Parabolic PDE

Connor Mooney

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1 Introduction

In these notes we discuss aspects of regularity theory for parabolic equations, and some applications to fluids and geometry. They are growing from an informal series of talks given by the author at ETH Zürich in 2017.

2 Representation Formulae

We consider the heat equation

$$u_t - \Delta u = 0. \tag{1}$$

Here $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. In this section we discuss a classical approach based on the regularity and decay properties of the fundamental solution.

2.1 Derivation and Scaling

We give two derivations of the heat equation, one giving divergence-structure and the other non-divergence structure. We then discuss the symmetries and scaling of the equation.

Divergence: Say $u(x, t)$ represents temperature. In the absence of heat sources, the amount of heat lost in a domain agrees with the heat flux through the boundary:

$$\int_{B_r} u \, dx \Big|_T^{T+\tau} + \int_T^{T+\tau} \int_{\partial B_r} Q \cdot \nu \, ds \, dt = 0.$$

Heat tends to flow from regions of high temperature to low temperature; taking $Q = -\nabla u$ (a physical assumption) and integrating by parts yields the divergence-form version of the equation: $u_t = \text{div}(\nabla u)$.

More general divergence-form equations arise by taking $-Q$ to be a monotone operator of ∇u (e.g. $-Q = A(x)\nabla u$ for A a positive matrix, or $Q = -\nabla F$ for F convex).

Non-Divergence: Another viewpoint comes from the observation that

$$\Delta u(x) \sim \frac{c}{r^2} \int_{\partial B_r(x)} (u - u(x)) \, ds$$

for r small measures average deviation from tangent plane. (To see this, Taylor expand and use symmetry). This gives the non-divergence interpretation: solutions to (1) tend toward their average. To see how the equation arises naturally, consider the following discrete model. At location and time $(m, k) \in \mathbb{Z} \times \mathbb{Z}$, there are Dirac masses of size $f(m, k)$. These masses move with the following law: the mass at (m, k) is split evenly between $(m \pm 1, k + 1)$:

$$f(m, k+1) - f(m, k) = \frac{1}{2}(f(m+1, k) + f(m-1), k) - f(m, k).$$

(If one starts with particles of mass $f(m, 0)$ at points m, and lets them move by random walk in k, then $f(m, k)$ is the expected value of mass at location m and time k). The law above is a discrete version of $f_t = \frac{1}{2}f_{xx}$.

Symmetries: The heat equation is invariant under spatial rotations, space and time translations, and multiplication by constants. In particular, derivatives are solutions. However, space and time behave differently under dilation. Say we want to find a solution to the heat equation by rescaling a solution to the discrete model above. If we quadratically rescale in space so the right side looks like a second derivative, then we must rescale time by the

square of the spatial dilation to get a time derivative on the left side: $f \to \epsilon^2 f(m/\epsilon,\, k/\epsilon^2)$. This is the parabolic scaling:

$$\tilde{u}(x) = u(\lambda x,\, \lambda^2 t)$$

is a solution. The scaling says that if we dilate an object by a large factor, it takes much longer to change temperature. Differentiating in λ we see that $r u_r + 2 t u_t$ is also a solution.

It is useful to work in a geometry that is easily normalized to unit scale by parabolic scaling. In this case, the natural objects are the parabolic cylinders

$$Q_r = B_r \times (-r^2, 0].$$

2.2   The Fundamental Solution

The fundamental solution to the heat equation is

$$\Gamma(x,\, t) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \chi_{\{t>0\}}.$$

It solves the heat equation for $t > 0$, with initial data a Dirac mass. It is a distributional solution to

$$(\partial_t - \Delta)\Gamma = \delta(0,\, 0).$$

We justify these interpretations below. Note that Γ is not locally analytic in time (the initial disturbance immediately propagates everywhere), but it is analytic spatially and decays rapidly in x. These properties will allow us to write representation formulae that are very useful for understanding both the local behavior of caloric functions and existence and uniqueness questions for the Cauchy problem.

We now give several derivations.

Scaling and Divergence Structure: The simplest derivation is based on scaling. The equation $(\partial_t - \Delta)\Gamma = \delta$ is invariant under the scaling

$$\Gamma \to \lambda^n \Gamma(\lambda x,\, \lambda^2 t).$$

(This scaling preserves mass). By searching for solutions invariant under this scaling and imposing radial invariance $(\Gamma(x,\, 1) = g(r))$ we obtain

$$\Gamma = t^{-n/2} g(r/t^{1/2}).$$

This reduces the problem to an ODE, which one can easily solve. The factor of $(4\pi)^{-n/2}$ ensures that Γ has mass 1 on each time slice $t = t_0 \geq 0$.

