SOBOLEV REGULARITY FOR OPTIMAL TRANSPORT MAPS OF NON-CONVEX PLANAR DOMAINS

CONNOR MOONEY AND ARGHYA RAKSHIT

Abstract. We prove a sharp global $W^{2,p}$ estimate for potentials of optimal transport maps that take a certain class of non-convex planar domains to convex ones.

1. Introduction

Optimal transport maps play an important role in physics, geometry, economics, and meteorology. The regularity of optimal transport maps is a delicate matter that for the most part has focused on the case that the source and target domains are convex. However, this condition is not satisfied in many applications. In this paper we initiate the study of the Sobolev regularity of optimal transport maps in the plane, where the source domain is non-convex.

The setting is as follows. Let Ω_1 and Ω_2 be bounded domains in $\mathbb{R}^2$ of unit area. We assume that Ω_2 is convex. Then the optimal transport map from Ω_1 to Ω_2 is the gradient map of a convex function u on $\mathbb{R}^2$ which satisfies (see [7]):

$$\det D^2 u = \chi_{\Omega_1} \tag{1}$$

in the Alexandrov sense, u is smooth and locally uniformly convex in Ω_1, and

$$\nabla u(\Omega_1) = \Omega_2. \tag{2}$$

We assume further that Ω_1 is a convex domain Ω_0 with a finite number of disjoint, $C^{1,1}$, uniformly convex holes a positive distance δ from $\partial\Omega_0$ and from each other removed. Our main result is:

Theorem 1.1. *We have $u \in C^{1,1/2}\left(\overline{\Omega_1}\right)$ with norm depending only on the diameters of Ω_1 and Ω_2, δ, and the lower and upper bounds for the boundary curvatures of the holes in Ω_1. We also have $u \in W^{2,p}(\Omega_1)$ for any $p < 2$, with norm depending only on the same quantities and p.*

Theorem 1.1 is sharp. To see this, consider the radially symmetric example where Ω_1 is an annulus with inner radius r, Ω_2 is a disk, and the potential is

$$u(x) = \int_0^{|x|} (s^2 - r^2)_+^{1/2}\, ds. \tag{3}$$

Below we will refer to (3) as the model example.

One motivation for Theorem 1.1 comes from the semigeostrophic equations (SGEs) from meteorology. The SGEs lead one to consider optimal transport maps that take a bounded density on the torus to the uniform one (see e.g. [15]). When

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the source density is bounded between positive constants, $W^{2,1}$ estimates for the potential are available ([13], [14], [24]), which lead to long-time existence results for the SGEs ([2], [3], [15]). However, in physically interesting cases, the SGEs involve optimal transport maps where the source density is allowed to vanish. In this case, $W^{2,1}$ estimates for the potential do not always hold (see [22]). An important special situation is when the source density is the characteristic function of a domain (which need not be convex), as in the situation of Theorem 1.1. For the SGEs, this corresponds to a fully nonlinear analogue of the vortex patch problem for the 2D Euler equations. Global $W^{2,1}$ estimates for optimal transport maps of non-convex domains may be useful for extending long-time existence results for the SGEs to this situation.

More generally, Theorem 1.1 can be viewed as a step towards obtaining global regularity results for optimal transport maps of general non-convex source domains. The global regularity of optimal transport maps in the case of convex source and target domains is well-studied. Caffarelli proved that, in this case, the potentials are $C^{1,\alpha}$ up to the boundary, and $C^{2,\alpha}$ up to the boundary provided the domains are C^2 and uniformly convex [6]. Here α is small. The conditions on the domains required for global $C^{2,\alpha}$ regularity of the potential were recently relaxed to $C^{1,1}$ and merely convex [9], and even slightly non-convex but close to convex in the $C^{1,1}$ sense [10]. In two dimensions, Savin and Yu showed that convexity of the domains is enough to get global $W^{2,p}$ regularity for any $p < \infty$ [23]. As for the case of non-convex source domain, in [4] the authors obtain global $C^{1,\alpha}$ estimates for potentials of optimal transport maps in any dimension when the densities are bounded between positive constants, the target domain is convex, and the source domain is a convex set with finitely many convex holes removed, using ideas from [5]. (Again, here α is small). Our methods (described below) are quite different from those in [4], and the smoothness of the densities and the regularity properties of the holes play a delicate role in our analysis. We remark that our methods in fact apply near any "uniformly concave" part of the boundary of a general smooth planar source domain.

Our strategy is as follows. First, we may focus our attention on a neighborhood of the holes in Ω_1 (the "concave part" of the boundary of Ω_1), thanks to work of Savin-Yu which shows the $W^{2,p}$ regularity of u (for any p) near the "convex part" $\partial\Omega_0$ of the boundary of Ω_1 [23]. We carefully analyze the geometry of the sections of u (defined in Section 2) which are centered at concave boundary points. We show that there are three possible cases, all of which are "good." The first case is that the complement of Ω_1 fills only a tiny fraction of the section. In this case, morally speaking u solves $\det D^2u = 1$ in the whole section and we can control section geometry at smaller scales using the regularity theory for the Monge-Ampère equation. The second case is that the complement of Ω_1 fills a positive universal fraction of the section and the long axis of the section is transversal to the boundary. In this case we show that renormalization by an affine transformation flattens the boundary, and we are again in a good situation where section geometry can be controlled at smaller scales using Pogorelov-type estimates for the Monge-Ampère equation. The last case is that the complement of Ω_1 fills a positive universal fraction of the section, and the long axis of the section is roughly tangent to the boundary. In this case we are in a situation that resembles what happens at every inner boundary point for the model example, which has the desired regularity properties. Our analysis near the holes is valid for any solution of the Monge-Ampère equation, and does not use the convexity of Ω_2.

In the course of the proof we also prove a new interior second derivative estimate for solutions to $\det D^2 w = \chi_{\{x_2>0\}}$, which is special to two dimensions. The classical Pogorelov estimate bounds the tangential second derivative w_{11}. Although this suffices for our application, using the partial Legendre transform we are also able to bound the ratio $|w_{12}|/w_{11}$ from above (see Proposition 2.3). As a result, the sections of w centered on $\{x_2 = 0\}$ are well-approximated by ellipsoids whose axes are aligned with the coordinate axes. This result simplifies our proof, and may be useful for future applications.

The paper is organized as follows. In Section 2 we discuss some preliminary results about the geometry of centered sections, as well as some Pogorelov-type estimates (including the one mentioned in the previous paragraph). In Section 3 we prove several key lemmas, corresponding to the three scenarios mentioned above. In Section 4 we prove Theorem 1.1. In Section 5 we discuss some future directions. Finally, in the appendix Section 6 we prove some of the preliminary results.

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2. Preliminaries

For the remainder of the paper we fix a constant $\delta > 0$. We let $\mathcal{F}_\delta$ denote the space of convex functions on $\mathbb{R}^2$ that satisfy (1) and (2), where Ω_1 and Ω_2 have unit area and are contained in $B_{\delta^{-1}}$, Ω_2 is convex, and the source domain Ω_1 consists of a convex domain Ω_0 with convex holes removed, where the holes are separated a distance at least δ from one another and from the boundary of Ω_0, and the boundaries of the holes have lower and upper bounds δ, δ^{-1} on their curvature. (Here Ω_i are not fixed, they are any domains satisfying the above conditions). We note that $\mathcal{F}_\delta$ is a compact family, namely, any sequence in $\mathcal{F}_\delta$ contains (after possibly adding constants) a subsequence that converges locally uniformly on $\mathbb{R}^2$ to a function in $\mathcal{F}_\delta$. The local uniform convergence follows from the fact that the gradients lie in $B_{\delta^{-1}}$ and the Arzelà-Ascoli theorem. The fact that the limit lies in $\mathcal{F}_\delta$ uses the weak convergence of Monge-Ampère measures under local uniform convergence [20]. We call constants depending only on δ universal, and we say that positive quantities a and b satisfy $a \sim b$ if their ratio is trapped between positive universal constants. We call constants "absolute" if they are fixed positive numbers independent of δ. (Generally these are dimensional constants, which become absolute in view of the fact that we are working in the plane). Finally, we say that $a \approx b$ if their ratio is trapped between absolute constants.

