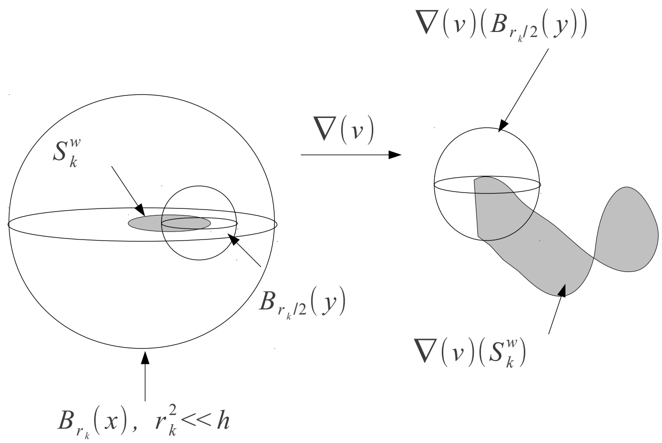

Figure 4.2: $\nabla v(B_{r_k}(x))$ contains an $r_k/2$-neighborhood of the surface $\nabla v(S_k^w)$, which projects in the e_1 direction (down) to a set of $\mathcal{H}^2$ measure at least $\frac{c}{\delta}r_k$.

4.2 Example showing optimality of Theorem 1.4.1

In this section we construct examples of solutions to $\det D^2u = 1$ in $\mathbb{R}^3$ such that Σ has Hausdorff dimension as close to 2 as we like. A small modification produces the analagous examples in $\mathbb{R}^n$.

For this section, fix $\delta > 0$ small. We construct our examples in several steps, which we briefly describe:

1. First, we construct a function w such that

$$\det D^2 w \geq 1$$

in $\Omega := \{|x'| < 1\} \times (-1, 1) \subset \mathbb{R}^3$ that degenerates along $\{x_1 = x_2 = 0\}$ and behaves like $x_1^{2-\delta}$ along the x_1 axis.

2. Next, we construct a standard $S \subset [-1, 1]$ with Hausdorff dimension close to 1 and a convex function v on $[-1, 1]$ such that for any $x \in S$, there is a tangent line such that v separates from this line faster than $r^{2-\delta}$.

3. Finally, we get our example by solving the Dirichlet problem

$$\det D^2u = 1 \quad \text{in } \Omega = \{|x'| < 1\} \times (-1, 1), \qquad u|_{\partial\Omega} = C(\delta)(v(x_1) + |x_2|)$$

and comparing with w at points in $S \times \{0\} \times \{\pm 1\}$.

In the following analysis c and C will denote small and large constants depending on δ.

Construction of w: Let w be the subsolution constructed at the end of chapter 2 with $\alpha = 2 - \delta$. (Recall that w is a rescaling of the function $C(x_1^\alpha + x_2^\beta)(1 + x_3^2)$ with $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{2}$). Then $\det D^2 w \geq 1$ in Ω and has the desired growth in the x_1 direction.

Construction of S: Let $\epsilon > 0$ be a small constant we will choose shortly depending on δ. Construct a self-similar set in $[-1/2, 1/2]$ as follows: First, remove an open interval of length $\gamma = 1 - 2^{-3\epsilon}$ from the center. Proceed inductively by removing intervals a fraction γ of each of those that remains. Denote the centers of the intervals removed at stage k by $\{x_{i,k}\}_{i=1}^{2^{k-1}}$, and the intervals by $I_{i,k}$. Finally, let

$$S = [-1/2, 1/2] - \cup_{i,k} I_{i,k}.$$

It is easy to check that $|I_{i,k+1}| = \gamma 2^{-(1+3\epsilon)k}$ and that S has Hausdorff dimension $\frac{1}{1+3\epsilon}$.

Construction of v: Let

$$v_0(x) = \begin{cases} |x| & |x| \leq 1 \\ 2|x| - 1 & |x| > 1 \end{cases}$$

We add rescalings of v_0 together to produce the desired function:

$$v(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} 2^{-2(1+2\epsilon)k} v_0(2\gamma^{-1} 2^{(1+3\epsilon)k}(x - x_{i,k})).$$

We now check that v satisfies the desired properties:

1. v is convex, as the sum of convex functions. Furthermore,

$$\begin{aligned} |v(x)| &\leq C \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} 2^{-(1+\epsilon)k} \\ &\leq C \sum_{k=1}^{\infty} 2^{-\epsilon k}, \end{aligned}$$

so v is bounded.

2. Let $x \in S$. We aim to show that v separates from a tangent line more than $r^{2-\delta}$ a distance r from x. By subtracting a line assume that $v(x) = 0$ and that 0 is a subgradient at x. Assume further that $x + r < 1/2$ and that $2^{-(1+3\epsilon)k} < r \leq 2^{-(1+3\epsilon)(k-1)}$. There are two cases to examine:

Case 1: There is some $y \in (x + r/2, x + r) \cap S$. Then by the construction of S it is easy to see that there is some interval $I_{i,k+2}$ such that $I_{i,k+2} \subset (x, x + r)$. On this interval, v grows by

$$2^{-2(1+2\epsilon)(k+2)} \geq cr^{2\frac{1+2\epsilon}{1+3\epsilon}} = cr^{2-\delta},$$

where we choose ϵ so that

$$\delta = \frac{2\epsilon}{1+3\epsilon}.$$

Case 2: Otherwise, there is an interval $I_{i,j}$ of length exceeding $r/2$ such that $(x+r/2, x+r) \subset I_{i,j}$. In particular, $j \leq k + 2$. Then at the left point of $I_{i,j}$, the slope of v jumps by at least $2^{-(1+\epsilon)(k+2)}$. It follows that at $x + r$, v is at least

$$\frac{r}{2}2^{-(1+\epsilon)(k+2)} \geq cr^{2-\delta}.$$

Thus, v has the desired properties.

Construction of u: We recall the following lemma on the solvability of the Monge-Ampère equation (see [Gut],[Har]).

Lemma 4.2.1. *If Ω is open, bounded and convex, μ is a finite Borel measure on Ω and g is continuous and convex in $\overline{\Omega}$ then there exists a unique convex solution $u \in C(\overline{\Omega})$ to the Dirichlet problem*

$$\det D^2 u = \mu, \quad u|_{\partial\Omega} = g.$$

Let $g(x_1, x_2, x_3) = C(v(x_1) + |x_2|)$ for a constant C depending on δ we will choose shortly, and obtain u by solving the Dirichlet problem

$$\det D^2 u = 1 \quad \text{ in } \Omega = \{|x'| < 1\} \times [-1, 1], \qquad u|_{\partial\Omega} = g.$$

Take $z = (z_1, 0, 0)$ for $z_1 \in S$, and let a_z be a subgradient of v at z_1. Let

$$w_z(x) = g(z) + a_z(x_1 - z_1) + w(x - z).$$

Since

$$w(x - z) \leq C_0(|x_1 - z_1|^{2-\delta} + |x_2|^\beta)$$

for some C_0, we can take C large so that

$$g(x_1, x_2, \pm 1) \geq g(z) + a_z(x_1 - z_1) + C(|x_1 - z_1|^{2-\delta} + |x_2|) \geq w_z(x_1, x_2, \pm 1)$$

on the top and bottom of Ω. Furthermore, since g is independent of x_3 and for any fixed x' we know w_z takes its maxima at $(x', \pm 1)$, we have $g \geq w_z$ on all of $\partial\Omega$. Thus, $u \geq w_z$ in all of Ω. Since u takes the value $g(z)$ at $(z_1, 0, \pm 1)$ and $w_z(z_1, 0, x_3) = g(z)$ for all $|x_3| < 1$, we have by convexity that $u = g(z)$ along $(z_1, 0, x_3)$.

We conclude that Σ contains $S \times \{0\} \times (-1, 1)$, which has Hausdorff dimension $1 + \frac{1}{1+3\epsilon} = 2 - \frac{3}{2}\delta$.

Remark 4.2.2. To get the analagous example in $\mathbb{R}^n$, take

$$u(x_1, x_2, x_3) + x_4^2 + ... + x_n^2.$$

Observe that this solution has exactly the behavior described by Lemma 4.1.2, which says that u must grow faster than quadratically in two directions. In chapter 6 we show that for any ϵ, we can take δ small enough that these examples are not in $W^{2,1+\epsilon}$.

Chapter 5

Unique Continuation

In this chapter we show that if two (Alexandrov) solutions to $\det D^2u = 1$ on B_1 agree on a set with nonempty interior, then they agree everywhere. The difference between the solutions solves a linear equation with coefficients depending on their second derivatives. If the solutions are smooth then the coefficients are smooth, and the result follows from a classical unique continuation theorem for linear equations. However, solutions to the Monge-Ampère equation are not smooth everywhere. The idea is that by theorem 1.4.1, solutions are smooth on an open dense connected set, so the classical unique continuation theorem still suffices.

In the first section we discuss the unique continuation theorem for linear equations and prove one version via Carleman estimates. The key hypothesis is that the coefficients are Lipschitz. In the next section we use it to prove theorem 1.4.2. Finally, in the last section we construct a counterexample to linear unique continuation when the coefficients are not Lipschitz.

The first and third sections of this chapter are expository. They included to keep this paper as self-contained as possible.

5.1 Unique continuation for linear equations

In the first section we discuss a classical unique continuation theorem for linear equations (see e.g. [H]).

Theorem 5.1.1. *Assume that Ω is a connected open domain in $\mathbb{R}^n$ and that u solves*

$$a_{ij}(x)u_{ij} = 0$$

in Ω, where $0 < \lambda I \leq (a_{ij}) \leq \frac{1}{\lambda} I$. Assume further that a_{ij} are Lipschitz. Then if $\{u = 0\}$ has nonempty interior, then u vanishes identically.

Remark 5.1.2. In $\mathbb{R}^2$, the theorem holds true without any regularity hypotheses on a_{ij}. However, there are counterexamples in higher dimensions (see the last section of this chapter). Unique continuation also holds when there are lower-order terms with for example bounded coefficients.

Remark 5.1.3. The result is also true if $\sup_{\partial B_r} |u| = O(r^k)$ for all k. This property is known as strong unique continuation. It is not hard to modify the discussion below to obtain this result.

Remark 5.1.4. For linear equations with Lipschitz coefficients, any two solutions which agree on a set with nonempty interior must agree everywhere. For fully nonlinear uniformly elliptic equations of the form

$$F(D^2 u) = 0$$

with F in $C^{1,1}$, the same result is known provided we know one of the two solutions is smooth ([AS]). The idea is to reduce problem to the linear case by showing that the solution is very close to a paraboloid near some point in the boundary of the agreement set. One can then use a small-perturbations regularity result of Savin ([S]) to conclude that the solution is smooth in a neighborhood of this point, and apply theorem 5.1.1 to the linearized equation.

We will discuss a version of unique continuation in which the solution vanishes in a half space to emphasize ideas and simplify computations. These ideas are not hard to adapt to a radial geometry to prove strong unique continuation (see remark 5.1.8).

To motivate the idea, consider a harmonic function u on $\mathbb{R}^2$ vanishing on the boundary of the strip $\{0 \leq x_1 \leq \pi\}$. We can "decompose u into frequencies," writing

$$u(x', x_n) = \sum_k a_k \sin(kx_1) e^{kx_2}.$$

If $|u|$ is very small for $x_2 < 0$ then the dominant terms oscillate rapidly in x_1 and grow rapidly in x_2 (i.e. k is large), and if u vanishes near $\{x_2 = 0\}$ then there are no nonzero terms in the expansion. We aim to adapt these ideas to a proof that does not use analyticity.