Remark 1. *In a similar way one can find "backwards" self-similar solutions, e.g. of the form $\frac{1}{(-t)^{n/2}} h(r/\sqrt{-t})$. This gives an ODE of the form*

$$(r^{n-1} h')' - \frac{1}{2}(r^n h)' = 0,$$

yielding the solution

$$\frac{1}{(-t)^{n/2}} e^{|x|^2/(-4t)}.$$

This solution blows up rapidly as $t \to 0^-$.

To see that Γ solves $(\partial_t - \Delta)\Gamma = \delta(0, 0)$, let φ be a compactly supported function in $\mathbb{R}^{n+1}$ and compute

$$-\int_\mathbb{R} \int_{\mathbb{R}^n} \Gamma(\partial_t + \Delta)\varphi \, dx \, dt.$$

Integrating by parts we are left with $\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \Gamma(x, \epsilon)\varphi(x, \epsilon) \, dx$. It is easy to verify that $\Gamma(\cdot, \epsilon)$ is an approximation to the identity as $\epsilon \to 0$, and we obtain $\varphi(0, 0)$.

Remark 2. *The same argument gives the simple formula*

$$\varphi = \Gamma * (\partial_t - \Delta)\varphi,$$

for φ compactly supported in $\mathbb{R}^{n+1}$. If we integrate starting at $t = 0$ we obtain the useful representation formula

$$\varphi(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)\varphi(y, 0) \, dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y, t - s)(\partial_s - \Delta_y)\varphi(y, s) \, dy \, ds.$$

This formula is physically meaningful; the first term is caloric and captures the initial data, and the second term captures the "heat source" $(\partial_t - \Delta)\varphi$.

Combinatorial: The analogue of the fundamental solution in the discrete model is the solution with $f(m, 0) = \delta_0(m)$. Roughly, the mass redistributes by Pascal's triangle:

$$f(m, k) = 2^{-k} \binom{k}{\frac{k-m}{2}}.$$

Using Stirling's formula $n! \sim \sqrt{2\pi n} \, n^n e^{-n}$ one sees that

$$f(m, k) \sim (2\pi k)^{-1/2} e^{-m^2/2k}.$$

This is the fundamental solution for the model equation $\partial_t - \frac{1}{2}\partial_{xx}$. As expected, speeding up time by a factor of 2 we get the fundamental solution to $\partial_t - \partial_{xx}$: $f(m, 2k) \sim \Gamma(m, k)$.

Linearity: Another simple derivation uses the linearity of the heat operator. This makes the Fourier transform a powerful tool. If we take the spatial Fourier transform of the equation for Γ we get

$$\partial_\tau \hat{\Gamma} + |\xi|^2 \hat{\Gamma} = 0, \quad \hat{\Gamma}(\cdot, 0) = 1.$$

Solving the ODE and inverting the Fourier transform gives the solution.

2.3 Local Representation Formula

By the analysis above, if ψ is compactly supported in $\mathbb{R}^{n+1}$ then we have the representation formula

$$\psi(x, t) = \Gamma * (\partial_t - \Delta)\psi.$$

In particular, if u is caloric, then taking $\psi = \varphi u$ where φ is a cutoff that is constant in Q_1 we obtain a representation formula of the form

$$u(x, t) = \Gamma * (u(\partial_t + \Delta)\varphi - 2\mathrm{div}(u\nabla\varphi))$$

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for $(x, t) \in Q_1$. Since $(\partial_t - \Delta)(u\varphi)$ vanishes in Q_1, this formula in a sense shows how to compute u given its values on the sides and bottom of a parabolic cylinder. Furthermore, we immediately obtain that u is locally smooth, and in fact analytic in x (but not t).