Let $u \in \mathcal{F}_\delta$. For any $x \in \overline{\Omega}_1$ and $h > 0$ there exists an affine function $L_{x,h}$ such that

$$L_{x,h}(x) = u(x) + h$$

and such that the set $\{u < L_{x,h}\}$ is bounded and has center of mass x (see [6]). We call $\{u < L_{x,h}\}$ the centered section of height h at x, and we denote it by $S_h^u(x)$. One can show that u is not linear when restricted to any line segment centered at a point in $\overline{\Omega}_1$ (see Appendix). Combined with a compactness argument, this shows that there exists a universal modulus of continuity ω such that for any $x \in \overline{\Omega_1}$ and any $h < 1$,

$$\text{diam}(S_h^u(x)) \leq \omega(h). \tag{4}$$

In particular, for $h < c_0$ universal, we have that $S_h^u(x)$ intersects at most one connected component of Ω_1^c for any $x \in \overline{\Omega_1}$. Below we will always assume that $h \in (0,\, c_0)$, and we will only consider sections centered in $\overline{\Omega_1}$ that are either contained in Ω_1 or intersect a hole in Ω_1.

Such centered sections satisfy the area estimate

$$|S_h^u(x)| \approx h. \tag{5}$$

This estimate follows from the absolute positive density of Ω_1 in such sections; see Lemma 6.3 in the Appendix for a proof of (5). By a version of John's Lemma, there exist rectangles $R_h^u(x)$ centered at 0 of area $4h$ and $d_0 \approx 1$ such that

$$x + d_0 R_h^u(x) \subset S_h^u(x) \subset x + (d_0)^{-1} R_h^u(x). \tag{6}$$

We denote the short and long side lengths of $R_h^u(x)$ by $2\lambda_h^u(x)$ and $2\Lambda_h^u(x)$ respectively (note that $\lambda_h^u(x)\Lambda_h^u(x) = h$), and we define the eccentricity of $R_h^u(x)$ by the quantity

$$\eta_h^u(x) = \frac{\Lambda_h^u(x)}{\lambda_h^u(x)}.$$

Finally, we have the following engulfing property (see Appendix), which allows us to compare sections in Ω_1 tangent to a hole with a section centered on the boundary of the hole:

Proposition 2.1. *If $u \in \mathcal{F}_\delta$, $y \in \overline{S_h^u(x)} \cap (\partial\Omega_1 \backslash \partial\Omega_0)$ and $S_h^u(x) \subset \Omega_1$, then*

$$S_h^u(x) \subset y + R_{c^{-1}h}^u(y),$$

for some universal $c > 0$.

We now state some Pogorelov-type estimates. The first result is Pogorelov's interior C^2 estimate (see e.g. [20]):

Proposition 2.2. *If $\det D^2 w = 1$ in $S_1^w(0)$ and $B_\lambda \subset S_1^w(0)$ for some $\lambda > 0$, then $|D^2 w| < \kappa^{-1}$ in $\frac{1}{2}S_1^w(0)$ and*

$$B_{\kappa t^{1/2}} \subset S_h^w(0) \subset B_{\kappa^{-1} t^{1/2}}$$

for all $t < 1$ and some $\kappa(\lambda) > 0$.

Combining Proposition 2.2 with the affine invariance of the Monge-Ampère equation and the area estimate (5) we have

$$|D^2 u(x)| \approx \eta_h(x) \tag{7}$$

whenever $S_h^u(x) \subset \Omega_1$. To see this, assume after a translation and a rotation that $x = 0$ and that the long side of $R_h^u(0)$ is horizontal. Then

$$u(x_1,\, x_2) := hw(x_1/\Lambda_h^u(0),\, x_2/\lambda_h^u(0)),$$

where w satisfies the conditions of Proposition 2.2 with λ absolute. Thus, $|D^2 u(0)| \leq h\lambda_h^u(0)^{-2}|D^2 w(0)| \leq \kappa^{-1} h\lambda_h^u(0)^{-2}$ for some $\kappa > 0$ absolute. Using that $h =$

$\lambda_h^u(0)\Lambda_h^u(0)$ gives the upper bound in (7). The lower bound in (7) follows similarly using that $w_{22}(0) \approx 1$, which comes from the equation solved by w and the absolute upper bound on its Hessian at 0.

The next estimate is a variant of Pogorelov's interior C^2 estimate with flat boundary, which to our knowledge is new:

Proposition 2.3. *If* $\det D^2 w = \chi_{\{x_2>0\}}$ *in* $S_1^w(0)$ *and* $B_\lambda \subset S_1^w(0) \subset B_{\lambda^{-1}}$ *for some* $\lambda > 0$*, then*

$$\sup_{\frac{1}{2}S_1^w(0)\cap\{x_2>0\}} \left(w_{11} + \frac{|w_{12}|}{w_{11}} \right) \leq \kappa^{-1}$$

for some $\kappa(\lambda) > 0$.

The upper bound on w_{11} is the classical Pogorelov estimate (see [6]) and doesn't use that we are working in the plane. The upper bound on $|w_{12}|/w_{11}$ uses that we are in the plane.

Proof. We may assume after subtracting a linear function that $w|_{\partial S_1^w(0)} = 0$. Let w^* denote the partial Legendre transform of w (the Preliminaries section of [11] discusses the definition and relevant properties, and we recall these in Section 6.5 of the Appendix for the reader's convenience). The function w^* is convex in the first variable, concave in the second, and formally solves

$$\chi_{\{x_2>0\}} w_{11}^* + w_{22}^* = 0$$

in $B_{\kappa_1}(0)$ for some $\kappa_1(\lambda) > 0$. More precisely, w^* is harmonic in $\{x_2 > 0\}$, linear on vertical segments in $\{x_2 < 0\}$, and moreover w_2^* has the same limit from above and below on $\{x_2 = 0\}$ along vertical lines. It is not hard to verify the first two properties by approximating $\chi_{\{x_2>0\}}$ with smooth positive functions of x_2. The third property can be verified using that $w \in C^1$ ([1], [18]). Since w^* is linear on vertical segments in $\{x_2 \leq 0\}$ we have

$$w_2^*(x_1, 0) = a^{-1}(w^*(x_1, 0) - w^*(x_1, -a))$$

for any $a > 0$. Choosing $a = \kappa_2(\lambda) > 0$ and using that w^* is convex (hence locally Lipschitz with Lipschitz constant bounded by a constant depending only on λ) in the horizontal directions, we conclude that w_2^* is Lipschitz on $\{x_2 = 0\}$. In particular, w_2^* is harmonic in $B_{\kappa_1} \cap \{x_2 > 0\}$ and Lipschitz on $\{x_2 = 0\}$ (with Lipschitz constant bounded above by a constant depending only on λ). It follows from harmonic function theory that $|w_{12}^*| < \kappa_3^{-1}$ in $B_{\kappa_1/2} \cap \{x_2 > 0\}$ for some $\kappa_3(\lambda) > 0$. To see this one can apply Lemma 6.6 from the Appendix, appropriately rescaled. Using the relation