Assume now that u is harmonic on $\mathbb{R}^n$. The heuristic idea of the Carleman estimates is to show that the dominant frequency in u is "increasing with x_n". If u vanishes in $\{x_n \leq 0\}$ then its frequency is "infinite" near $\{x_n = 0\}$, so this estimate would imply that u also vanishes for $x_n > 0$.

To that end, we multiply u by the weights $e^{-\lambda x_n}$, giving a function v whose amplitude (roughly) represents how much u oscillates with frequency λ, and compute the equation for v. A short computation gives

$$0 = e^{-\lambda x_n}\Delta u = \Delta v + \lambda^2 v + 2\lambda v_n.$$

Let Q denote the cylinder $\{|x'| < 1\} \times (-1, 1)$ and assume that u vanishes in $Q \cap \{x_n \leq 0\}$. Assume further that u is supported away from $\partial Q \cap \{x_n \leq a\}$ for some $a > 0$. We will show that u vanishes for $x_n \leq a$, i.e. we can push the zero set a little further up. The strategy is to square the equation for v and integrate by parts. Pairing v_n with the other terms gives only boundary terms involving some quadratic (vector-valued) polynomial P in $\lambda^{3/2}v$ and $\lambda^{1/2}\nabla v$. We thus have

$$\lambda^4 \int_Q v^2\,dx + \int_Q (\Delta v)^2\,dx + 2\lambda^2 \int_Q v\Delta v\,dx + 4\lambda^2 \int_Q v_n^2\,dx = \int_{\partial Q} P(\lambda^{3/2}v, \lambda^{1/2}\nabla v)\cdot \nu\,ds.$$

Observe that if $n = 1$ then the last two terms are positive plus a boundary term of the same form. This gives the inequality

$$\lambda^4 \int_{-1}^{1} e^{-2\lambda x}u^2\,dx \leq C\lambda^3 e^{-2\lambda}$$

where C depends on u and its derivative, but not λ. Taking λ to ∞ implies unique continuation because $e^{-2\lambda x}$ is much larger than $e^{-2\lambda}$ for $x < 1$.

In higher dimensions, if we somehow squeezed another $\mu\lambda^4 \int_Q v^2\,dx$ in this computation, we could swallow the third term by the first two using that

$$2\lambda^2|v\Delta v| \leq (1-\mu/2)(\Delta v)^2 + \frac{\lambda^4}{1-\mu/2}v^2.$$

This would give the inequality

$$\lambda^4 \int_Q e^{-2\lambda x_n}u^2\,dx \leq C\lambda^3 e^{-2\lambda a}$$

where C depends on u, its derivatives, and μ. Again, taking λ to ∞ would give continuation of the zero set to $\{x_n \leq a\}$ because

$$e^{-2\lambda x_n} >> e^{-2\lambda a}$$

where $x_n < a$.

We can in fact obtain such an estimate by making the weight slightly concave: we replace $e^{-\lambda x_n}$ by $e^{-\lambda\phi(x_n)}$ where

$$\phi(x_n) = x_n - \frac{x_n^2}{2}.$$

This is the main subtlety in the Carleman estimate: to modify the weight so that we get a favorable term in our integration by parts.

We now state and prove the precise inequality motivated above, for harmonic functions. It is straightforward to adapt the argument to Lipschitz coefficients and to a radial geometry. We indicate how in remarks 5.1.7 and 5.1.8.

Proposition 5.1.5. *Assume that* $\Delta u = 0$ *in* $Q := \{|x'| < 1\} \times (-1, 1)$ *and let* ϕ *be defined as above. Then for all* λ *large we have*

$$\lambda^3 \int_Q e^{-2\lambda\phi} u^2 \, dx \leq \int_{\partial Q} e^{-2\lambda\phi} P(\lambda^{3/2} u, \lambda^{1/2} \nabla u) \cdot \nu \, ds$$

where P *is some quadratic polynomial.*

Note that the homogeneity in λ on the left side is one less than what appears in the discussion above. If u vanishes in $Q \cap \{x_n \leq 0\}$ and is supported away from $\partial Q \cap \{x_n \leq a\}$, then taking λ to ∞, we get continuation of the zero set to $\{x_n \leq a\}$.

The strategy to prove proposition 5.1.5 is the same as that outlined above. We compute the equation for $v = e^{-\lambda\phi} u$, square and integrate by parts. A short computation gives

$$e^{-\lambda\phi} \Delta u = \Delta v + \lambda^2 \phi'^2 v^2 + \lambda(2\phi' v_n - v).$$

Proposition 5.1.5 follows easily from the following key computation.

Lemma 5.1.6. *The following identity holds for any smooth function* v*, with* ϕ *defined as above:*

$$\begin{aligned} \int_Q (\Delta v + \lambda^2 \phi'^2 v + \lambda(2\phi' v_n - v))^2 \, dx &= \int_Q (\Delta v + \lambda^2 \phi'^2 v - \lambda(2\phi' v_n - v))^2 \, dx \\ &\quad + 8\lambda \int_Q (\lambda^2 \phi'^2 v^2 + v_n^2) \, dx \\ &\quad + 4\lambda \int_{\partial Q} [(2\phi' v_n - v)\nabla v + (\lambda^2 \phi'^3 v^2 - \phi' |\nabla v|^2) e_n)] \cdot \nu \, ds. \end{aligned}$$

Proof. We compute

$$\int_Q (\Delta v + \lambda^2 \phi'^2 v)(2\phi' v_n - v).$$

Pairing the first term in each expression and moving a derivative from Δv gives

$$\begin{aligned} 2\int_Q \phi' v_n \Delta v \, dx &= 2\int_{\partial Q} \phi' v_n \nabla v \cdot \nu \, dx - \int_Q \phi' \partial_n |\nabla v|^2 \, dx + 2\int_Q v_n^2 \, dx \\ &= \int_{\partial Q} (2\phi' v_n \nabla v - \phi' |\nabla v|^2 e_n) \cdot \nu \, ds + 2\int_Q v_n^2 \, dx - \int_Q |\nabla v|^2 \, dx. \end{aligned}$$

Pairing the first term from the first expression and the last from the second gives

$$-\int_Q v\Delta v\,dx = -\int_{\partial Q} v\nabla v\cdot\nu + \int_Q |\nabla v|^2\,dx.$$

Pairing the second term from the first expression with the first term from the second expression gives

$$\lambda^2\int_Q \phi'^3\partial_n(v^2)\,dx = \lambda^2\int_{\partial Q}\phi'^3 v^2 e_n\cdot\nu\,ds + 3\lambda^2\int_Q \phi'^2 v^2\,dx.$$

Finally, the last term is

$$-\lambda^2\int_Q \phi'^2 v^2\,dx.$$

Summing them all gives the identity. □

Proof of proposition 5.1.5. Let $v(x) = e^{-\lambda\phi(x_n)}u(x)$. Then the left hand side of identity 5.1.6 is zero, and the boundary term has the desired form. □

Remark 5.1.7. The above argument is easy to adapt to the case that a_{ij} are Lipschitz. By scaling we may assume that $|a_{ij}-\delta_{ij}| < \epsilon$ and that $|\nabla a_{ij}| < \epsilon$ for some small ϵ. Computing the equation for $v = e^{-\lambda\phi(x_n)}u$, squaring and integrating by parts, we get the same thing as before plus a small error of the form

$$O(\epsilon)\int_Q(\lambda^3v^2 + \lambda|\nabla v|^2)\,dx.$$

The reason we don't have second derivative errors appearing is because terms of the form $AD^2v\cdot Dv$ can be rewritten as $AD(Dv\cdot Dv)$ via integration by parts, up to an error of $O(\epsilon)|\nabla v|^2$. By using the favorable terms in the key computation a little more carefully we can cancel this error to get proposition 5.1.5 for Lipschitz coefficients.

Remark 5.1.8. The previous discussion can be adapted to a radial geometry, and in fact to give strong unique continuation. The idea is that if u is harmonic then it can be decomposed into homogeneous harmonic polynomials $r^kY_k(\omega)$ (with $r=|x|$ and $\omega\in\partial B_1$) whose radial growth rate corresponds to their "frequency on the sphere." To measure how much u oscillates spherically with frequency λ, examine the function

$$v = \varphi(r)^{-\lambda}u$$

where $\varphi(r)$ is a perturbation of r chosen to give the correct terms when integrating the equation for v by parts. One computes

$$0 = \frac{\varphi}{\varphi'}\phi^{-\lambda}\Delta u = \frac{\varphi}{\varphi'}\Delta v + \lambda^2\frac{\varphi'}{\varphi}v + 2\alpha\frac{\varphi'}{\varphi}w$$

where

$$w = \frac{\varphi}{\varphi'} v_r + \frac{1}{2}\left(\frac{\varphi''}{\varphi} + \frac{(n-1)\varphi}{r\varphi'} - 1\right) v.$$

Taking φ such that $\frac{\varphi}{r\varphi'} = e^r$, and provided $\sup_{\partial B_r} |u| = O(r^k)$ for some $k > \lambda$, we can square the equation and integrate by parts to get an analogue to our key computation. This gives a Carleman estimate of the form

$$\int_{B_1} \varphi^{-2\lambda} u^2 \, dx \le C \int_{\partial B_1} \varphi^{-2\lambda} P(\lambda, u, \nabla u) \, ds$$

where P is some polynomial. If u vanishes to infinite order at 0 we can take λ to ∞, and conclude that u vanishes identically.

5.2 Proof of Theorem 1.4.2

Assume that u, v satisfy the hypotheses of Theorem 1.4.2. For our proof of unique continuation we apply theorem 5.1.1 to the difference of u and v, which solves a linear equation where u and v are sufficiently regular. Indeed, suppose u and v are C^2 in a neighborhood of x and let w_t be the convex combination $tu + (1-t)v$. Let $(W_t)^{ij}$ be the matrix of cofactors for $D^2 w_t$. Then by expanding $0 = \int_0^1 \frac{d}{dt} \det D^2 w_t dt$ we get

$$a^{ij}(x)(u - v)_{ij} = 0,$$

where

$$a^{ij}(x) = \int_0^1 (W_t)^{ij}(x) dt.$$

If the right side is sufficiently regular, then the coefficients are Lipschitz. In particular, $f \in C^{1,\alpha}$ suffices by Caffarelli's perturbation theory for strictly convex solutions:

Theorem 5.2.1. *Assume*

$$\det D^2 u = f \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0$$

where $f \in C^{1,\alpha}(\Omega)$ is strictly positive. Then

$$u \in C^{3,\alpha}(\Omega).$$

Finally, we observe that open sets whose complements have zero Hausdorff $n-1$ dimensional measure are connected.

Lemma 5.2.2. *Assume $K \subset \mathbb{R}^n$ is closed, and assume further that $\mathcal{H}^{n-1}(K) = 0$. Then $\mathbb{R}^n - K$ is pathwise connected.*

Proof. Assume by way of contradiction that $D = \mathbb{R}^n - K$ is not pathwise connected. Since D is open, by rotation, translation and scaling we can assume that the points $\pm Re_n \in D$ cannot be connected by any continuous path through D and that

$$\{|x'| < 1\} \times \{\pm R\} \subset D.$$

Let K' be the projection of K onto $\{x_n = 0\}$ and let $B_1' = B_1 \cap \{x_n = 0\}$. If $B_1' - K' \neq \emptyset$, this would violate the contradiction hypothesis because then we could find a point $x' \in B_1'$ such that $(x', t) \in D$ for all $t \in \mathbb{R}$ and take our path to be the straight lines from $-Re_n$ to $(x', -R)$ to (x', R) to Re_n.