By integration by parts we have $u = \Gamma * (u\, f) + \nabla\Gamma * (u\, h)$ in Q_1, where f and h are supported away from Q_1. Differentiating gives a derivative estimate of the form

$$|\nabla u|_{Q_{1/2}} < C(n) \sup_{Q_1} |u|.$$

Since the equation is translation-invariant, we obtain derivative estimates of all orders in $Q_{1/2}$ in terms of the oscillation of u in Q_1. (The equation gives the estimates of time derivatives). Rescaling these, we obtain:

$$|D_x^k D_t^l u|_{Q_{r/2}} < C(k, l, n) r^{-(k+2l)} \sup_{Q_r} |u|. \tag{2}$$

Remark 3. *Note that the derivative estimates only use the representation formula and the regularity properties of Γ. Such estimates also hold e.g. for higher-order PDE and systems, as long as the fundamental solution is well-behaved. However, in that case the maximum principle and Harnack inequality (see below) don't hold.*

The derivative estimates give an analogue of the Liouville theorem for harmonic functions with polynomial growth: if a solution in $\{t < 0\}$ grows no faster than r^k in parabolic cylinders, then it is a polynomial in x and t consisting of terms of the form $x^a t^b$ with $a + 2b \leq k$. To see this use that the rescalings $r^{-k} u(rx, r^2 t)$ remain bounded in Q_1 for r large.

Remark 4. *That solutions to the heat equation are locally analytic in space, but not time, is suggested by the difference in scaling. Indeed, the spatial derivative estimate, iterated in nested cylinders $Q_{i/k}$, gives (using that $k^k < C^k k!$)*

$$|D_x^k u|_{Q_{1/2}} < C^k k!,$$

so the spatial Taylor series converges in B_r for r small. On the other hand, the same estimate for the time derivatives gives

$$|D_t^k u| < C^k (k!)^2,$$

which is not good enough for convergence. We in fact see this behavior for the fundamental solution away from the origin; the derivatives of $e^{-1/t}$ grow like $2^k k! t^{-k} e^{-1/t}$. At $t_k = 1/k$ this is of order $C^k (k!)^2$.

2.4 Mean Value Property

Above we used the representation formula for compactly supported functions to write a caloric function in terms of its "parabolic boundary values." By cutting off we avoid boundary terms. The philosophy for finding a mean value formula is to instead kill the boundary terms by considering test functions that have matching boundary data. Indeed, we have the general formula

$$\int\int_\Omega v(\partial_t - \Delta)u\, dx\, dt = \int\int_\Omega (-\partial_t - \Delta)v\, u\, dx\, dt + \text{ boundary terms },$$

where the boundary terms involve the values and gradients of u and v. In particular, if u is caloric then the first term on the right is the same for any v with the same values and spatial gradient on the boundary. Taking the natural choices $v = \Gamma_B$ the backwards fundamental solution (which solves $(-\partial_t - \Delta)\Gamma_B = \delta_{0,0}$) and $\Omega = \{\Gamma_B > 1\}$ we conclude that

$$\int\int_{\{\Gamma_B > 1\}} (-\partial_t - \Delta)w\, u\, dx\, dt = u(0,\, 0)$$

for any w that has the same value and spatial derivative as Γ_B on the level set.

If we construct w by replacing Γ_B in $\{\Gamma_B > 1\}$ at each time by the quadratic with the same values and normal derivative on the boundary of the super-level set, a short computation gives the mean value formula:

$$u(0,\, 0) = \frac{1}{4}\int\int_{\{\Gamma_B > 1\}} u(y,\, s)\frac{|y|^2}{s^2}\, dy\, ds.$$

From this one obtains a mean value property in super-level sets of any height by scaling:

$$u(0,\, 0) = \frac{r}{4}\int\int_{\{\Gamma_B > r\}} u(y,\, s)\frac{|y|^2}{s^2}\, dy\, ds. \tag{3}$$

Remark 5. *Comparing averages in slightly different level sets, we get a surface version of the mean value property:*

$$u(0,\, 0) = \frac{1}{2}\int_{\partial\{\Gamma_B = 1\}} u(y,\, s)\frac{r^2}{|s|(r^2 + (r^2/2|s| - n)^2)^{1/2}}\, dA.$$

We use that the distance between the $1 + \epsilon$-level set and the 1-level set is order $\epsilon/|\nabla_{x,t}\Gamma_B|$. One can further simplify using the identity $r^2/2s = n\log(-4\pi s)$ on $\{\Gamma_B = 1\}$.