$$w_{12} = -w_{12}^* w_{11}$$

(see (18)) we obtain the desired estimate on $|w_{12}|/w_{11}$. □

As a result of Proposition 2.3, under the same assumptions we can say that $S_t^w(0)$ is approximated by (contains and is contained in dilations by constants depending only on λ of) a rectangle with axes that are aligned with the coordinate axes for all $t < 1$. Indeed, if not, then $S_t^w(0)$ is approximated (contains and is contained in dilations by absolute constants times) an ellipsoid of the form $A_t B_1$, where

$$A_t(x_1, x_2) = \left(A\sqrt{t}(x_1 + Kx_2),\, A^{-1}\sqrt{t}x_2 \right),$$

$A \geq \kappa > 0$ (this follows from the upper bound on w_{11}), and $|K| >> 1$. (Here and for the rest of this paragraph, κ denotes a small positive constant depending on λ). The function

$$v(x) = \frac{1}{t} w(A_t x)$$

satisfies the conditions of Proposition 2.3 with λ absolute (call it $2r$). In addition, we have

$$w_{12} = v_{12} - K v_{11}.$$

In $B_r(re_2)$ we can find points where $v_{11} \geq k$ and $|v_{12}| < k^{-1}$ for some absolute constant $k > 0$ to arrive at a contradiction of Proposition 2.3 when $|K|$ is sufficiently large depending on λ.

3. Key Lemmas

In this section we prove some lemmas that control the geometry of sections of $u \in \mathcal{F}_\delta$ centered at concave boundary points in various scenarios. We will use several times below the standard fact that if w_k are convex functions with $B_\lambda \subset S_1^{w_k}(0) \subset B_{\lambda^{-1}}$ for some $\lambda > 0$, $w_k|_{\partial S_1^{w_k}(0)} = 0$ and $\det D^2 w_k$ are uniformly bounded above, then a subsequence of w_k converges uniformly to a convex function w satisfying the same properties, and the Monge-Ampère measures converge weakly to that of the limit.

The first lemma deals with the case that Ω_1^c bites only a small fraction of the section.

Lemma 3.1. *There is an absolute constant $c_1 > 0$ such that the following holds. Given $M > 1$, there exists $\epsilon > 0$ depending on δ and M such that if $u \in \mathcal{F}_\delta$, $x \in \partial\Omega_1 \backslash \partial\Omega_0$ and*

$$\frac{|(x + R_h^u(x)) \cap \Omega_1^c|}{|R_h^u(x)|} \leq \epsilon,$$

then

$$\eta_{h/M}^u(x) \leq c_1^{-1} \eta_h^u(x)$$

and

$$R_{h/M}^u(x) \subset c_1^{-1} M^{-1/2} R_h^u(x).$$

Proof. Assume by way of contradiction that the lemma is false. Then there is a sequence $u_k \in \mathcal{F}_\delta$, points x_k on the boundaries of the holes in the source domains, and $h_k > 0$ such that the area fraction ϵ_k of the complements of the source domains in $x_k + R_{h_k}^{u_k}(x_k)$ tends to zero but the conclusions don't hold for a choice of absolute constant c_1 to be explained below and all k. After performing a rigid motion we may assume that $x_k = 0$ and that that $R_{h_k}^{u_k}(0)$ have short side vertical. Then up to adding affine functions and taking a subsequence, the rescalings $w_k := h_k^{-1} u_k(\Lambda_{h_k}^{u_k}(0) x_1, \lambda_{h_k}^{u_k}(0) x_2)$ converge locally uniformly to a function w that satisfies the conditions of Proposition 2.2, with $\lambda = a_0 > 0$ absolute. Applying Proposition 2.2 we conclude that $B_{2a_1 M^{-1/2}} \subset S_{1/M}^w(0) \subset B_{(2a_1)^{-1} M^{-1/2}}$ for some $a_1 > 0$ absolute. In particular, $B_{a_1 M^{-1/2}} \subset S_{1/M}^{w_k}(0) \subset B_{(a_1)^{-1} M^{-1/2}}$ for k large. It follows after scaling back that

$$a_2 M^{-1/2} R_{h_k}^{u_k}(0) \subset R_{h_k/M}^{u_k}(0) \subset (a_2)^{-1} M^{-1/2} R_{h_k}^{u_k}(0)$$

for some $a_2 > 0$ absolute and all k large, hence

$$\lambda_{h_k/M}^{u_k}(0) \geq a_2 M^{-1/2} \lambda_{h_k}^{u_k}(0) \text{ and } \Lambda_{h_k/M}^{u_k}(0) \leq 2(a_2)^{-1} M^{-1/2} \Lambda_{h_k}^{u_k}(0).$$

Define $c_1 := a_2^2/4$, and note that c_1 is an absolute positive constant. For this choice, we get the desired contradiction when k is large. □

We define

$$M_1 := c_1^{-6}, \tag{8}$$

where c_1 is the absolute constant in Lemma 3.1, and we let ϵ_1 be the volume fraction from the lemma corresponding to the choice $M = M_1$. (In particular, ϵ_1 is universal).

The next lemma deals with the case that Ω_1^c bites a positive fraction of the section, and the long axis is transversal to the boundary. We let $l_h^u(x)$ denote the length of the intersection of the tangent line to $\partial\Omega_1$ at $x \in \partial\Omega_1$ with $x + R_h^u(x)$. This lemma uses the regularity and uniform convexity of the boundary.

Lemma 3.2. *There exists a universal constant $c_2 > 0$ such that the following holds. Given $M > 1$, there exists $\epsilon > 0$ depending on δ and M such that if $u \in \mathcal{F}_\delta$, $x \in \partial\Omega_1 \backslash \partial\Omega_0$, and in addition*

$$\eta_h^u(x) > c_2^{-1}M, \quad \frac{|(x + R_h^u(x)) \cap \Omega_1^c|}{|R_h^u(x)|} > \epsilon_1, \text{ and } \quad \frac{l_h^u(x)}{\Lambda_h^u(x)} \leq \epsilon,$$

then

$$\eta_{h/M}^u(x) \leq c_2^{-1}\eta_h^u(x).$$

Furthermore, if

$$\eta_{h/M}^u(x) > \eta_h^u(x),$$

then

$$R_{h/M}^u(x) \subset c_2^{-1}M^{-1/2}R_h^u(x).$$

Proof. Assume by way of contradiction that the lemma is false, for a choice of universal constant c_2 explained below. Then there is a sequence $u_k \in \mathcal{F}_\delta$, points x_k on the boundaries of the holes in the source domains Ω_{1k}, and $h_k > 0$ such that the first two inequalities hold for $R_{h_k}^{u_k}(x_k)$ and

$$l_{h_k}^{u_k}(x_k)/\Lambda_{h_k}^{u_k}(x_k) < 1/k, \tag{9}$$

but the conclusion is false for all k. After performing a rigid motion we may assume that $x_k = 0$, that $R_{h_k}^{u_k}(0)$ have short side vertical, and that te_2 and se_1 are contained in the source domain for $t \in (0, \lambda_{h_k}^{u_k}(0))$ and $s \in (-\Lambda_{h_k}^{u_k}(0), 0)$. From hereon out we write $R_k = R_{h_k}^{u_k}(0)$, $\lambda_k = \lambda_{h_k}^{u_k}(0)$, and $\Lambda_k = \Lambda_{h_k}^{u_k}(0)$. We claim that up to taking subsequence and adding affine functions, the rescalings $h_k^{-1}u_k(\Lambda_k x_1, \lambda_k x_2)$ converge locally uniformly to a function w such that $S_1^w(0)$ contains and is contained in balls of absolute radius, and Proposition 2.3 applies to w, up to swapping x_1 and x_2 (see Figure 1).