We conclude that for any cover of K by balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$, we have

$$\sum_i r_i^{n-1} \geq 1,$$

contradicting that $\mathcal{H}^{n-1}(K) = 0$. □

The proof of unique continuation follows easily from these observations and our partial regularity result theorem 1.4.1.

Proof of Theorem 1.4.2: Let Σ_u and Σ_v be the singular sets of u and v respectively, and let $A = \Omega - (\Sigma_u \cup \Sigma_v)$. Since A is dense in Ω, it suffices to show that $u = v$ on A.

By Caffarelli's theory ([C1]), A is an open set. Indeed, for $x \in A$ we can find some p in $\mathbb{R}^n$ and h small such that $S^u_{h,p}(x) \subset\subset \Omega$, and since f is bounded in this section Caffarelli gives that u is strictly convex in a neighborhood of x. (The same reasoning gives that v is strictly convex in a neighborhood of x.) By Theorem 1.4.1, the complement of A has Hausdorff $n-1$ dimensional measure zero, so by Lemma 5.2.2, A is connected. By Theorem 5.2.1, the difference $u - v$ satisfies the linear equation

$$a^{ij}(x)(u - v)_{ij} = 0$$

on A, where a^{ij} are locally uniformly elliptic and $C^{1,\alpha}$ in A. The conclusion follows from Theorem 5.1.1. □

5.3 Counterexample

In this last section we indicate how to build a counterexample to unique continuation for the equation

$$a_{ij}(\mathbf{x})u_{ij} = 0$$

when a_{ij} are C^α for any $\alpha < 1$ but not Lipschitz. We construct the example in $\mathbb{R}^3$ and we write $\mathbf{x} = (x, y, t)$.

The first counterexample to unique continuation with Hölder coefficients is due to Pliš ([Pl]). The example described below is similar to the counterexample for divergence form equations from [Mi].

Remark 5.3.1. In $\mathbb{R}^2$, unique continuation holds with bounded measurable coefficients. We discuss in remark 5.3.3 the reasons that our strategy to build examples of nonunique continuation fail in $\mathbb{R}^2$.

Strategy:

We first outline the idea of the construction. Suppose we can construct a "building block" solution u_0 to a uniformly elliptic equation in $\mathbb{R}^2 \times [0, D]$ that starts off as the harmonic function $e^{-t}\cos(x)$ for t small and ends up as the harmonic function $e^{-2t}\cos(2x)$ for t close to D. Then it is easy to build an example of nonunique continuation with discontinuous coefficients by stacking rescalings of u_0 on top of each other in t. Indeed, let u_k be the shifted rescalings of u_0 given by

$$u_k(\mathbf{x}) = e^{-2kD}u_0(2^k(\mathbf{x} - (0, 0, (2 - 2^{1-k})D))).$$

Let $u = u_k$ for $(2 - 2^{1-k})D \le t < (2 - 2^{-k})D$, and let $u = 0$ for $t \ge 2D$. Then u decays to zero for $t \ge 2D$, and

$$|D^l u| \le Ce^{-k(2D-l)}$$

a distance $2^{-k}D$ from $t = 2D$. Thus, u is smooth away from $\{t = 2D\}$ and $u \in C^M(\mathbb{R}^n)$ for some M large depending on D. Furthermore, u solves an elliptic equation whose coefficients oscillate by the same amount during each time we stack a new rescaling of u_0, so the coefficients are bounded and measurable. (In fact, they are discontinuous on $\{t = 2D\}$ and smooth otherwise).

We can improve the regularity of the coefficients and of u with some easy modifications, with the price that the solutions aren't self-similar. We need the oscillation of the coefficients to decay each time we stack. To that end, we construct u_k whose frequency increases from λ_k to λ_{k+1} over a distance D_k. Rescaling by factor λ_k, this is equivalent to constructing a building block $\bar{u}_k$ that boosts frequency from 1 to $\mu_k = \frac{\lambda_{k+1}}{\lambda_k}$ in distance $\lambda_k D_k$. Heuristically, if $\bar{u}_k$ goes from frequency 1 to μ_k over a distance $\lambda_k D_k$, then the second derivatives oscillate by $a_k = \max\{\mu_k^2 - 1, \frac{1}{\lambda_k D_k}\}$. Thus, we expect to pay for increasing frequency by making the coefficients oscillate by a_k.

Taking D_k as large as possible so that the sum converges, like

$$D_k = \frac{1}{k \log^2(k)},$$

and taking

$$\lambda_k = k^3,$$

we get that $\mu_k^2 - 1 \sim \frac{1}{k}$ and $\frac{1}{\lambda_k D_k} < \mu_k^2 - 1$ for k large, so

$$a_k \sim \frac{1}{k}.$$

Thus, the coefficients for the solution built by "stacking" u_k oscillate by roughly $1/k$ over a distance $\frac{1}{k \log^2 k}$, giving Lipschitz·Log(Lipschitz) regularity for the coefficients. Furthermore, the solution and its derivatives are exponentially decaying in k, but the distance from the vanishing set at stage k is $\sim \frac{1}{\log(k)}$, so it is smooth.

Remark 5.3.2. It is natural to ask why we cannot get Lipschitz regularity of the coefficients with this strategy. The point is that with Lipschitz coefficients we would need infinite space to boost the frequency to ∞. Indeed, if λ_k to go to ∞, we must have

$$\sum_k \log(\mu_k^2) = 2 \sum_k (\log(\lambda_{k+1}) - \log(\lambda_k)) = \infty$$

and the left side is dominated by $\sum_k (\mu_k^2 - 1)$, i.e. the total variation of the coefficients in t must blow up, which precludes the possibility that u vanishes in finite distance with Lipschitz coefficients.

Construction of building block:

As noted above, we can construct the desired example provided we can build a solution $\bar{u}$ that starts off as the harmonic function $e^{-t}\cos(x)$ near $t = 0$ and ends as $e^{-\mu t}\cos(\mu x)$ where t is larger than $\frac{1}{\mu^2 - 1}$, with $\mu - 1 > 0$ small.

For the first step, take

$$\bar{u} = e^{-t}(cos(x) + \eta(t)\cos(\mu y))$$

where η starts of identically 0 and ends up identically 1, with $|\eta'|, |\eta''| = O(\mu^2 - 1)$. The idea is to slowly introduce a rotated component oscillating with a slightly higher frequency. The rotation is important; see remark 5.3.3.

While η is zero, modify the coefficients from δ_{ij} to $(a^0_{ij}) = \text{diag}\left(1, 1, \frac{1}{\mu^2}\right)$. Then let

$$(a_{ij}) = (a^0_{ij}) + ae_x \otimes e_x + b(e_x \otimes e_t + e_t \otimes e_x).$$

We compute

$$a_{ij}u_{ij} = e^{-t}\left(-a\cos(x) + 2b\sin(x) + O(\mu^2 - 1)\right).$$

Since $\cos(x)$ and $\sin(x)$ don't simultaneously vanish we can choose a and b of order $\mu^2 - 1$ so that (a_{ij}) are uniformly elliptic and the equation holds. Furthermore, we can choose them to vanish for t large.

It is easy to slowly remove the component oscillating in x in a similar fashion, and finally to slowly increase the decay rate to $e^{-\mu t}$ to complete the construction of the building block (up to a rotation) and complete the counterexample.

Remark 5.3.3. We require a rotation to construct the building block. This is related to the reason we cannot build counterexamples in two dimensions. We examine why the natural attempts fail in this remark.

First imagine working in just two dimensions and introducing a $\cos(\mu x)$ component instead. It is easy to compute that $\det D^2\bar{u} > 0$ at some points, so that $\bar{u}$ cannot possibly solve an elliptic equation. Geometrically, the level lines $\{\bar{u} = 0\}$ start off spaced evenly at intervals of length π at $t = 0$, but eventually branch off and have higher frequency for larger t, and continue to do so. Thus, the level lines close around bounded regions, precluding the maximum principle.

Another attempt to increase the frequency in two dimensions while preventing branching of the level lines might take the form $e^{-\phi(t)}\sin(\phi(t)x)$ where $\phi(t)$ increases to ∞ over finite distance in t. Then the zero set consists of paths fanning out from some point $(0, t_0)$ intersecting horizontal lines at evenly spaced points, which indicates rapid oscillation in t as x gets large. Indeed, one computes that this function cannot solve an elliptic equation for x large, and this example only works in (roughly) a triangle.

Finally, one might think of taking a building block that is positive in the slab $0 \leq t \leq 1$ and 0 on the boundary, with smaller normal derivative at $t = 1$ than at $t = 0$, and then stack rescalings.

However, the Harnack inequality implies exponential growth in the horizontal directions, and we cannot match the horizontal growth rates of the rescalings.

Chapter 6

$W^{2,1}$ Estimate

If u is convex then it is locally Lipschitz, and $\|D^2u\| \leq \Delta u$. It follows that D^2u is a locally finite measure, but it doesn't rule out the possibility that D^2u concentrates on sets of measure zero. Take for example the function $|x_n|$, whose second derivatives concentrate on the hyperplane $\{x_n = 0\}$.

In this chapter we examine the integrability of second derivatives for solutions to

$$0 < \lambda \leq \det D^2u \leq \frac{1}{\lambda}.$$

The first result in this direction, due to De Philippis and Figalli ([DF]), shows that if u is strictly convex then D^2u is in $L\log^k L$ for all $k > 0$. The idea is to show that, in any section, Δu is controlled above and below in measure by quantities depending only section geometry. A variant of this technique shows that D^2u is in fact $L^{1+\epsilon}$ for some ϵ depending on n, λ ([DFS]). This result is optimal in light of counterexamples due to Wang ([W]) obtained by seeking solutions on $\mathbb{R}^2$ with the homogeneity

$$u(x, y) = \frac{1}{\lambda^{2+\alpha}}u(\lambda x, \lambda^{1+\alpha}y).$$

We prove the $W^{2,1+\epsilon}$ estimate for strictly convex solutions in the first section.

A natural question is whether these estimates extend to singular solutions which are not strictly convex. In the second section we show using theorem 1.4.1 that the second derivatives are L^1. The idea is that the singular set is too small for the second derivatives to concentrate there, and the solutions are $W^{2,1}_{loc}$ away from the singular set. However, we also show that the solutions to $\det D^2u = 1$ constructed in chapter 4 are not in $W^{2,1+\epsilon}$ for ϵ as small as we like. Thus, we cannot extend the strongest result for strictly convex solutions to singular solutions.

We show in the third section that the second derivatives are in fact $L\log^{\epsilon} L$ for some $\epsilon(n)$. The key idea is to refine the techniques from chapter 4 to show that solutions grow logarithmically faster than quadratic in at least two directions perpendicular to a singularity. We combine this estimate in long, thin sections near singularities with the estimate for strictly convex solutions in large sections to obtain theorem 1.4.3.

Finally, in the last section we construct examples of singular solutions to $\det D^2u = 1$ such that Δu is not $L\log^M L$ for M large, showing the optimality of our result.

For simplicity of notation, and since we will not need to "tilt planes" in this chapter, we denote $S^u_{h,p}(x)$ by $S_h(x)$.