Remark 6. *An important observation is that the weight in the mean value formula is positive; as a consequence, we will conclude a maximum principle and a Harnack inequality (see below), which are not shared by higher-order equations or systems.*

Remark 7. *An instructive exercise may be to understand the Fourier transform of e^{-x^k} for k even. (This is related to computing the fundamental solution of $u_t - \Delta^{k/2}u = 0$.) It solves the ODE*
$$g^{(k-1)}(\xi) - i^k\xi g = 0.$$

The scaling of the ODE suggests the change of variable $f(z) = g(z^{1-1/k})$. For z large, the equation looks roughly like
$$f^{(k-1)}(z) - i^k f = 0,$$

which has some oscillating/decaying solutions corresponding to the roots of $\lambda^{k-1} - i^k = 0$ with negative real part. This explains the asymptotic behavior

$$g(\xi) \sim e^{-c\,|\xi|^{k/(k-1)}}\cos(c\,|\xi|^{k/(k-1)}).$$

The same heuristic can be used to understand the Airy ODE $g'' - \xi g = 0$. An important difference is that the equation is not invariant under reflection (it changes to $h'' + \xi h =$

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0). *The change of variable $f(z) = g(z^{2/3})$ makes the equation look like $f'' = f$, whose bounded solution has exponential decay. The asymptotics for $\xi < 0$ correspond to the equation $f'' = -f$. The bounded solutions in fact transition from oscillation to exponential decay. (However, is slow polynomial decay in the oscillatory direction; one must do more precise computations to capture this behavior.)*

2.5 Maximum Principle and Harnack Inequality

For harmonic functions, the strong maximum principle says that if one solution touches another from above, then they agree identically. This is false for the heat equation, since one can change the boundary data of one solution after the touching time. (Take e.g. 0 and the fundamental solution away from the spatial origin). However, if one solution touches another from above in space-time, then they agree for all previous times:

Theorem 1. *Assume that $\partial_t u - \Delta u = 0$ in Q_1 and that $u \geq 0$. Then if $u = 0$ at some point in $B_1 \times \{0\}$, then $u \equiv 0$.*

Another way to view it is: if there are two solutions such that one is larger at the initial time and on the boundary, then they can never cross. Or: if a non-negative solution is positive at some point, the positivity immediately propagates to the boundary.

The proof of Theorem 1 is immediate from the mean value formula; the solution vanishes identically in a small spacetime ball behind any vanishing point, and if there is a vanishing point at $t = 0$ then we can fill Q_1 with these balls.

The Harnack inequality is a quantitative version of the strong maximum principle. For harmonic functions, it says that if a positive solution is close to 0 somewhere, it is close to 0 in a neighborhood. For the heat equation, there is again a time lapse; if it is close to 0 at some time, then it is close to 0 in a neighborhood at a previous time:

Theorem 2. *Assume that u is caloric and $u \geq 0$ in Q_1. Then $u|_{B_{1/2} \times \{-1/2\}} < C(n) u(0, 0)$.*

By scaling we see that if a positive solution is 1 at $(0, 0)$, then we have an upper bound of $C(n)$ in a backwards spacetime parabola $\{t < -|x|^2\}$ and a lower bound of $\delta(n) > 0$ in a forward spacetime parabola $\{t > |x|^2\}$. In particular, if a nonnegative solution is strictly positive somewhere, the positivity "spreads" to future times, and the values are controlled at past times.

Observe that positive harmonic functions are globally bounded by the Harnack inequality; this is not the case for caloric functions. To see the sharpness of the above result, take for example the self-similar solution $u(x, t) = \frac{1}{\sqrt{1-t}} e^{x^2/4(1-t)}$ for $t < 0$. This is not globally bounded, but it is bounded in backwards spacetime parabolas.

On the other hand, if we know the solution is bounded we have the parabolic Liouville theorem:

Theorem 3. *If u is caloric and bounded in $\{t < 0\}$, then it is constant.*

Indeed, if not, then by adding a constant and multiplying we may assume that $\inf u = 0$ and $\sup u = 1$. Then $u < \epsilon(n)$ small at some point and $u > 1 - \epsilon(n)$ at another point. By

the Harnack inequality, $u < 1/2$ in the spacetime parabola behind the first, and $u > 1/2$ in that behind the second. These eventually intersect for t very negative, a contradiction.

A closely related observation is that the Harnack inequality gives oscillation decay in parabolic cylinders:

$$\mathrm{osc}_{Q_{1/2}} u < (1 - \delta(n)) \mathrm{osc}_{Q_1} u.$$

The proof is the same as for Liouville; assume $0 \le u \le 1$ in Q_1. If $u < \delta(n)$ and $u > 1 - \delta(n)$ at two points in $Q_{1/2}$, then $u < 1/2$ and $u > 1/2$ in spacetime parabolas that eventually intersect. By iterating this procedure, we obtain:

$$\mathrm{osc}_{Q_r} u < C(n) \mathrm{osc}_{Q_1} u \, r^\alpha$$

for some $\alpha(n)$. Combining this with scaling gives another short proof of the Liouville theorem.