Before proceeding we note that (9) implies that $\lambda_k\Lambda_k^{-1} \to 0$, which in turn implies that $h_k \to 0$ (by the area estimate for centered sections) and thus that $\Lambda_k \to 0$ (by the universal modulus of continuity for the diameters of centered sections).

Let $s_k \geq 0$ be the slope of the tangent line to the boundary of the source domain at 0 (we allow $s_k = \infty$). Inequality (9) implies that

$$s_k\Lambda_k\lambda_k^{-1} > k. \tag{10}$$

We first treat the case that $s_k > 1$ for some subsequence. Then for k large, the fact that Λ_k tend to zero and elementary geometry imply that

$$\{x_2 < |x_1|/2,\, x_1 > 0\} \cap S^{u_k}_{h_k}(0) \subset \Omega_1^c. \tag{11}$$

After the change of variable $x_1 := \Lambda_k \tilde{x}_1,\ x_2 := \lambda_k \tilde{x_2}$, the domains $\{x_2 < |x_1|/2,\, x_1 > 0\}$ become $\{\tilde{x}_2 < \Lambda_k\lambda_k^{-1}|\tilde{x}_1|/2,\, \tilde{x}_1 > 0\}$. Since the latter domains converge to the right half-space, we conclude that w solves $\det D^2 w = \chi_{\{x_2<0\}}$ in $S^w_1(0)$, as desired.

The alternative is that $s_k \leq 1$ for all k large. Uniform concavity of the boundary implies that $\{x_2 < s_k x_1 - a_0 x_1^2\}$ contain the holes with 0 in their boundary for some $a_0 > 0$ universal. Since

$$R_k \cap \{x_2 < s_k x_1 - a_0 x_1^2\} \subset R_k \cap \{|x_1| < a_1^{-1}(s_k^2 + \lambda_k)^{1/2}\}$$

for some $a_1 > 0$ universal, the lower bound on the volume of the complement implies that

$$\Lambda_k^2 \leq a_2^{-1}(s_k^2 + \lambda_k)$$

for some $a_2 > 0$ universal. We claim that $s_k^2 \geq \lambda_k$ for k large universal. Indeed, if not, then the previous inequality implies that $\Lambda_k^2 \leq 2a_2^{-1}\lambda_k$, which combined with (10) gives $k < \sqrt{2}a_2^{-1/2}s_k\lambda_k^{-1/2}$, and since we assumed that $s_k\lambda_k^{-1/2} < 1$ we get a contradiction for k large universal. We conclude that

$$\Lambda_k \leq \sqrt{2}a_2^{-1/2}s_k := a_3^{-1}s_k \tag{12}$$

when k is large universal. By the $C^{1,1}$ regularity of the holes, the complements of the domains contain $\{x_2 < s_k x_1 - a_4^{-1}x_1^2\} \cap S^{u_k}_{h_k}(0)$ for some $a_4 > 0$ universal. In the new coordinates defined above, the parabolic domains $\{x_2 < s_k x_1 - a_4^{-1}x_1^2\}$ become

$$\{\tilde{x}_2 < s_k\Lambda_k\lambda_k^{-1}\tilde{x}_1 - a_4^{-1}\Lambda_k^2\lambda_k^{-1}\tilde{x}_1^2\}.$$

Using the bound (12) on Λ_k, we see that these domains contain

$$\{\tilde{x}_2 < s_k\Lambda_k\lambda_k^{-1}(\tilde{x}_1 - a_5^{-1}\tilde{x}_1^2)\}$$

for some $a_5 > 0$ universal. Using (10) we see that these domains converge to the slab $\{0 < \tilde{x}_1 < a_5\}$ as k tends to infinity. We conclude that w satisfies $\det D^2 w = 1$ in $\{x_1 < 0\} \cap S^w_1(0)$ and that $\det D^2 w = 0$ in $\{0 < x_1 < a_5\} \cap S^w_1(0)$, as desired.

In either case, a small modification of Proposition 2.3 implies that $S^w_{1/M}(0)$ contains and is contained in dilations by a universal constant of a rectangle with axes aligned with the coordinate axes. Moreover, the upper bound on the vertical second derivative of w implies that the horizontal length l and vertical length L of this rectangle satisfy $l \leq a_6^{-1}L$ for $a_6 > 0$ universal. For k large we conclude that

$$\eta^{u_k}_{h_k/M}(0) \sim \eta^{u_k}_{h_k}(0)l/L,$$

provided $\eta^{u_k}_{h_k}(0)l/L > 1$. (Here and below we use that if a rectangle R_1 centered at 0 is approximated by (contains and is contained in dilations by universal constants times) another rectangle R_2 centered at 0, then their side lengths $\lambda_i,\ \Lambda_i\ (i = 1, 2)$ satisfy $\lambda_1 \sim \lambda_2,\ \Lambda_1 \sim \Lambda_2$). Since $L/l \leq a_7^{-1}M$ for some $a_7 > 0$ universal by the local universal Lipschitz regularity of w, the first inequality we assumed about $\eta^{u_k}_{h_k}(0)$ guarantees this is satisfied (for a sufficiently small universal choice of c_2). Thus, the eccentricity $\eta^{u_k}_{h_k/M}(0)$ increased by at most a universal factor compared to $\eta^{u_k}_{h_k}(0)$. Moreover, if $l < a_8 L$ for $a_8 > 0$ universal sufficiently small, then we have $\eta^{u_k}_{h_k/M}(0) < \eta^{u_k}_{h_k}(0)$. The alternative is that $S^w_{1/M}(0)$ is approximated by (contains

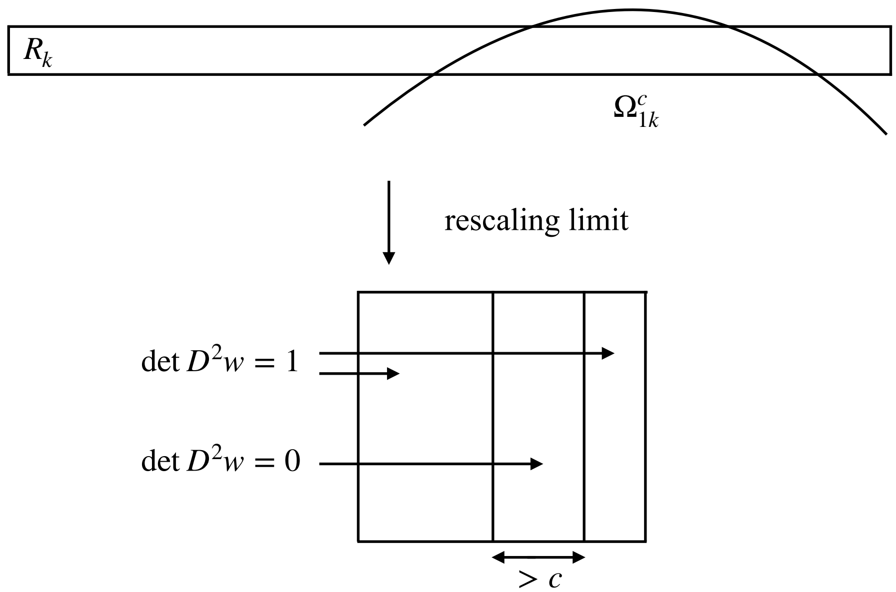

FIGURE 1. Rescaling limit in the case of nontrivial exterior area and transversal boundary.

and is contained in dilations by universal constants times) $B_{M^{-1/2}}$, thus for k large the rectangles $R^{u_k}_{h_k/M}(0)$ are contained in universal dilations of $M^{-1/2}R_k$. This gives the desired contradiction (namely, the conclusion is true for k large), provided c_2 was chosen sufficiently small (but still universal). $\square$

We now define

$$M_2 := c_2^{-6}, \tag{13}$$

where c_2 is the universal constant from Lemma 3.2, and we let $\epsilon_2 > 0$ be the length ratio from that lemma corresponding to the choice $M = M_2$. Note that ϵ_2 is universal.