6.1 $W^{2,1+\epsilon}$ estimate for strictly convex solutions

In this section we prove the following estimate of De Philippis, Figalli and Savin ([DFS]):

Theorem 6.1.1. *Assume that $\lambda < \det D^2u < \frac{1}{\lambda}$ in $S_1(0)$ and that $B_1 \subset S_1(0) \subset B_n$. Then there exist $\epsilon > 0$ and C depending only on n, λ such that*

$$\int_{S_{1/2}(0)} (\Delta u)^{1+\epsilon}\,dx \leq C.$$

The strategy of the proof is to show polynomial decay of the truncated L^1-norms of u:

$$\int_{\{\Delta u > t\}\cap S_{1/2}(0)} \Delta u < Ct^{-\delta}$$

for $t > 1$. The theorem then follows easily from Fubini:

$$\begin{aligned}\int_{S_{1/2}(0)} (\Delta u)^{1+\epsilon}\,dx &= \frac{1}{\epsilon}\int_{S_{1/2}(0)} \Delta u\left(\int_0^{\Delta u} t^{\epsilon-1}\,dt\right)\,dx \\ &\leq C\left(1 + \int_1^{\infty} t^{\epsilon-1}\left(\int_{\{\Delta u>t\}\cap S_{1/2}(0)} \Delta u\,dx\right)\,dt\right),\end{aligned}$$

which is bounded provided $\epsilon < \delta$.

The key estimate controls the mass of Δu in a section by the mass in a smaller section where Δu is controlled, in terms of the geometry of the section. Below δ is the universal constant from lemma 3.3.3. Recall that any compactly contained section $S_h(x)$ is equivalent to an ellipsoid of the form $A_hB_{h^{1/2}}$ where $|\det A_h| = 1$. Let d_h be the smallest eigenvalue of A_h.

Proposition 6.1.2. *Assume that $\lambda < \det D^2u < \frac{1}{\lambda}$ in a section $S_2(0)$. Then*

$$\int_{S_1(0)} \Delta u\, dx \leq C \int_{\{c_0 d_1^{-2} < \Delta u < c_0^{-1} d_1^{-2}\} \cap S_\delta(0)} \Delta u\, dx$$

where C and c_0 are large and small constants depending only on n, λ.

Proof. In the case that $A_1 = I$ we have

$$\int_{S_1(0)} \Delta u\, dx < C$$

by integration by parts and the fact that u is convex, hence locally Lipschitz. It follows that for some small c_0,

$$\frac{|\{\Delta u > c_0^{-1}\} \cap S_\delta|}{|S_\delta|} < \frac{1}{2}.$$

Thus, for c_0 small we have $D^2u < c_0^{-1}I$ in at least half of S_δ. Using the equation and taking c_0 smaller if necessary we have the inequality $D^2u > c_0 I$ in this set as well. Thus,

$$\int_{S_\delta \cap \{c_0 I \leq D^2u \leq c_0^{-1} I\}} \Delta u\, dx \geq c \int_{S_1} \Delta u\, dx. \tag{6.1}$$

By affine invariance we can reduce to the case $A_1 = I$, treated above. Indeed, let

$$\tilde{u}(x) = u(A_1 x).$$

Since $(A_1^{-1})^T D^2\tilde{u}(A_1^{-1}) = D^2u$, we have

$$A_1\left(\{c_0 I < D^2\tilde{u} < c_0^{-1} I\}\right) \subset \{c_0 d_1^{-2} < \Delta u < c_0^{-1} d_1^{-2}\}$$

(up to shrinking c_0 on the right side) and A_1 is volume preserving. It follows that

$$\int_{\{c_0 I < D^2\tilde{u} < c_0^{-1} I\} \cap A_1^{-1}(S_\delta)} \Delta\tilde{u}\, dx \leq C d_1^2 \int_{\{c_0 d_1^{-2} < \Delta u < c_0^{-1} d_1^{-2}\} \cap S_\delta} \Delta u\, dx$$

and

$$d_1^2 \int_{S_1(0)} \Delta u\, dx \leq \int_{A_1^{-1}(S_1(0))} \Delta\tilde{u}\, dx.$$

Applying inequality 6.1 to $\tilde{u}$ gives the proposition. □

It is clear that under the rescaling $\tilde{u}(x) = \frac{1}{h} u(h^{1/2}x)$ (which preserves second derivatives) that the above estimate also holds in smaller sections:

Proposition 6.1.3. *Assume that* $\lambda < \det D^2u < \frac{1}{\lambda}$ *in* Ω *with* $S_{2h}(x) \subset\subset \Omega$. *Then*

$$\int_{S_h(x)} \Delta u \, dx \leq C \int_{\{c_0 d_h^{-2} < \Delta u < c_0^{-1} d_h^{-2}\} \cap S_{\delta h}(x)} \Delta u \, dx.$$

Finally, we can prove the desired decay of truncated L^1-norms of Δu by using a covering argument at just the right scale.

Proof of theorem 6.1.1. Let M be large to be chosen and let $A_k = \{\Delta u > M^k\} \cap S_{1/2}(0)$ with $k \geq 1$. By definition, for each $x \in A_k$ we have that $d_h^{-2} \sim M^k$ for h small, and we have d_δ uniformly bounded above and below for all $x \in S_{1/2}$. Thus, we can choose h at each x so that $c_0 d_h^{-2} > M^{k-1}$ and $c_0^{-1} d_h^{-2} < M^k$, provided M is large compared to c_0^{-1}.

Take a Vitali subcover as in the statement of lemma 3.3.3. Applying proposition 6.1.3 in each of these sections and summing we obtain

$$\int_{A_{k-1} - A_k} \Delta u \, dx \geq c \int_{A_k} \Delta u \, dx$$

and the theorem follows. □

6.2 $W^{2,1}$ regularity

In this section we obtain $W^{2,1}$ regularity for singular solutions to the Monge-Ampère equation. Furthermore, by examining the examples from the last section of chapter 4 we show that we cannot improve this result to $W^{2,1+\epsilon}$ regularity for an ϵ depending on λ, Λ and n.

Let u solve $0 < \lambda \leq \det D^2u \leq \Lambda$ in B_1. Theorem 6.1.1 gives local $W^{2,1}$ regularity on $B_1 - \Sigma$. By Theorem 1.4.1, for any $\eta > 0$ we can cover $\Sigma \cap B_{1/2}$ by balls $\{B_{r_i}(x_i)\}$ with $r_i < 1/4$ such that

$$\sum_{i=1}^{\infty} r_i^{n-1} < \eta.$$

Let $A = \cup_{i=1}^{\infty} B_{r_i}(x_i)$. Since u is a convex function, the second derivatives are controlled by Δu. It follows that

$$\begin{aligned}\int_A \|D^2u\|\,dx &\leq \int_A \Delta u\,dx \\ &\leq \sum_{i=1}^{\infty}\int_{\partial B_{r_i}} u_\nu\,ds \\ &\leq C\sum_{i=1}^{\infty} r_i^{n-1} \\ &\leq C\eta,\end{aligned}$$

where C is the Lipschitz constant of u in $B_{3/4}$. This shows that the second derivatives cannot concentrate on Σ.

We now examine the integrability of Δu for the singular examples from chapter 4. Fix a small δ, where δ is the parameter used in constructing those examples. We will show that for some ϵ small depending on δ, we have $\Delta u \notin L^{1+\epsilon}$.

On any ball B_r, by Hölder's inequality we have

$$\int_{B_r} (\Delta u)^{1+\epsilon}\,dx \geq c(n)r^{-\epsilon n}\left(\int_{B_r}\Delta u\,dx\right)^{1+\epsilon}.$$

Recall from the construction that at points in Σ, u grows from its tangent plane faster than $x_2^{\beta} = x_2^{1+\frac{\delta}{4-3\delta}}$ in the x_2 direction (at singular points, a translation and modification of w by a linear function touches u by below). It follows that for $x \in \Sigma$ and l_x a tangent plane to u at x, we have

$$\sup_{\partial B_r(x)} (u - l_x) \geq r^{\beta}.$$

Applying convexity,

$$\begin{aligned}\int_{B_r(x)} (\Delta u)^{1+\epsilon}\,dx &\geq c(n)r^{-\epsilon n}\left(\int_{\partial B_r} u_\nu\,ds\right)^{1+\epsilon} \\ &\geq c(n)r^{(n+\beta-2)(1+\epsilon)-\epsilon n} \\ &\geq c(n)r^{n-1-\epsilon+(1+\epsilon)\frac{\delta}{3}}.\end{aligned}$$

Fix η small and cover $S \times \{0\} \times (-1,1)^{n-2}$ with balls of radius $r_i < \eta$. Take a Vitali subcover $\{B_{r_i}\}_{i=1}^{\infty}$. It follows that

$$\int_{B_1} (\Delta u)^{1+\epsilon}\,dx \geq c(n)\sum_{i=1}^{\infty} r_i^{n-1-\epsilon+(1+\epsilon)\frac{\delta}{3}}.$$

Taking $\epsilon = 4\delta$ above, we conclude that

$$\int_{B_1} (\Delta u)^{1+\epsilon}\, dx \geq c(n) \sum_{i=1}^{\infty} r_i^{n-1-3\delta},$$

where the expression on the right goes to ∞ as $\eta \to 0$ because the Hausdorff dimension of $S \times \{0\} \times (-1,1)^{n-2}$ is $n - 1 - \frac{3}{2}\delta$. Thus, Δu is not $L^{1+\epsilon}$ for $\epsilon \geq 4\delta$.

Remark 6.2.1. In the next section we present a more precise version of Theorem 1.4.1 which gives $L \log^{\epsilon} L$ regularity of second derivatives of singular solutions to

$$\lambda \leq \det D^2 u \leq \Lambda.$$

6.3 Proof of Theorem 1.4.3

We first state the key proposition and use it to prove Theorem 1.4.3. In chapter 4 we show that the Monge-Ampere mass of $u + \frac{1}{2}|x|^2$ in small balls around singular points is large compared to the mass of Δu. The proposition is a more precise, quantitative version of this statement for long, thin sections. Let $\bar{h}(x) \geq 0$ be the largest h such that $S_h(x) \subset\subset B_1$. We say that $S_{\bar{h}(x)}(x)$ is the maximal section at x. If $\bar{h}(x) = 0$ then x is a singular point.

Proposition 6.3.1. *If $u \in D_{n,\lambda,\Lambda,K}$, $v = u + \frac{1}{2}|x|^2$, $x \in B_{1/2}$ and $h > \bar{h}(x)$ then there exist $\eta(n)$ and c universal such that for some r with*

$$|\log r| > c|\log h|^{1/2},$$

we have

$$Mv(B_r(x)) > cr^{n-1}|\log r|^{\eta}.$$

Remark 6.3.2. Let Σ denote the singular set of u, where $\bar{h} = 0$. It follows from proposition 6.3.1 and a covering argument that

$$\inf_{\delta > 0} \left\{ \sum_{i=1}^{\infty} r_i^{n-1} |\log r_i|^{\eta} : \{B_{r_i}(x_i)\}_{i=1}^{\infty} \text{ cover } \Sigma, r_i < \delta \right\} = 0$$

for some small $\eta(n)$, giving a quantitative version of the main theorem in chapter 4 for solutions to $\lambda \leq \det D^2 u \leq \Lambda$.

We will give a proof of Proposition 6.3.1 later in this section by closely examining the geometric properties of maximal sections.