The proof of Theorem 2 again follows from the mean value formula. The idea is as follows. Multiplying by a constant we can assume that $u(0, 0) = 1$. If the value at some point in $B_{1/2}$ at time $t = -1/2$ is very large, then a certain average is very large in a spacetime ball of universal size behind this point. Since $u \ge 0$, this forces the mean-value formula average in a (larger) spacetime ball behind $(0, 0)$ to be large, contradicting that $u(0, 0) = 1$.

It is also instructive to argue by compactness. Take a sequence of positive solutions in Q_1, and assume that they converge to 0 at $(0, 0)$ but are all larger than 1 at some point in a cylinder strictly in the past. By the mean value formula, the solutions converge to 0 locally in L^1. Observe that the above derivative estimates from the local representation formula hold with the L^1 norm of u replacing $\sup |u|$ on the right side; thus, the solutions converge to 0 uniformly (along with all of their derivatives), giving the desired contradiction.

2.6 Global Representation Formulae

Above we discussed local results. We now discuss the Cauchy problem: we prescribe (say, bounded) initial data $u(\cdot, 0) = \psi$ and search for solutions for $t > 0$. One heuristic that this is the correct problem is that initial data plus the equation determine all the derivatives of u at $(0, 0)$. (In contrast, to obtain this information for the Laplace or wave equations $\Delta u \pm u_{tt} = 0$, one needs both initial data and normal derivative). However, we have seen that solutions are not generally analytic in time, so this heuristic can be misleading.

Since $(\partial_t - \Delta)\Gamma = 0$ for $t > 0$ and Γ is an approximation of the identity for $t > 0$ small, it is easy to see that

$$u(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)\psi(y) \, dy \tag{4}$$

is a caloric function. If ψ is continuous and bounded then $u(\cdot, t)$ approaches ψ continuously as $t \to 0$. The solution (4) also makes sense for discontinuous initial data (for example the Heaviside function). In this case the solution approaches its initial data in an integral sense, and the data are immediately smoothed out. These illustrate the principles that for the heat equation, disturbances propagate instantaneously and get smoothed out.

We can understand the inhomogeneous problem $v_t - \Delta v = f$ in the same way as for ODE. Intuitively, if we extend the above solution to be zero for $t < 0$, then $(\partial_t - \Delta)u = \delta_{t=0}\psi(x)$. Thus, if u_i are solutions to the initial value problems with data $f(\cdot, s_i)$, then $\sum_i u_i(x, t - s_i)$

looks like a solution to the inhomogeneous problem with initial data 0 when there are lots of s_i closely spaced. This guides us to the representation formula

$$v(x,\,t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y,\, t - s) f(y,\, s)\, dy\, ds \tag{5}$$

e.g. for f compactly supported and smooth. To see that this is correct, change variables, differentiate, and use that $\Gamma(\cdot, \epsilon)$ is an approximation to the identity.

2.7 Uniqueness Questions for the Cauchy Problem

The integral formulae above give a solution to the Cauchy problem $(\partial_t - \Delta)u = f$, $u|_{t=0} = \psi$. The question of whether this is the only solution to the Cauchy problem is more delicate. In fact, without growth hypotheses, we cannot conclude uniqueness.

Recall that if φ is compactly supported in $\mathbb{R}^{n+1}$ then integration by parts gives

$$\varphi(x,\,t) = \int_{\mathbb{R}^n} \Gamma(x - y,\, t)\varphi(y,\, 0)\, dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y,\, t - s)(\partial_s - \Delta)\varphi(y,\, s)\, dy\, ds.$$

Observe that to justify this formula, all we need is that φ and φ_r grow much more slowly than Γ decays. In particular, if u is caloric and $|u| < Ce^{A|x|^2}$ for some constants C and A, independently of time, then above formula is valid for t small depending on A, so u agrees with the representation formula. By iterating in small chunks of time we obtain uniqueness for the Cauchy problem with this spatial growth assumed.