Finally, the remaining lemma is purely geometric, and also uses the regularity and convexity properties of the holes in Ω_1. Below, d denotes the distance function from Ω_1^c.

Lemma 3.3. *There exists $c_3 > 0$ universal such that the following holds. Assume that $u \in \mathcal{F}_\delta$, that $x \in \partial\Omega_1 \backslash \partial\Omega_0$ and that*

$$\frac{|(x + R_h^u(x)) \cap \Omega_1^c|}{|R_h^u(x)|} > \epsilon_1, \quad \frac{l_h^u(x)}{\Lambda_h^u(x)} > \epsilon_2.$$

Then

$$[\Lambda_h^u(x)]^2 + \sup_{x+R_h^u(x)} d \leq c_3^{-1}\lambda_h^u(x).$$

Proof. Perform a rigid motion as in the proof of Lemma 3.2 so that $x = 0$, the short side of $R_h^u(0)$ is vertical, and the hole lies beneath its tangent line at 0 which has slope $s \geq 0$. Below we will denote $R_h^u(0)$ by R_h, and we will similarly drop the superscript u and the point (0) in the notation from the other relevant quantities.

Elementary geometry implies that $l_h \leq 4\max\{\lambda_h,\, \lambda_h/s\}$, so the second inequality in the hypothesis gives $\epsilon_2 < 4\max\{\lambda_h/\Lambda_h,\, \lambda_h/(s\Lambda_h)\}$. In the case that $\Lambda_h < 4\epsilon_2^{-1}\lambda_h$ the lemma is obvious, since then $\lambda_h \sim \Lambda_h$. We can thus assume otherwise. This gives

$$s \leq 4\epsilon_2^{-1}\lambda_h/\Lambda_h \leq 1. \tag{14}$$

We may further assume h is small enough that in $\{|x_1| \leq \Lambda_h\}$, the top part of the boundary of the hole is a graph with slope bounded by 2. Indeed, this is guaranteed for $h < c$ universal by the universal diameter bound on sections, and for $h \geq c$ we have $\lambda_h \sim \Lambda_h$ (in particular, that the lemma holds) by the area estimate for sections and the universal Lipschitz bound on u. Here and below, c denotes a positive universal constant that may change from line to line.

The uniform concavity of $\partial\Omega_1$ implies that

$$R_h \cap \Omega_1^c \subset R_h \cap \{x_2 < sx_1 - cx_1^2\} \subset R_h \cap \{|x_1| < c^{-1}(s^2 + \lambda_h)^{1/2}\}.$$

Using the first inequality in the hypothesis we conclude that

$$\Lambda_h^2 < c^{-1}(s^2 + \lambda_h).$$

Using (14) in the previous inequality gives

$$\Lambda_h^2 \leq c^{-1}(\lambda_h^2\Lambda_h^{-2} + \lambda_h),$$

and it follows that

$$\Lambda_h^2 \leq c^{-1}\lambda_h. \tag{15}$$

Furthermore, the $C^{1,1}$ regularity of $\partial\Omega_1$ implies that the hole has a boundary portion that lies above $\{x_2 = sx_1 - c^{-1}x_1^2\}$ in $\{|x_1| < \Lambda_h\}$. The distance of points in R_h from Ω_1^c is thus bounded above by

$$c^{-1}(\lambda_h + |s|\Lambda_h + c^{-1}\Lambda_h^2).$$

Using (14) and (15) we arrive at the desired estimate. □

See Figure 2 for a summary of the results from Lemma 3.3.

4. Proof of Theorem 1.1

Proof of Theorem 1.1. For each $x \in \Omega_1$, let $S^u_{\bar{h}(x)}(x)$ be the "maximal section contained in Ω_1" centered at x, so that $S^u_{\bar{h}(x)}(x)$ is contained in Ω_1 and tangent to $\partial\Omega_1$. The existence of such a section follows from the continuity of the sections in h, see [8]. By (4) and the universal Lipschitz bound on u, there exists $\bar{c} > 0$ universal such that for all x in the $\bar{c}$-neighborhood $\mathcal{N}_{\bar{c}}$ of the union of the holes in Ω_1, the section $S^u_{\bar{h}(x)}(x)$ is tangent to a hole and not $\partial\Omega_0$.

The arguments in the proof of Theorem 1.1 from [23] show that

$$u \in W^{2,p}(\Omega_1\backslash\mathcal{N}_{\bar{c}}) \cap C^{1,\alpha}(\overline{\Omega_1\backslash\mathcal{N}_{\bar{c}}})$$

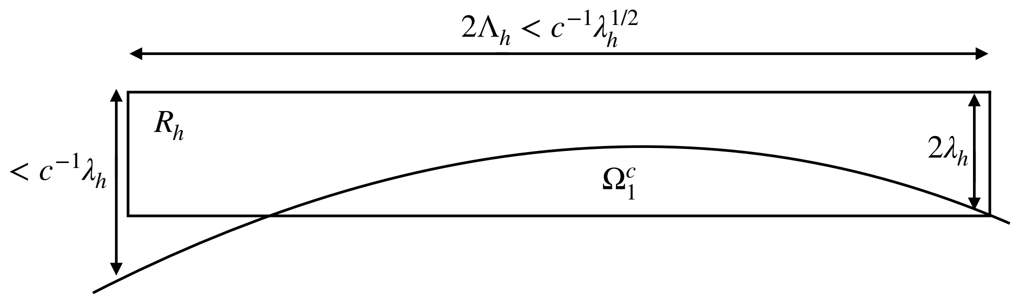

FIGURE 2. "Model example geometry" in the case of nontrivial exterior area and roughly tangential boundary.

for any $p > 1$ and $\alpha \in (0, 1)$, with corresponding estimates in these spaces depending on δ, p and α. More precisely, the estimates used to prove Theorem 1.1 in [23] are local in nature, and thus apply in our setting to sections centered in $\overline{\Omega_1} \backslash \mathcal{N}_{\bar{c}}$ of height smaller than some $h_1 > 0$ universal chosen so that these sections do not intersect $\mathcal{N}_{\bar{c}/2}$. In particular, in our setting, Theorem 1.1 from [23] has the form $|D^2u| \leq C(\epsilon, \delta)d^{-\epsilon}$ in $\Omega_1 \backslash \mathcal{N}_{\bar{c}}$ for any $\epsilon > 0$, from which the claim follows. Here and below d denotes distance from Ω_1^c. (Alternatively, one can note that all arguments in [23] work equally well if the source density is smooth and positive in the closure of the source domain, and 1 in a neighborhood of the boundary. In our setting, we can reduce to this case by considering the function that is u near $\partial\Omega_0$, glued to $\rho_\tau * (u + \tau|x|^2)$ for $\tau > 0$ small universal, a positive universal distance from $\partial\Omega_0$. Here ρ_τ is a standard mollifier).