The idea of the proof of Theorem 1.4.3 is to apply Proposition 6.3.1 in the thin maximal sections, and then apply the $W^{2,1+\epsilon}$ estimate of [DFS] in the larger sections to show the following decay of the integral of Δu over its level sets:

$$\int_{\{\Delta u>t\}} \Delta u \, dx \leq \frac{C}{|\log t|^{\epsilon}}, \tag{6.2}$$

for some $\epsilon(n)$. Assuming this is true, theorem 1.4.3 follows easily by Fubini:

$$\begin{aligned}
\int_{B_{1/2}} \Delta u(\log(1+\Delta u))^{\epsilon/2} \, dx &\leq C \int_{B_{1/2}} \Delta u \int_1^{1+\Delta u} \frac{1}{t(\log t)^{1-\epsilon/2}} \, dt \, dx \\
&\leq C + C \int_2^{\infty} \frac{1}{t(\log t)^{1-\epsilon/2}} \int_{\{\Delta u>t\}} \Delta u \, dx \, dt \\
&\leq C + C \int_2^{\infty} \frac{1}{t(\log t)^{1+\epsilon/2}} \, dt \\
&\leq C(\epsilon).
\end{aligned}$$

To prove (6.2), we use the rescaled version of theorem 6.1.1 in the larger maximal sections.

Lemma 6.3.3. *If $u \in D_{n,\lambda,\Lambda,K}$ with $x \in B_{1/2}$ and $S_h(x) \subset\subset B_1$, then for C universal and $\epsilon(n, \lambda, \Lambda)$ we have*

$$\int_{S_{h/2}(x)\cap\{\Delta u>t\}} \Delta u \, dx < C h^{n/2-1-\epsilon} t^{-\epsilon}.$$

Proof. By subtracting a linear function and translating assume that $x = 0$ and $u|_{\partial S_h(0)} = 0$. Let

$$u(x) = (\det A)^{2/n} \tilde{u}(A^{-1}x)$$

where A normalizes $S_h(x)$ and $\tilde{u}$ has height H. Then

$$D^2 u(x) = C|S_h(0)|^{2/n} (A^{-1}) D^2 \tilde{u}(A^{-1}x)(A^{-1})^T.$$

Applying the estimate on $|S_h(0)|$ from Lemma 3.3.2 and letting d denote the length of the smallest axis for the John ellipsoid of $S_h(0)$, it follows that

$$\Delta u(x) \leq C \left(\frac{h}{d^2}\right) \Delta \tilde{u}(A^{-1}x).$$

Using change of variables and Theorem 6.1.1 we obtain that

$$\int_{S_{h/2}(0)\cap\{\Delta u>t\}} \Delta u\,dx \le C(\det A)\left(\frac{h}{d^2}\right)\int_{S_{H/2}(0)\cap\{\Delta\tilde{u}>c\frac{d^2}{h}t\}} \Delta\tilde{u}(y)\,dy$$
$$\le C(\det A)\left(\frac{h}{d^2}\right)^{1+\epsilon} t^{-\epsilon}.$$

Since $\det A = h^{n/2}$ up to a universal constants and $d > ch$ since u is locally Lipschitz, the conclusion follows. □

Let $F_\gamma = \{x \in B_{1/2} : \frac{\gamma}{2} \le \bar{h}(x) < \gamma\}$.

Lemma 6.3.4. *Let $u \in D_{n,\lambda,\Lambda,K}$. Then there is some C universal and $\epsilon(n,\lambda,\Lambda)$ such that*

$$\int_{F_\gamma\cap\{\Delta u>t\}} \Delta u\,dx < C\gamma^{-\epsilon}t^{-\epsilon}$$

Proof. By Lemma 3.3.3 we can take a cover of F_γ by sections $\{S_{\bar{h_i}(x_i)/2}(x_i)\}_{i=1}^{M_\gamma}$ with $x_i \in F_\gamma$ and $S_{\delta\bar{h_i}(x_i)}(x_i)$ disjoint for some universal δ. Then

$$\int_{F_\gamma\cap\{\Delta u>t\}} \Delta u\,dx \le CM_\gamma\gamma^{n/2-1-\epsilon}t^{-\epsilon}$$

by Lemma 6.3.3. We need to estimate the number of sections M_γ in our Vitali cover of F_γ.

Take $x \in F_\gamma$ and consider $S_{\bar{h}(x)}(x)$, which touches ∂B_1. By translation and subtracting a linear function assume that $x = 0$ and $u|_{\partial S_{\delta^2\bar{h}(0)}(0)} = 0$. By rotating and applying Lemma 3.3.2 assume that $S_{\delta^2\bar{h}(0)}(0)$ contains the line segment from $-ce_n$ to ce_n, with c universal.

Let w_t be the restriction of u to $\{x_n = t\}$ and let

$$S^{w_t} = S_{\delta^2\bar{h}(0)}(0)\cap\{x_n = t\}$$

be the slice of $S_{\delta^2\bar{h}(0)}(0)$ at $x_n = t$. Since $|S_{\delta^2\bar{h}(0)}(0)| \le C\gamma^{n/2}$ and this section has length $2c$ in the e_n direction, it follows from convexity that

$$|S^{w_t}|_{\mathcal{H}^{n-1}} \le C\gamma^{n/2}.$$

By convexity, $u(te_n) < -\delta^2\bar{h}(0)/2$ for $-c/2 \le t \le c/2$. Applying Lemma 3.2.5, we conclude that for $t \in [-c/2, c/2]$,

$$Mw_t(S^{w_t}) > c\gamma^{n/2-1}.$$

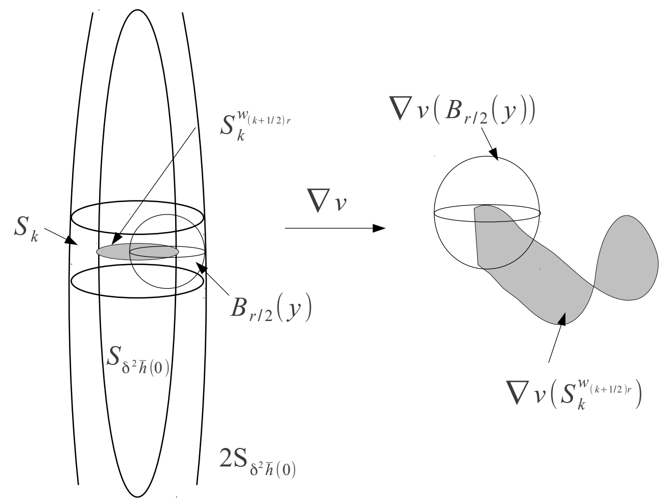

Figure 6.1: $\nabla v(S_k)$ contains an $r/2$-neighborhood of the surface $\nabla v(S^{w_{(k+1/2)r}})$, which projects in the x_n direction to a set of $\mathcal{H}^{n-1}$ measure at least $c\gamma^{n/2-1}$.

Let r be the distance between $\partial S_{\delta^2\bar{h}(0)}(0)$ and $\partial(2S_{\delta^2\bar{h}(0)}(0))$. Divide $2S_{\delta^2\bar{h}(0)}(0)$ into the slices

$$S_k = 2S_{\delta^2\bar{h}(0)}(0) \cap \{kr < x_n < (k+1)r\}$$

for $k = -\frac{c}{2r}$ to $\frac{c}{2r}$. Let $v = u + \frac{1}{2}|x|^2$. Then $\nabla v(S_k)$ contains a ball of radius $r/2$ around each point in $\nabla v(S^{w_{(k+1/2)r}})$ (see Figure 6.1), so

$$Mv(S_k) \geq crMv(S^{w_{(k+1/2)r}}) \geq cr\gamma^{n/2-1}.$$

Summing from $k = -\frac{c}{2r}$ to $\frac{c}{2r}$ we obtain that

$$|\nabla v(2S_{\delta^2\bar{h}(0)}(0))| \geq c\gamma^{n/2-1}.$$

Using that $2S_{\delta^2\bar{h}_i}(x_i) \subset S_{\delta\bar{h}_i}(x_i)$ are disjoint and summing over i we obtain that

$$M_\gamma \gamma^{n/2-1} < C$$

and the conclusion follows.

□

Proof of Theorem 1.4.3. We first consider the set where $\bar{h}(x) \leq \frac{1}{t^{1/2}}$. At any point in this set, by Proposition 6.3.1, we can find some $r > 0$ such that $|\log r| > c|\log t|^{1/2}$ and

$$Mv(B_r(x)) > cr^{n-1}(\log t)^{\eta/2}.$$

We conclude that

$$\int_{B_r(x)} \Delta u \, dx \leq Cr^{n-1} \leq \frac{C}{(\log t)^{\eta/2}} Mv(B_r(x)).$$

Covering $\{\Delta u > t\} \cap \{\bar{h}(x) \leq \frac{1}{t^{1/2}}\}$ with these balls and taking a Vitali subcover $\{B_{r_i}(x_i)\}$, we obtain that

$$\int_{\{\Delta u > t\} \cap \{\bar{h}(x) < \frac{1}{t^{1/2}}\}} \Delta u \, dx \leq \frac{C}{(\log t)^{\eta/2}} \sum_i Mv(B_{r_i}(x_i)) \leq \frac{C}{(\log t)^{\eta/2}},$$

giving the desired bound over the "near-singular" points.

We now study the integral of Δu over the remaining subset of $\{\Delta u > t\}$. Take k_0 so that

$$2^{k_0 - 1} \leq t^{1/2} < 2^{k_0}.$$

Applying Lemma 6.3.4 we obtain that

$$\begin{aligned}
\int_{\{\Delta u > t\} \cap \{\bar{h}(x) > \frac{1}{t^{1/2}}\}} \Delta u \, dx &\leq \sum_{i=0}^{k_0} \int_{\{\Delta u > t\} \cap F_{2^{-i}}} \Delta u \, dx \\
&\leq Ct^{-\epsilon} \sum_{i=1}^{k_0} 2^{\epsilon i} \\
&\leq Ct^{-\epsilon/2},
\end{aligned}$$

giving the desired bound. □

We next closely examine the geometric properties of maximal sections of solutions in $D_{n,\lambda,\Lambda,K}$ to prove Proposition 6.3.1.

Let $u \in D_{n,\lambda,\Lambda,K}$ and fix $x \in B_{1/2}$. Then for any $h > \bar{h}(x)$, $S_h(x)$ is not compactly contained in ∂B_1. If $\bar{h}(x) > 0$, then by Lemma 3.3.2, $S_{\bar{h}(x)}(x)$ contains an ellipsoid E centered at x with a long axis of universal length $2c$.

If $\bar{h}(x) = 0$ and L is the tangent to u at x then $\{u = L\}$ has no extremal points (see chapter 3), and in particular for any $h > 0$ we know $S_h(x)$ contains a line segment (independent of h) exiting ∂B_1 at both ends.

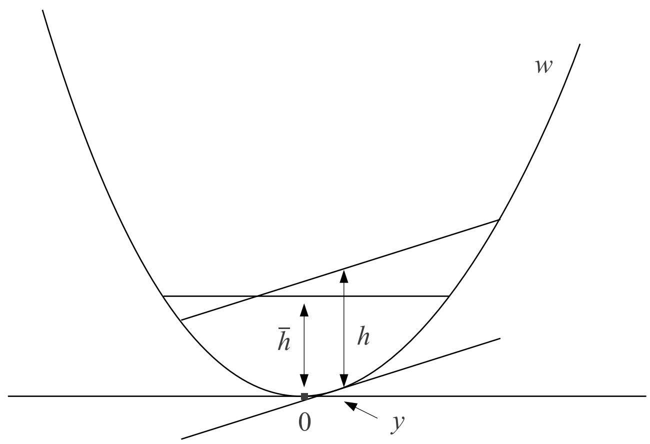

Figure 6.2: $S_h^w(y)$ satisfies property F if the tangent plane at y, lifted by h, lies above $\bar{h}$ at 0.