This result is ultimately a consequence of the decay properties of the fundamental solution, so uniqueness for higher-order problems may be approached in this way. One can also use the maximum principle. Let v_R be the solution to the Cauchy problem with initial data $\chi_{B_{R+1} \setminus B_R}$ given by the representation formula. Then $Ce^{AR^2}v_R$ lies above u in B_1 for $t < 1$ by the maximum principle. However, its value in B_1 is order $Ct^{-n/2}R^{C_1(n)}e^{AR^2 - R^2/4t}$, which goes to 0 as $R \to \infty$ when $t \in (0, t_0(A)]$.

2.8 Tychonoff Example

The growth hypotheses above are necessary. Tychonoff constructed an example of the form

$$u = \sum_{k=0}^{\infty} F_k(t)x^{2k},$$

with $F_k(0) = 0$ for all k. We obtain a solution to the equation if we take $F_k(t) = F_0^{(k)}(t)/[(2k)!]$, provided the series converges. The key point is the $(2k)!$ decay as a result of differentiating twice in space, versus only once in time (giving k derivatives of F_0). Since this decay is much better than the usual $k!$, it is not hard to construct F_0. Roughly, if we try $F_0(t) = e^{-1/t}$, then the derivatives grow like $2^k k! t^{-k} e^{-1/t}$. The series for u has terms that look like $(x^{2k}t^{-k}/k!)e^{-1/t}$ with coefficients controlled by $2^k(k!)^2/(2k)! < C$, so the series looks like $e^{x^2/t - 1/t}$, which almost gives the example. It is easy to see now that $F_0 = e^{-1/t^\alpha}$ for $\alpha > 1$ works.

Remark 8. *One can think of such examples as limits of solutions whose initial data drift to infinity, but have rapidly growing energy. By infinite speed of propagation and freedom in the shape of the initial distribution, the limit can be made nontrivial. The scheme goes roughly as follows. First, take u_1 to be the fundamental solution centered at the origin. Now assume that we can find a linear combination of fundamental solutions with initial data supported away from 0 (say in $\{x > 100\}$), such that the value of the combination at $x = -1$ closely approximates $u_1(-1, t)$. Call this combination u_2. We claim that u_1 and u_2 are very close for $x < -1$. This follows from the representation formula for bounded solutions of the problem $v_t = v_{xx}$ in a half-space, with initial data 0 and boundary data $g(t)$, which has the form*

$$v = cx \int_0^t \frac{g(s)}{(t-s)^{3/2}} e^{-x^2/4(t-s)}.$$

(Exercise. Derive by subtracting g, taking the odd reflection, and using the representation formula for bounded solutions to the non-homogeneous Cauchy problem.) Now assume we can take u_3 a combination of fundamental solutions with initial data supported in $\{x > 1000\}$ that approximates u_2 well at $x = 100$, and so on. In this way we can remove the initial data but "inject energy from ∞" in just the right way that we get a nontrivial limit.

It is interesting that we can in fact approximate continuous functions well by combinations of functions of the form $e^{-a/t}$ for $a > a_0$ arbitrarily large. This is a consequence of the Stone-Weierstrass theorem.

3   Viscosity Approach

Here we discuss an elementary approach to regularity based only on a simple "parabolic second-order" inequality: If $v > u$ in B_1 for all times $t < 0$ and $v = u$ at some point in $B_1 \times \{0\}$, then $D^2 v \geq D^2 u$ and $\partial_t v \leq \partial_t u$ at the contact point. If either inequality is strict, these functions can't both solve the heat equation.

More generally, take any smooth

$$F(M, p, z, (x, t)) : (Sym)^{n \times n} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$$

that is increasing in the matrix variable. (That is, if we fix all the other coordinates and add a nonnegative symmetric matrix to M, then F increases). If u and v satisfy

$$v_t - F(D^2 v, \nabla v, v, (x, t)) > 0, \quad u_t - F(D^2 u, \nabla u, u, (x, t)) < 0,$$

and $v > u$ on the sides and bottom of a parabolic cylinder, then they can never cross in the interior. (Here use that at the first time they touch, the values and gradients agree at the contact point). We call v a super-solution to the equation $w_t - F(D^2 w, \nabla w, w, x)$, and u a sub-solution to the equation. In the viscosity approach, one constructs super- or sub-solutions ("barriers") to control quantities that solve equations of the above type.

Some simple examples of how to apply this principle include:

- Say u solves

$$u_t - A(x, t) \cdot D^2 u - b(x, t) \cdot \nabla u = 0$$

  for some positive matrix field A and vector field b. Use the sub-solutions $-\epsilon(1 + t)$ to prove the maximum principle (nonnegative on the parabolic boundary implies nonnegative in the interior).