We may thus focus our attention on $\mathcal{N}_{\bar{c}}$. To that end, let $x \in \mathcal{N}_{\bar{c}}$, and let d be the distance from x to the boundary. We will prove that

$$|D^2u(x)| \leq c^{-1}d^{-1/2}. \tag{16}$$

Here and for the remainder of the proof, c denotes a positive universal constant that may change from line to line. The $W^{2,p}$ estimate from Theorem 1.1 follows immediately from (16), and the $C^{1,1/2}$ estimate comes from integrating (16) along line segments.

Assume after a translation that $S^u_{\bar{h}(x)}(x)$ is tangent to a hole at the origin. Applying Proposition 2.1 we engulf $S^u_{\bar{h}(x)}(x)$ by $R^u_h(0)$, with $h \sim \bar{h}(x)$. We will prove that

$$\eta^u_h(0) \leq c^{-1}d^{-1/2}.$$

Using that $|S^u_{\bar{h}(x)}(x)| \sim |R^u_h(0)|$ it is easy to see that $\eta^u_{\bar{h}(x)}(x) \leq c^{-1}\eta^u_h(0)$. Combining this with the above inequality and (7) gives (16).

In what follows we will use that if $h_1 \sim h_2$ then $R^u_{h_1}(0)$ is approximated by (contains and is contained in dilations by universal constants times) $R^u_{h_2}(0)$ (see Appendix, Section 6.4), whence $\lambda^u_{h_1}(0) \sim \lambda^u_{h_2}(0)$, $\Lambda^u_{h_1}(0) \sim \Lambda^u_{h_2}(0)$, and $\eta^u_{h_1}(0) \sim \eta^u_{h_2}(0)$. We will also denote $\eta^u_h(0)$ by η_h, and we will similarly drop the notation (0) and the superscript u from the other relevant quantities.

If either $\eta_h \leq c_2^{-1} M_2$ or the conditions of Lemma 3.3 are satisfied, then we are done. Indeed, in the first case, use that $c_2^{-1} M_2$ is universal, hence the desired bound on η_h follows provided c universal is sufficiently small. In the second case, Lemma 3.3 gives $\Lambda_h^2 \leq c^{-1}\lambda_h$, hence $\eta_h \leq c^{-1}\lambda_h^{-1/2}$. Moreover, the distance from boundary in R_h is at most $c^{-1}\lambda_h$. In particular, $\lambda_h^{-1/2} \leq c^{-1} d^{-1/2}$, and we are also done in this case.

So assume that neither is satisfied, and let $\hat{t}$ be the supremum of heights t such that $\eta_s > c_2^{-1} M_2$ and the conditions of Lemma 3.3 are not satisfied at height s for all $s \in [h,\, t]$.

We first assume that $\hat{t} \leq 1$. Then there exist $\hat{h} \in [h,\, \hat{t}]$ and $\kappa \in (1,\, 2)$ such that $\kappa\hat{h} \geq \hat{t}$ and either $\eta_{\kappa\hat{h}} \leq c_2^{-1} M_2$ or the conditions of Lemma 3.3 are satisfied at height $\kappa\hat{h}$, but neither holds at height $\hat{h}$. At height $\hat{h}$, we can either apply Lemma 3.1 with the choice $M = M_1$ or 3.2 with the choice $M = M_2$, giving a conclusion at height $M_1^{-1}\hat{h}$ or $M_2^{-1}\hat{h}$. Repeat applying Lemma 3.1 or Lemma 3.2 (say the former k times and the latter l times) until the first time $M_1^{-k} M_2^{-l}\hat{h} < h$. Assume that eccentricity increased in the application of Lemma 3.2 $l' \leq l$ times. We then have

$$\eta_h \leq c^{-1} c_1^{-k} c_2^{-l'} \eta_{\hat{h}} \leq c^{-1} r^{-1/2} \eta_{\kappa\hat{h}},$$

where

$$r := c_1^{-k} M_1^{-k/2} c_2^{-l'} M_2^{-l'/2} = c_1^{2k} c_2^{2l'} < 1.$$

By Lemmas 3.1 and 3.2, we also have that

$$R_h \subset c^{-1} r R_{\kappa\hat{h}}.$$

We now consider the case that at height $\kappa\hat{h}$, the conditions of Lemma 3.3 are satisfied. The lemma implies that

$$\eta_{\kappa\hat{h}} \leq c^{-1} \lambda_{\kappa\hat{h}}^{-1/2},$$

thus

$$\eta_h \leq c^{-1} r^{-1/2} \lambda_{\kappa\hat{h}}^{-1/2}. \tag{17}$$

Assume after a rigid motion that the picture is oriented as in the proof of Lemma 3.3, so that $R_{\kappa\hat{h}}$ has vertical short axis. Arguing as in the proof of Lemma 3.3 we either have that $\lambda_{\kappa\hat{h}} \sim \Lambda_{\kappa\hat{h}}$, in which case the desired estimate $\eta_h \leq c^{-1} r^{-1/2} \lambda_{\kappa\hat{h}}^{-1/2} \leq c^{-1} d^{-1/2}$ follows from the inclusion $R_h \subset c^{-1} r R_{\kappa\hat{h}}$, or the boundary of Ω_1 contains a portion that lies above $\{x_2 = s x_1 - c^{-1} x_1^2\}$ in $\{|x_1| < \Lambda_{\kappa\hat{h}}\}$, where $0 \leq s \leq c^{-1}\lambda_{\kappa\hat{h}}/\Lambda_{\kappa\hat{h}}$ and $\Lambda^2_{\kappa\hat{h}} \leq c^{-1}\lambda_{\kappa\hat{h}}$. Recall that R_h is contained in the $c^{-1} r$ times dilation of $R_{\kappa\hat{h}}$. Thus, in R_h, the distance from boundary is at most

$$c^{-1}(r\lambda_{\kappa\hat{h}} + s r \Lambda_{\kappa\hat{h}} + r^2 \Lambda^2_{\kappa\hat{h}}).$$

Using the previous inequalities for s and $\Lambda_{\kappa\hat{h}}$ we see that the second and third terms are bounded by $c^{-1} r \lambda_{\kappa\hat{h}}$ and $c^{-1} r^2 \lambda_{\kappa\hat{h}}$, respectively, giving a bound on the distance between boundary in R_h of the size

$$d \leq c^{-1} r \lambda_{\kappa \hat{h}}.$$

Rearranging gives

$$\lambda_{\kappa \hat{h}}^{-1/2} \leq c^{-1} r^{1/2} d^{-1/2}.$$

Using this in (17) gives

$$\eta_h \leq c^{-1} d^{-1/2},$$

and we are done with this case.

In the case that $\eta_{\kappa \hat{h}} \leq c_2^{-1} M_2$, we have

$$\eta_h \leq c^{-1} r^{-1/2}.$$

Furthermore, we have that R_h is contained in a universal dilation of B_r since $R_{\hat{h}}$ is contained in $B_{c^{-1}}$, thus in R_h the distance from the boundary is at most $c^{-1}r$, hence

$$\eta_h \leq c^{-1} d^{-1/2}$$

in this case as well.

Finally, we deal with the case that $\hat{t} > 1$. Since η_1 is still bounded by a universal constant, we can take $\hat{h} = 1$ and repeat exactly the same arguments as above. More precisely, by repeated application of Lemma 3.1 or Lemma 3.2 starting from height $\hat{h} = 1$, we get

$$\eta_h \leq c^{-1} r^{-1/2}$$

and

$$R_h \subset B_{c^{-1} r},$$

where r is defined in the same way as above. Combining these two conclusions we get $\eta_h \leq c^{-1} d^{-1/2}$, and this completes the proof. □

5. Future Directions

To conclude the paper we list a few questions to be investigated in future work.