By translating and subtracting a linear function assume that $x = 0$ and $\nabla u(0) = 0$. By rotating assume that $S_h(0)$ contains the line segment from $-ce_n$ to ce_n for all $h > \bar{h}(0)$. For the rest of the section denote $\bar{h}(0)$ by just $\bar{h}$.

Let w be the restriction of u to $\{x_n = 0\}$ with sections S_h^w. Since $|S_h(0)| < Ch^{n/2}$ for all h and $S_{\bar{h}}(0)$ contains a line segment of universal length in the e_n direction, we have

$$|S_h^w(0)|_{\mathcal{H}^{n-1}} < Ch^{n/2}$$

for $h \geq \bar{h}$. In the following analysis we need to focus on those sections of w with the same volume bound. The following property is sufficient:

Property F: We say $S_h^w(y)$ satisfies property F if

$$w(y) + \nabla w(y) \cdot (-y) + h \geq \bar{h}.$$

(See Figure 6.2).

Lemma 6.3.5. *If $S_h^w(y)$ satisfies property F then*

$$|S_h^w(y)| < Ch^{n/2}.$$

Proof. The plane $u(y)+\nabla u(y)\cdot(z-y)+h$ is greater than $\bar{h}$ along $z=te_n$ for either $t>0$ or $t<0$. Since $u<\bar{h}$ on the segment from $-ce_n$ to ce_n, it follows that $S_h(y)$ contains the line segment from 0 to ce_n or $-ce_n$. Since $|S_h(y)|<Ch^{n/2}$ the conclusion follows. □

The first key lemma says that w grows logarithmically faster than quadratic in at least two directions at a level comparable to $\bar{h}$. Let

$$d_1^y(h)\geq d_2^y(h)\geq ...\geq d_{n-1}^y(h)$$

denote the axis lengths of the John ellisoid for $S_h^w(y)$.

Lemma 6.3.6. *For any $h>\bar{h}$ there exist $\epsilon(n)$, C_0 universal, $h_0<e^{-|\log h|^{1/2}}$ and y such that $S_{h_0}^w(y)$ satisfies property F and*

$$d_{n-2}^y(h_0)<C_0h_0^{1/2}|\log h_0|^{-\epsilon}.$$

The next lemma says that if w grows logarithmically faster than quadratic in at least two directions up to height h then the Monge-Ampère mass of $u+\frac{1}{2}|x|^2$ is logarithmically larger than the mass of Δu in a ball with radius comparable to $h^{1/2}$.

Lemma 6.3.7. *Fix $\epsilon>0$ and assume that for some $h>0$, $S_h^w(y)$ satisfies property F. Then there exist $\eta_1,\eta_2(n,\epsilon)$ and C depending on universal constants and ϵ such that if*

$$d_{n-2}^y(h)<h^{1/2}|\log h|^{-\epsilon}$$

then for some $r<Ch^{1/2}|\log h|^{-\eta_1}$ we have

$$M\left(u+\frac{1}{2}|x|^2\right)(B_r(0))>C^{-1}r^{n-1}|\log r|^{\eta_2}.$$

These lemmas combine to give the key proposition:

Proof of Proposision 6.3.1: By Lemma 6.3.6, there is some $S_h(y)$ satisfying property F with

$$d_{n-2}^y(h)<C_0h^{1/2}|\log h|^{-\epsilon},$$

with $\epsilon(n)$, C_0 universal and $h<e^{-|\log(\delta+\bar{h}(x))|^{1/2}}$ for any δ. The conclusion follows from Lemma 6.3.7. □

We now turn to the proofs of Lemmas 6.3.6 and 6.3.7.

Proof of Lemma 6.3.6. Assume by way of contradiction that for all $h < h_0$ and $S_h^w(y)$ satisfying property F we have

$$d_{n-2}^y(h) > C_0 h^{1/2} |\log h|^{-\epsilon},$$

for h_0 depending on $\bar{h}$ and C_0, ϵ we will choose later. We divide the proof into two steps.

Step 1: Define the breadth $b(h)$ as the minimum distance between two parallel tangent hyperplanes to $\partial S_h^w(0)$. We show that for $\bar{h}|\log \bar{h}| < h < h_0$ we have

$$b(h/2) > \left(\frac{1}{2} + \frac{C_1}{|\log h|}\right) b(h)$$

for some C_1 large depending on C_0. Let x_0 be the center of mass of $S_h^w(0)$ and rotate so that the John ellipsoid for $S_h^w(0)$ is $A(B_1) + x_0$, where

$$A = \mathrm{diag}(d_1^0(h), ..., d_{n-1}^0(h)).$$

Let P_1, P_2 be the tangent hyperplanes to $\partial S_{h/2}^w(0)$ a distance $b(h/2)$ apart. Let x_1, x_2 be points where P_1 and P_2 become tangent to $\partial S_h^w(0)$ when we slide them out. Assume that the distance between 0 and the plane tangent at x_1 is larger than that between 0 and the plane tangent at x_2. (See Figure 6.3).

Let $\tilde{x}_1$ be the image of x_1 under A^{-1} and let

$$\tilde{w}(x) = (\det A)^{-2/n} w(Ax).$$

Observe that $\tilde{w}$ is the restriction of $\tilde{u}(x) = (\det A)^{-2/n} u(Ax', x_n)$ which solves $\lambda \leq \det D^2 u \leq \Lambda$, so that sections $S_h^{\tilde{w}}$ of $\tilde{w}$ satisfying property F with $\bar{h}$ replaced by $(\det A)^{-2/n}\bar{h}$ have volume bounded above by $Ch^{n/2}$. Furthermore, since the distance between 0 and the plane tangent at x_1 was larger and the images of the tangent planes under A^{-1} are separated by distance at least 2, we have $|\tilde{x}_1| \geq 1$.

By convexity we can find $\tilde{y}$ on the line segment connecting 0 to $\tilde{x}_1$ such that

$$\nabla \tilde{w}(\tilde{y}) \cdot \frac{\tilde{x}_1}{|\tilde{x}_1|} = \frac{H}{|\tilde{x}_1|},$$

where $H = \det A^{-2/n} h$ is the height of $\tilde{w}$. Let $\tilde{h}$ be the smallest t such that $0 \in S_t^{\tilde{w}}(\tilde{y})$. We aim to bound $\tilde{h}$ below, which heuristically rules out cone-like behavior in the $\tilde{x}_1$ direction. Let

$$h^* = \tilde{h} + (\det A)^{-2/n}\bar{h}.$$

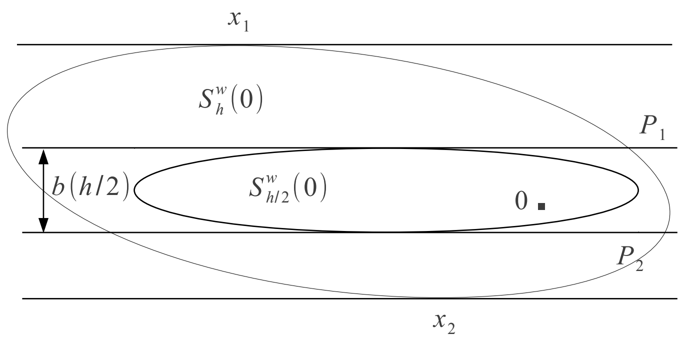

Figure 6.3:

We have chosen h^* so that $S_{h^*}^{\tilde{w}}(\tilde{y})$ and $S_\delta^w(y) = A(S_{h^*}^{\tilde{w}}(\tilde{y}))$ satisfy property F, where $\delta = (\det A)^{2/n}h^*$. (See Figure 6.4). It follows that

$$|S_{h^*}^{\tilde{w}}(\tilde{y})| < C(h^*)^{n/2}.$$

We now bound the volume of $S_{h^*}^{\tilde{w}}(\tilde{y})$ by below. Since $0, \tilde{x}_1$ are in this section, it has diameter at least 1. Since $\tilde{w}$ has height H it has interior Lipschitz constant $\frac{C}{H}$, so the smallest axis of the John ellipsoid for $S_{h^*}^{\tilde{w}}(\tilde{y})$ has length at least $c\frac{h^*}{H}$. We turn to the remaining axes.

Let E_y be the John ellipsoid for $S_\delta^w(y)$. By contradiction hypothesis for any $n-2$ dimensional plane P passing through the center of E_y, we can find a $n-3$ dimensional plane P' contained in P such that $P' \cap E_y$ is an $n-3$ dimensional ellipsoid with axes $d_{1,P'}^y \geq ... \geq d_{n-3,P'}^y$ satisfying

$$d_{n-3,P'}^y > C_0\delta^{1/2}|\log\delta|^{-\epsilon}.$$

Take P such that $A^{-1}(P)$ is perpendicular to the segment connecting 0 and $\tilde{x}_1$. By using the hypothesis and that w is locally Lipschitz we have

$$d_{n-2}^0(h)d_{n-1}^0(h) > cC_0h^{3/2}|\log h|^{-\epsilon}.$$

Since

$$d_1^0(h)...d_{n-1}^0(h) < Ch^{\frac{n}{2}},$$

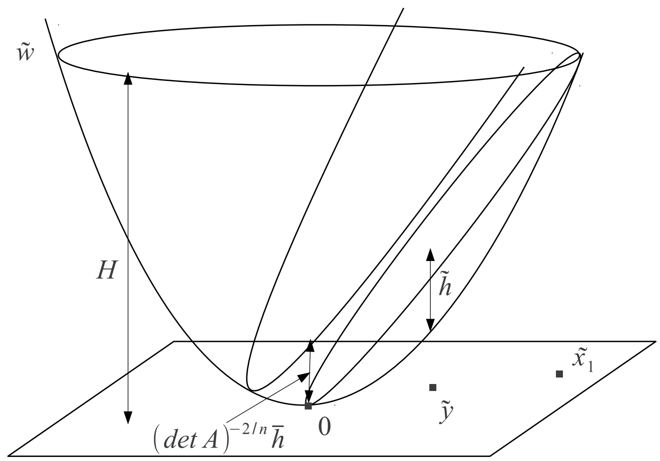

Figure 6.4: Lifting the tangent plane at $\tilde{y}$ by $h^* = \tilde{h} + \det(A)^{-2/n}\bar{h}$ we obtain a section of $\tilde{w}$ satisfying property F.

this gives

$$d_1^0(h)...d_{n-3}^0(h) < \frac{C}{C_0} h^{\frac{n-3}{2}} |\log h|^{\epsilon}.$$

It follows that A^{-1} changes the $n-3$ dimensional volume of $P' \cap E_y$ by a factor of at least

$$\frac{c(n)}{d_1^0(h)...d_{n-3}^0(h)} \geq cC_0 h^{-\frac{n-3}{2}} |\log h|^{-\epsilon}.$$