(1) Establish Sobolev regularity for optimal transport maps of a natural class of non-convex domains in higher dimensions. As noted above, the Pogorelov-type estimate Proposition 2.3 is convenient but not required for the result in this paper, so there is hope for such an extension.

(2) In two dimensions, enlarge the class of source domains being considered e.g. to arbitrary smooth domains. Our arguments handle concave parts of the boundary since we only use the equation and not the convexity of the target domain. However, convex parts of the boundary that lie inside the convex hull of the source domain may be tricky to handle, since at such points we do not have duality, which played an important role in the works of Caffarelli [6] and Savin-Yu [23].

(3) Investigate applications of our results to the existence theory for the semi-geostrophic equations in cases where the source density is allowed to vanish. In previous works dealing with the case where the source density is bounded from below by a positive constant, $W^{2,1}$ estimates played a central role ([2], [3], [15]).

(4) Investigate applications of the ideas in this paper to the partial regularity theory of optimal transport maps when the domains are not convex. In this case, optimal transport maps can have singularities, and interesting results have been obtained about the size of the singular set ([12], [17], [16], [19]). However, the fine geometric measure-theoretic structure of the singular set is not well-understood, even in two dimensions with smooth domains and quadratic cost. In that case, a reasonable conjecture seems to be that the one-dimensional Hausdorff measure of the singular set is bounded.

6. Appendix

In this section we provide some of the details that we omitted for simplicity of presentation above.

6.1. **No Segments.** We start with a simple lemma that will be used in some of the subsequent proofs.

Lemma 6.1. *There is no convex function w on $B_1 \subset \mathbb{R}^2$ that satisfies*

$$\det D^2 w \geq \chi_{\{x_2>0\}}, \quad w|_{\{x_2=0\}} \text{ linear.}$$

Proof. After subtracting an affine function we may assume that $w \geq 0$ and $w|_{\{x_2=0\}} = 0$. After subtracting a multiple of x_2 we may assume further that $w(0,\, t) = o(t)$ for $t > 0$. It follows that, for any $k > 0$, we can choose $h > 0$ small such that

$$R := [-1/2,\, 1/2] \times [0,\, 2kh] \subset \{w < h\}.$$

The quadratic polynomial $Q = 8hx_1^2 + 2k^{-2}h^{-1}(x_2 - kh)^2$ thus lies above w on the boundary of R, and for k large satisfies $\det D^2 Q < 1$. The comparison principle implies that $Q > w \geq 0$ in R, contradicting that Q vanishes at the center of R. □

We now prove the claim that functions in F_δ are not linear along line segments centered in the closure of the source domain, from which the universal bound on diameters of sections followed.

Lemma 6.2. *Let $u \in \mathcal{F}_\delta$. Then u is not linear along any line segment centered at a point in $\overline{\Omega_1}$.*

Proof. By Lemma 6.1 (appropriately rescaled), u is not linear along any segment in $\overline{\Omega_1}$. The only remaining possibility is that u is linear along a segment that is tangent to $\partial\Omega_0$ at a single point that lies inside the segment. The function u cannot be linear along the whole line l containing this segment. Indeed, if it was, then the monotonicity of the sub-gradient of u implies that $\partial u(\mathbb{R}^2)$ (the subgradient image of u on the whole plane) lies in some line orthogonal to l, whence $\det D^2 u \equiv 0$, a contradiction. We conclude that the agreement set between u and a linear function has extremal points outside of Ω_0. This contradicts that $\det D^2 u = 0$ outside of Ω_0. □

6.2. **Area Estimate.** Next we prove the area estimate (5). Note that if $S_h^u(x)$ is centered at a point $x \in \overline{\Omega_1}$ and intersects at most one hole, then the hole lies on one side of some line through x. Since x is the center of mass of $S_h^u(x)$, it follows that $|S_h^u(x) \cap \Omega_1| \geq d|S_h^u(x)|$ for some $d > 0$ absolute. The area estimate (5) is thus a consequence of the following general result:

Lemma 6.3. *Assume that $S_t^w(x)$ is a bounded domain in $\mathbb{R}^n$, that $\det D^2 w = \chi_U$ for some $U \subset S_t^w(x)$, and that $|U| \geq \lambda |S_t^w(x)|$ for some $\lambda > 0$. Then there exists $\kappa(n, \lambda) > 0$ such that*

$$\kappa t^{n/2} \leq |S_t^w(x)| \leq \kappa^{-1} t^{n/2}.$$

Proof. After translating and subtracting a linear function we may assume that $x = 0$ and that $w = 0$ on $\partial S_t^w(0)$. By John's lemma, there exists an ellipsoid E_t centered at zero such that $c(n)E_t \subset S_t^w(0) \subset E_t$. In this proof, $c(n)$ will denote a constant depending on n that may change from line to line. Write $E_t = A_t(B_1)$ where A_t is a linear map, and let $\tilde{w}(x) = |\det A_t|^{-2/n} w(A_t x)$, $\tilde{U} = A_t^{-1} U$, and $H = |\det A_t|^{-2/n} t$. Then $c(n)B_1 \subset S_H^{\tilde{w}}(0) \subset B_1$, $\det D^2 \tilde{w} = \chi_{\tilde{U}}$, and $|\tilde{U}| \geq c(n)\lambda$. In particular, provided $\alpha(n, \lambda)$ is sufficiently close to one, we have $|S_H^{\tilde{w}}(0) \backslash \alpha S_H^{\tilde{w}}(0)| \leq |\tilde{U}|/2$, giving $|\tilde{U}| \leq 2|\tilde{U} \cap \alpha S_H^{\tilde{w}}(0)|$. Proposition 1.1 from [21] implies that $H \sim 1$, up to multiplication by constants depending on n, λ. Since $|\det A_t|$ is equivalent to $|S_t^w(0)|$ up to multiplication by dimensional constants, we conclude that $t^{n/2}$ is equivalent to $|S_t^w(0)|$ up to multiplication by constants depending on n and λ, as desired. □

6.3. **Engulfing Properties.** The engulfing property Proposition 2.1 follows from the following pair of lemmas. Before proceeding we recall a standard renormalization procedure. For $u \in \mathcal{F}_\delta$, $x \in \overline{\Omega_1}$ and $S_h^u(x)$ contained in Ω_1 or intersecting one hole, let

$$u_h(y) = \frac{1}{h} u(A_h y) + L_h,$$

where A_h is an affine transformation of determinant h that takes a square centered at 0 to $x + R_h(x)$, and L_h is a linear function chosen so that $u_h = 0$ on $\partial S_1^{u_h}(0)$. Then u_h solves $\det D^2 u_h = \chi_{\Omega_h}$ in the normalized domain $S_1^{u_h}(0)$ (equivalent to B_1 up to dilations by absolute constants), where Ω_h^c is convex and $\overline{\Omega_h}$ contains 0. We call this procedure "renormalizing in the section $S_h^u(x)$".

Lemma 6.4. *If $u \in \mathcal{F}_\delta$, $y \in \overline{S_h^u(x)} \cap (\partial\Omega_1 \backslash \partial\Omega_0)$ and $S_h^u(x) \subset \Omega_1$, then*

$$S_h^u(x) \subset S_{c^{-1}h}^u(y).$$

Here c is universal.