Since

$$\det A > ch^{n/2} |\log h|^{-C(n)\epsilon}$$

(by the contradiction hypothesis) and $\delta = (\det A)^{2/n} h^*$ we conclude that

$$\begin{aligned} |S_{h^*}^{\tilde{w}}(\tilde{y}) \cap A^{-1}(P')|_{\mathcal{H}^{n-3}} &> C_1 \frac{(\delta^{1/2} |\log \delta|^{-\epsilon})^{n-3}}{d_1^0(h)...d_{n-3}^0(h)} \\ &\geq C_1 (h^*)^{\frac{n-3}{2}} (\det A)^{\frac{n-3}{n}} h^{-\frac{n-3}{2}} (C|\log h| + |\log h^*|)^{-C(n)\epsilon} \end{aligned}$$

for some large C_1 depending on C_0. We also have

$$H = h(\det A)^{-2/n} \leq |\log h|^{C(n)\epsilon}.$$

Using that the remaining axes have lengths at least 1 and $c\frac{h^*}{H}$ we obtain

$$|S_{h^*}^{\tilde{w}}(\tilde{y})| > C_1 (h^*)^{\frac{n-1}{2}} |\log h|^{-C(n)\epsilon} (C|\log h| + |\log h^*|)^{-C(n)\epsilon}.$$

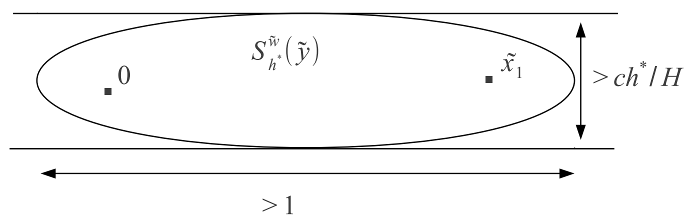

Figure 6.5: For the case $n = 3$, the above figure implies that $|S^{\tilde{w}}_{h^*}(\tilde{y})| > ch^*/H$. This, combined with the volume estimate $|S^{\tilde{w}}_{h^*}(\tilde{y})| < C(h^*)^{3/2}$ and the upper bound on H from the contradiction hypothesis give a lower bound of $c|\log h|^{-C\epsilon}$ for h^*.

Using that $|S^{\tilde{w}}_{h^*}(\tilde{y})| < C(h^*)^{n/2}$ we get a lower bound on h^*:

$$h^* > C_1|\log h|^{-C(n)\epsilon}.$$

(See Figure 6.5 for the simple case $n = 3$.)

Recalling the definition of h^* and using again the lower bound on $\det A$ it follows that

$$\tilde{h} + C\frac{\bar{h}}{h}|\log h|^{C(n)\epsilon} > C_1|\log h|^{-C(n)\epsilon}.$$

Taking ϵ to be small enough that $C(n)\epsilon = 1/2$ and using that $\bar{h}|\log \bar{h}| < h$ we get

$$\tilde{h} > C_1|\log h|^{-1/2}.$$

Finally, let $\left(\frac{1}{2} + \gamma\right)\tilde{x_1}$ be the point where $\tilde{w} = \frac{H}{2}$. It is clear from convexity (see Figure 6.6) that

$$2\gamma H \geq \tilde{h}.$$

Recalling that $H < c|\log h|^{C(n)\epsilon} < c|\log h|^{1/2}$, we obtain

$$\gamma \geq C_1|\log h|^{-1}.$$

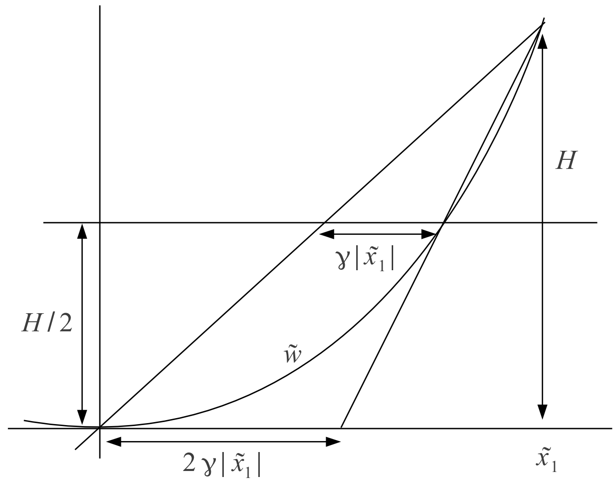

Figure 6.6: By convexity 2γ is at least $\tilde{h}/H$, giving a quantitative modulus of continuity for ∇w near 0 which we exploit in Step 2 to obtain a contradiction.

Let l_1, l_2 be the distances from 0 to the translations of P_1 and P_2 which are tangent to $\partial S_h^w(0)$ so that $b(h) \leq l_1 + l_2$. The previous analysis implies that P_1 and P_2 have distance at least $\left(\frac{1}{2} + \gamma\right) l_1$ and $\frac{1}{2} l_2$ from 0. Since $l_1 \geq l_2$ it follows that

$$b(h/2) \geq \left(\frac{1}{2} + \gamma\right) l_1 + \frac{1}{2} l_2 \geq \left(\frac{1+\gamma}{2}\right)(l_1 + l_2).$$

Since $\gamma \geq \frac{C_1}{|\log h|}$, step 1 is finished.

Step 2: We iterate step 1 to prove the lemma. First assume that $\bar{h} > 0$ and that $\bar{h}|\log \bar{h}| = 2^{-k}$ and $h_0 = 2^{-k_0}$. Note that $d_{n-1}^0(h) > c(n)b(h)$ and that $d_{n-1}^0(h_0) > c2^{-k_0}$ since u is locally Lipschitz. Iterating step 1 for C_1 large we obtain

$$\begin{aligned} d_{n-1}^0(2^{-k}) &\geq c(1/2 + C_1/k)(1/2 + C_1/(k-1))...(1/2 + C_1/k_0)2^{-k_0} \\ &\geq c2^{-k} \exp(C_1 \sum_{i=k_0}^{k} \frac{1}{i}) \\ &\geq 2^{-k} \frac{k}{k_0}, \end{aligned}$$

showing that

$$d_{n-1}^0(\bar{h}|\log \bar{h}|) \geq c\bar{h}|\log \bar{h}| \left(|\log \bar{h}||\log h_0|^{-1}\right).$$

Finally, take $|\log h_0| = |\log \bar{h}|^{1/2}$. We conclude using convexity that

$$d^0_{n-1}(\bar{h}) > |\log \bar{h}|^{-1} d(\bar{h}|\log \bar{h}|) > c\bar{h}|\log \bar{h}|^{1/2}.$$

Since

$$d^0_1(\bar{h})...d^0_{n-1}(\bar{h}) < C\bar{h}^{n/2}$$

we thus have

$$d^0_{n-2}(\bar{h}) < C\bar{h}^{1/2}|\log \bar{h}|^{-\epsilon(n)},$$

giving the desired contradiction.

In the case that $\bar{h} = 0$, we may run the above iteration for any $h > 0$ starting at height $h_0 = e^{-|\log h|^{1/2}}$ to obtain the contradiction. □

Proof of Lemma 6.3.7. First assume that $d^y_1(h) < h^{1/2}|\log h|^{-\alpha_1}$ for some α_1. Since $|S^w_h(y)| < Ch^{n/2}$, Lemma 3.2.5 gives

$$Mw(S^w_h(y)) > ch^{\frac{n-2}{2}}.$$

Take $C(n)$ large enough that for $r = C(n)h^{1/2}|\log h|^{-\alpha_1}$,

$$S^w_h(y) \subset B_{r/2}(0).$$

Clearly,

$$M\left(\frac{1}{2}|x|^2 + w\right)(S^w_h(y)) > Mw(S^w_h(y)).$$

Furthermore, $\nabla\left(u + \frac{1}{2}|x|^2\right)(B_r(0))$ contains a ball of radius $r/2$ around every point in $\nabla\left(u + \frac{1}{2}|x|^2\right)(S^w_h(y))$ (see Figure 6.7). We conclude that

$$\begin{aligned} M\left(u + \frac{1}{2}|x|^2\right)(B_r(0)) &> crMw(S^w_h(y)) \\ &\geq crh^{\frac{n-2}{2}} \\ &\geq cr^{n-1}|\log h|^{(n-2)\alpha_1} \\ &\geq cr^{n-1}|\log r|^{(n-2)\alpha_1}. \end{aligned}$$

We proceed inductively. Assume that $d^y_i(h) > h^{1/2}|\log h|^{-\alpha_i}$ for $i = 1, ..., k-1$ and that

$$d^y_k(h) < h^{1/2}|\log h|^{-\alpha_k}$$

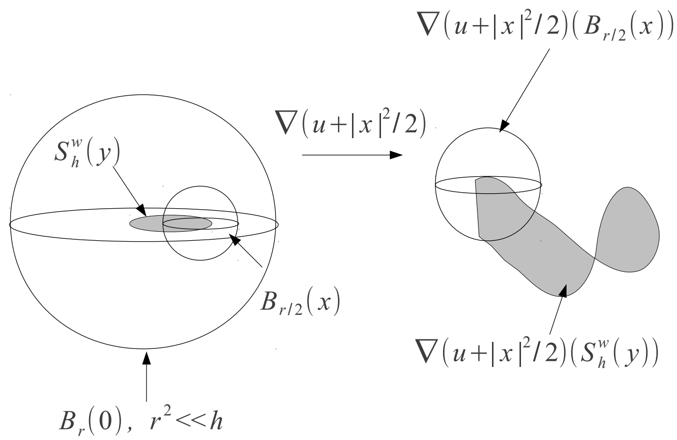

Figure 6.7: $\nabla(u+|x|^2/2)(B_r(0))$ contains an $r/2$-neighborhood of the surface $\nabla(u+|x|^2/2)(S_h^w(y))$, which projects in the x_n direction to a set of $\mathcal{H}^{n-1}$ measure at least $cr^{n-2}|\log r|^{(n-2)\alpha_1}$.

for some $\alpha_1, ..., \alpha_k$ to be chosen shortly. We aim to apply Lemma 3.2.5 to slices of the section $S_h^w(y)$ at 0, but we need the height of the plane $w(y) + \nabla w(y) \cdot (x - y) + h$ at 0 to be at least h. We thus consider $S_{2h}^w(y)$ instead. Note that $d_i^y(2h) > h^{1/2}|\log h|^{-\alpha_i}$ for $i \leq k-1$ and by convexity $d_k^y(2h) < 2h^{1/2}|\log h|^{-\alpha_k}$.

Rotate so that the axes align with those for the John ellipsoid of $S_{2h}^w(y)$. Take the restriction of w to the subspace spanned by $e_k, ..., e_{n-1}$, and call this restriction w_k. Let

$$S^{w_k} = S_{2h}^w(y) \cap \{x_1 = ... = x_{k-1} = 0\},$$

the slice of the section $S_{2h}^w(y)$ in this subspace. Then since

$$d_1^y(2h)...d_{n-1}^y(2h) \leq Ch^{\frac{n}{2}},$$

by hypothesis we have

$$|S^{w_k}|_{\mathcal{H}^{n-k}} \leq Ch^{\frac{n+1-k}{2}}|\log h|^{\alpha_1+...+\alpha_{k-1}}.$$

Since $S_h^w(y)$ contains 0 and S^{w_k} is the slice of $S_{2h}^w(y)$, we know that w_k has height at least h in S^{w_k}. Using this and Lemma 3.2.5,

$$Mw_k(S^{w_k}) \geq ch^{\frac{n-k-1}{2}}|\log h|^{-(\alpha_1+...+\alpha_{k-1})}.$$

Finally, take $C(n)$ large enough that for $r = C(n)h^{1/2}|\log h|^{-\alpha_k}$ we have

$$S^{w_k} \subset B_{r/2}(0).$$

By strict quadratic growth, $\nabla\left(u + \frac{1}{2}|x|^2\right)(B_r(0))$ contains a ball of radius $r/2$ around every point in $\nabla(u + \frac{1}{2}|x|^2)(S^{w_k})$. It follows that

$$\begin{aligned} M\left(u + \frac{1}{2}|x|^2\right)(B_r(0)) &\geq cMw_k(S^{w_k})r^k \\ &\geq ch^{\frac{n-k-1}{2}}|\log h|^{-(\alpha_1+...+\alpha_{k-1})}r^k \\ &\geq cr^{n-1}|\log r|^{(n-k-1)\alpha_k-(\alpha_1+...+\alpha_{k-1})}. \end{aligned}$$

Choose β_i so that $(n-k-1)\beta_k - (\beta_1 + ... + \beta_{k-1}) = 1$ and let $\alpha_i = c\beta_i$, with c chosen so that $\alpha_{n-2} = \epsilon$. If $d_1^y(h) < h^{1/2}|\log h|^{-\alpha_1}$, we are done by the first step, so assume not. Then apply the inductive step for $i = 2, ..., n-2$ to conclude the proof. □

6.4 Example showing optimality of Theorem 1.4.3

In this section we construct a solution to $\det D^2u = 1$ in $\mathbb{R}^3$ with a singular set of Hausdorff dimension exactly 2. A small modification gives the analagous example in $\mathbb{R}^n$ with a singular set of Hausdorff dimension $n-1$. This shows that the estimate on the Hausdorff dimension of the singular set in chapter 4 cannot be improved to $n-1-\delta$ for any δ.

We then show that the second derivatives of this solution are not in $L\log^M L$ for M large, showing that theorem 1.4.3 is optimal.

We proceed in several steps:

1. The key step is to construct a subsolution w in $\mathbb{R}^3$ satisfying $\det D^2w \geq 1$ that degenerates along $\{x_1 = x_2 = 0\}$ and grows logarithmically faster than quadratic in the x_1 direction, in particular like $x_1^2|\log x_1|^4$.

2. Next, we construct $S \subset [-1,1]$ of Hausdorff dimension 1 and a convex function v on $[-1,1]$ such that v separates from its tangent line faster than $r^2|\log r|^4$ at each point in S.

3. Finally, we obtain our example by solving the Dirichlet problem

$$\det D^2u = 1 \quad \text{in } \Omega = \{|x'| < 1\} \times (-1,1), \qquad u|_{\partial\Omega} = C(v(x_1) + |x_2|)$$

and comparing with w at points in $S \times \{0\} \times \{\pm 1\}$.

In the following analysis c, C will denote small and large constants respectively.

Construction of w: We first seek a function with just faster than quadratic growth in one direction and sections $S_h(0)$ with volume smaller than $h^{3/2}$. To that end, let

$$g(x_1, x_2) = x_1^2 |\log x_1|^\alpha + \frac{|x_2|}{|\log x_2|^\beta}$$

for some α, β to be chosen shortly. It is tempting to guess $w = g(x_1, x_2)(1 + x_3^2)$. However, the dominant terms in the determinant of the Hessian near the x_2 axis are

$$\frac{|\log x_1|^\alpha}{|\log x_2|^{2\beta}} \left(\frac{1}{|\log g|} - x_3^2 \right),$$

where the first comes from the diagonal entries and the second from the mixed derivatives. Thus, this function is not convex. This motivates the following modification:

$$w(x', x_3) = g(x') \left(1 + \frac{x_3^2}{|\log g(x')|} \right).$$

It is straightforward to check that the leading terms in the determinant of the Hessian (taking x_3 small) are

$$\frac{x_1^2 |\log x_1|^{2\alpha}}{|x_2 (\log x_2)^{\beta+1} \log g|} + \frac{|\log x_1|^\alpha}{|(\log x_2)^{1+2\beta} \log g|},$$

since now the mixed derivative terms have the same homogeneity in $\log(g)$ as the diagonal terms. For $|x'|$ small, the first term is large in $\{|x_2| < |x_1|^3\}$, and by taking $\alpha = 2 + 2\beta$ the second term is bounded below by a positive constant in $\{|x_2| \geq |x_1|^3\}$. Thus, up to rescaling and multiplying by a constant we have

$$\det D^2 w \geq 1$$

in $\Omega = \{|x'| < 1\} \times (-1, 1)$. For convenience, we take $\beta = 1$ and $\alpha = 4$ for the rest of the example.

Construction of S: Start with the interval $[-1/2, 1/2]$. For the first step remove an open interval of length $\frac{5}{6}$ from the center. At the k^{th} step, remove intervals a fraction $\frac{5}{k+5}$ of the length of the remaining 2^k intervals from their centers. Denote the centers of the removed intervals by $\{x_{i,k}\}_{i=1}^{2^k}$, and the intervals by $I_{i,k}$. Finally, let

$$S = [-1, 1] - \cup_{i,k} I_{i,k}.$$

Let $l_k = |I_{i,k}|$. It is easy to check

$$l_k = \frac{10}{k+5}2^{-k}\left(1-\frac{5}{k+4}\right)...\left(1-\frac{5}{6}\right)$$
$$\leq \frac{C}{k^6}2^{-k}.$$

One checks similarly that the length of the remaining intervals after the k^{th} step is at least

$$2^{-k}k^{-15}.$$

It follows that

$$\inf\left\{\sum_{i=1}^{\infty} r_i|\log(r_i)|^{15} : \{B_{r_i}(x_i)\} \text{ cover } S, r_i < \delta\right\} > c \tag{6.3}$$

for all $\delta > 0$. In particular, the Hausdorff dimension of S is exactly 1.

Construction of v: Let

$$f(x) = \begin{cases} |x| & |x| \leq 1 \\ 2|x|-1 & |x| > 1 \end{cases}$$

We add rescalings of f together to produce the desired function:

$$v(x) = \sum_{k=1}^{\infty} k^4 l_k^2 f(l_k^{-1}(x - x_{i,k})).$$

We now check that v satisfies the desired properties:

1. v is convex, as the sum of convex functions. Furthermore, using that $l_k < C2^{-k}k^{-6}$ we have

$$|v(x)| \leq C\sum_{k=1}^{\infty}\sum_{i=1}^{2^k} k^4 l_k$$
$$\leq C\sum_{k=1}^{\infty} k^{-2}$$
$$\leq C$$

so v is bounded.

2. Let $x \in S$. We aim to show that v separates from a tangent line more than $r^2|\log(r)|^4$ a distance r from x. By subtracting a line assume that $v(x) = 0$ and that 0 is a subgradient at x. Assume further that $x + r < 1/2$ and that $l_k < r \leq l_{k-1}$. There are two cases to examine:

Case 1: There is some $y \in (x + r/2, x + r) \cap S$. Then by the construction of S it is easy to see that there is some interval $I_{i,k}$ such that $I_{i,k} \subset (x, x + r)$. On this interval, v grows by

$$k^4 l_k^2 \geq c l_k^2 |\log(l_k)|^4 \geq cr^2 |\log(r)|^4.$$

Case 2: Otherwise, there is an interval $I_{i,j}$ of length exceeding $r/2$ such that $(x+r/2, x+r) \subset I_{i,j}$. Then at the left point of $I_{i,j}$, the slope of v jumps by at least $k^4 l_k$. It follows that at $x + r$, v is at least

$$crk^4 l_k \geq cr^2 |\log(r)|^4.$$

Thus, v has the desired properties.

Construction of u: Let $\varphi(x_1, x_2, x_3) = C(v(x_1) + |x_2|)$ for a constant C we will choose shortly, and obtain u by solving the Dirichlet problem

$$\det D^2 u = 1 \quad \text{ in } \Omega = \{|x'| < 1\} \times [-1, 1], \qquad u|_{\partial\Omega} = \varphi.$$

Take $x \in S \times \{0\} \times \{\pm 1\}$. By translating and subtracting a linear function assume that $x_1 = 0$ and 0 is a subgradient for φ at x. Taking C large we guarantee that

$$\varphi(x_1, x_2, \pm 1) > C(x_1^2 |\log(x_1)|^4 + |x_2|) > w(x_1, x_2, \pm 1)$$

for all x_1, x_2, and that that $\varphi > w$ on the sides of Ω. Thus, $u \geq w$ in all of Ω. Since $u = 0$ at both $(0, 0, \pm 1)$ and $w(0, 0, x_3) = 0$ for all $|x_3| < 1$, we have by convexity that $u = 0$ along $(0, 0, x_3)$.

This shows that for these examples

$$\Sigma \subset S \times \{0\} \times (-1, 1),$$

which has Hausdorff dimension exactly 2.

Remark 6.4.1. To get the analagous example in $\mathbb{R}^n$, take

$$u(x_1, x_2, x_3) + x_4^2 + ... + x_n^2.$$

We now examine the integrability of the second derivatives for the singular solution constructed above. In chapter 4 we construct, for any ϵ, solutions to $\det D^2u = 1$ in $\mathbb{R}^n$ that are not in $W^{2,1+\epsilon}$, but as $\epsilon \to 0$ these examples blow up. We aim to improve this by showing that the example above is not in $W^{2,1+\epsilon}$ for any ϵ, and in fact the second derivatives are not in $L\log^M L$ for M large.

Let $\phi(x) = (1+x)(\log(1+x))^M$ for some M large. Then ϕ is convex for $x \geq 0$, so for any nonnegative integrable function f and ball B_r we have by Jensen's inequality that

$$\int_{B_r} \phi(r^n f(x))\,dx \geq cr^n \phi\left(\int_{B_r} f(x)\,dx\right).$$

Taking $f(x) = r^{-n}\Delta u(x)$ we obtain

$$\int_{B_r} (1+\Delta u)(\log(1+\Delta u))^M\,dx \geq c\left(\int_{B_r} \Delta u\,dx\right)\left(\log\left(r^{-n}\int_{B_r} \Delta u\,dx\right)\right)^M.$$

Recall that at points $x \in S \times \{0\} \times (-1,1)^{n-2}$ the subsolutions w touch u by below, and that w grows like $|x_2||\log x_2|^{-1}$ at x. It follows that

$$\sup_{\partial B_r(x)} (u - u(x)) \geq cr|\log r|^{-1}.$$

Applying convexity we conclude that

$$\begin{aligned}\int_{B_r(x)} (1+\Delta u)\left(\log(1+\Delta u)\right)^M\,dx &\geq c\left(\int_{\partial B_r(x)} u_\nu\right)\left(\log\left(r^{-n}\int_{\partial B_r(x)} u_\nu\right)\right)^M \\ &\geq cr^{n-1}|\log r|^{-1}\left(\log(cr^{-1}|\log(r)|^{-1})\right)^M \\ &\geq cr^{n-1}|\log r|^{M-1}.\end{aligned}$$

Cover $\Sigma \cap B_{1/2}$ with balls of radius less than δ and take a Vitali subcover $\{B_{r_i}\}_{i=1}^N$. We then have

$$\int_{B_{1/2}} (1+\Delta u)\left(\log(1+\Delta u)\right)^M\,dx \geq c\sum_{i=1}^N r_i^{n-1}|\log r_i|^{M-1},$$

and for M large the right side goes to ∞ as $\delta \to 0$ by equation 6.3.

Thus, the second derivatives of u are not in $L\log^M L$ for M large, and in particular u is not in $W^{2,1+\epsilon}$ for any $\epsilon > 0$.

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