Proof. Assume by way of contradiction that the lemma is false. Then there exists a sequence $u_k \in \mathcal{F}_\delta$ such that $S_{h_k}^{u_k}(x_k)$ are contained in the source domains and y_k are in their closures and the boundary of a hole (say $y_k = 0$ after a translation), but $S_{kh_k}^{u_k}(0)$ do not contain $S_{h_k}^{u_k}(x_k)$. Note that we may assume that $h_k \leq c^{-1}k^{-1}$ by the uniform Lipschitz bound on u_k, so $h_k \to 0$ and the sections localize close to the holes. After renormalizing in the sections $S_{kh_k}^{u_k}(0)$, we get a subsequence of rescalings w_k of u_k that converge to a function w which satisfies that $S_1^w(0)$ is normalized, $\det D^2 w = 1$ in a domain Ω with Ω^c convex, and w is linear along a segment from 0 to $\partial S_1^w(0)$ contained in $\overline{\Omega}$. This last fact comes from the observation that the sections $S_{h_k}^{u_k}(x_k)$, after renormalization, become sections of height $1/k$ in which $\det D^2 w_k = 1$, which are not contained in the renormalized sections $S_1^{w_k}(0)$ but have 0 in their boundaries. These sections contain line segments with endpoints at 0 and $\partial S_1^{w_k}(0)$ that tend to a line segment from 0 to $\partial S_1^w(0)$ in $\overline{\Omega}$ on which w is linear, since the heights of the sections tend to zero. This contradicts Lemma 6.1, appropriately rescaled. □

Lemma 6.5. *Let $u \in \mathcal{F}_\delta$. Then for all $\alpha \in (0, 1)$, there exists $\beta(\delta, \alpha) > 0$ such that*

$$S^u_{\beta h}(y) \subset \alpha S^u_h(y)$$

for all $y \in \partial\Omega_1 \backslash \partial\Omega_0$, where $\alpha S^u_h(y)$ is the α-times dilation of $S^u_h(y)$ around y.

Proof. The argument is similar to the one above. If the lemma is false, there is a sequence $u_k \in \mathcal{F}_\delta$ such that (up to translations) 0 is in the boundary of a hole and $S^{u_k}_{h_k/k}(0)$ are not contained in $\alpha S^{u_k}_{h_k}(0)$. After renormalizing in the sections $S^{u_k}_{h_k}(0)$, we get a subsequence of rescalings w_k that converge to a function w which satisfies that $S^w_1(0)$ is normalized, $\det D^2 w = 1$ in a domain Ω with Ω^c convex and $0 \in \partial\Omega$, and w is linear along a line segment passing through the origin with an endpoint on $\partial(\alpha S^w_1(0))$. (The reasoning is similar to above; $S^{w_k}_{1/k}(0)$ are not contained in $\alpha S^{w_k}_1(0)$, thus they contain segments passing through 0 with one endpoint on $\partial(\alpha S^{w_k}_1(0))$ and the other, by the fact that the sections are centered at 0, a positive distance depending on α from 0. These segments converge to the one claimed above). Again, we contradict Lemma 6.1. □

6.4. **Rectangle Containments.** Now we sketch the proof that if $u \in \mathcal{F}_\delta$, $0 \in \partial\Omega_1 \backslash \partial\Omega_0$, and $h_1 \sim h_2$, then $R^u_{h_1}(0)$ is approximated by (contains and is contained in dilations by universal constants of) $R^u_{h_2}(0)$. This fact was used in the proof of Theorem 1.1 in Section 4. It suffices to show that $S^u_{h_1}(0)$ is approximated by $S^u_{h_2}(0)$. After renormalizing in $S^u_{h_1}(0)$ as in Section 6.3 we get a convex function w such that $S^w_1(0) = \{w < 0\}$ is normalized, and we need to show that $S^w_c(0)$ is approximated by B_1, where $c \sim 1$. Since

$$|S^w_c(0)| \sim 1$$

we just need to show that $S^w_c(0)$ contains a ball centered at 0 that has small universal radius. If not, then by the local universal Lipschitz bound on w, the slope of the linear function defining $S^w_c(0)$ is extremely large, say after a rotation, Ke_1 with $K >> 1$. But in this case the line segment in $S^w_c(0)$ through the origin parallel to e_1 would intersect $\partial S^w_c(0)$ at a distance much shorter from the origin on the left than on the right, contradicting that 0 is the center of mass of $S^w_c(0)$ and completing the proof.

6.5. **Partial Legendre Transform.** Let u be a convex function on a domain $\Omega \subset \mathbb{R}^2$. We define its partial Legendre transform by taking the usual Legendre transform along horizontal lines:

$$u^*(p, x_2) := \sup_{x_1}(px_1 - u(x_1, x_2)).$$

The partial Legendre transformed is defined on $\{((\partial_1 u(x_1, x_2), x_2) : (x_1, x_2) \in \Omega)\}$, where ∂_1 denotes the sub-gradient of u understood as a one-variable function on the horizontal line with x_2 constant. The function u^* is convex in the horizontal directions and concave in the vertical directions.

When u is smooth and uniformly convex, we have

$$u^*(u_1(x_1, x_2), x_2) = x_1 u_1(x_1, x_2) - u(x_1, x_2).$$

By taking various derivatives of this identity one obtains

$$u^*_{11} = \frac{1}{u_{11}}, \quad u^*_{12} = -\frac{u_{12}}{u_{11}}, \quad u^*_{22} = -\frac{\det D^2 u}{u_{11}}. \tag{18}$$

In particular, if $\det D^2u = 1$, then u^* is harmonic.

Finally, we recall that if u and v are convex on Ω, then $|u^* - v^*| \leq \|u - v\|_{L^\infty(\Omega)}$ on the common domain of definition of u^* and v^*. In particular, C^0 convergence of convex functions in Ω implies C^0 convergence of their partial Legendre transforms.

6.6. **Boundary Estimate for Harmonic Functions.** We denote by B_r^+ the half-ball $B_r \cap \{x_n > 0\}$ in $\mathbb{R}^n$.

Lemma 6.6. *Let $u \in C\left(\overline{B_1^+}\right)$ be a harmonic function such that $u|_{\{x_n=0\}}$ is Lipschitz, with Lipschitz constant L. Then*

$$\|u_i\|_{L^\infty(B_{1/2}^+)} \leq C(n)(L + \|u\|_{L^\infty(B_1^+)}),$$

for all $i < n$.

Proof. We may assume that $L + \|u\|_{L^\infty(B_1^+)} \leq 1$ after dividing by this quantity. We may also assume that $u \in C^\infty(\overline{B_1^+})$; the general result follows by approximation. Let ψ be a smooth cutoff function that is 1 in $B_{3/4}$ and 0 outside of $B_{7/8}$, and let g be the function on $\{x_n = 0\}$ defined by $u\psi$ on $\{x_n = 0\} \cap B_1$, and extended to be zero outside. Note that $|g_i|$, $i < n$ is bounded by an absolute constant. Let K be the upper half-space Poisson kernel and let $h = K * g$ be the harmonic function in the upper half-space with boundary data g. Since $h_i = g_i$ on $\{x_n = 0\}$ and h_i tends to zero at infinity, the maximum principle implies that on the upper half-space, $|h_i| \leq \|g_i\|_{L^\infty(\mathbb{R}^{n-1})}$, which in turn is bounded by an absolute constant. To conclude we note that in $B_{3/4}^+$, we have $u = h + h'$, where h' is a harmonic function vanishing on $\{x_n = 0\} \cap B_{3/4}$ and $|h'| \leq |u| + |h| \leq 2$. The Schwarz reflection principle and the interior gradient estimate for harmonic functions give $\|\nabla h'\|_{L^\infty(B_{1/2}^+)} \leq C(n)$, so the result follows. □

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Department of Mathematics, UC Irvine
Email address: mooneycr@math.uci.edu

Department of Mathematics, UC Irvine
Email address: arakshit@uci.edu