- Say that there is also a zeroth order term $c(x, t)u$ in the above equation, with c bounded. Use the sub-solutions ϵe^{-At} for A large to get the same result.

- Say that u and v solve the fully nonlinear equation $w_t - F(D^2 w, \nabla w, w, (x, t)) = 0$. Then $v - u$ solves a linear equation whose coefficients are certain averages of derivatives of F. (To see this expand $(v - u)_t = \int_0^1 \frac{d}{ds} F(D^2 u + s(D^2 v - D^2 u), ...) \, ds$). Conclude from the above that if F_z is bounded, then $v \geq u$ on the parabolic boundary implies that the solutions don't cross in the interior.

  In particular, solutions to $u_t - Lu = G(u)$ are trapped by solutions to the ODE $v'(t) = G(v)$ with initial data $\sup_{t=0} u$ and $\inf_{t=0} u$.

  (Here $L = A(x, t) \cdot D^2 + b(x, t) \cdot \nabla$, and G is e.g. Lipschitz).

- If u is caloric, then convex functions of ∇u are sub-solutions, e.g. $(\partial_t - \Delta)(|\nabla u|^2) = -2|D^2 u|^2$. Use this to show that $\varphi^2 |\nabla u|^2 + C(n)u^2$ is a subsolution for $C(n)$ large (here φ is a smooth cutoff in Q_1). Conclude the interior gradient estimate.

3.1 Useful Barriers

In the elliptic case, radial functions of the form $|x|^{-\beta} - 1$ are useful. They are negative outside B_1, cross horizontal planes at a positive angle, have positive radial second derivative of order $\beta^2 r^{-\beta-2}$, and have negative tangential second derivatives of order $-\beta r^{-\beta-2}$. It is simple to verify that when β is large, such functions are sub-solutions to equations of the form

$$F(D^2 u, \nabla u, u, x) = 0$$

where F is uniformly elliptic (if we add a positive symmetric matrix N to M, then F increases proportional to the maximum eigenvalue of N), and e.g. Lipschitz in p and $F(0, 0, z, x) = 0$ (constants are solutions). Here β depends on structural constants.

Here we construct the parabolic analogue. It is natural to look for barriers that are positive in a ball at $t = 1$ and invariant under parabolic scaling and rotation:

$$\varphi(x, t) = t^{-\alpha} g(r t^{-1/2}),$$

where g is positive in B_1 and negative outside. Taking $F = F(D^2 u)$ a uniformly elliptic operator with $F(0) = 0$, the sub-solution inequality becomes

$$- \left(\alpha g + \frac{r}{2} g' \right) < F(\operatorname{diag}(g'', g'/r, ..., g'/r)).$$

Taking $g(r) = e^{-\beta r^2} - e^{-\beta}$, the inequality roughly becomes

$$-\alpha g \leq F(\beta e^{-\beta r^2} \operatorname{diag}(\beta r^2 - 1, -1, ..., -1)) - \beta r^2 e^{-\beta r^2}.$$

Taking β large we can make the right side positive where $r > 1/2$, and for $\alpha > 0$ the desired inequality holds in $B_1 \backslash B_{1/2}$. To complete the construction, we take α large so that the inequality holds in $\{r \leq 1/2\}$ where g is strictly positive.

This function can be used to prove the strong maximum principle and a parabolic Hopf lemma. Indeed, the barrier immediately gives that if a nonnegative solution is positive at some point, then it is positive in a forward spacetime paraboloid centered at this point. In fact, it is larger than a small multiple of the barrier centered slightly behind this point (in time). Thus, the boundary of the positivity set can't touch the boundary of the paraboloid, since the barrier has negative radial derivative there.

To prove the strong maximum principle, assume e.g. that $u(0, 0) = 0$ and $u \geq 0$ in Q_1. By the above considerations, $u = 0$ in a backwards spacetime parabola. (The forwards parabolas inside this one contain the origin). The proof then goes the same way as the proof using mean value formula.

To prove the parabolic Hopf lemma, assume that a positive solution vanishes at some point on the boundary of the domain, and the domain has an interior sphere condition. We can find a forward paraboloid centered inside that first hits the boundary at this point. The barrier tells us that the normal derivative of the solution is strictly negative there. Think e.g. $e^{-t} \cos(x)$ in a strip.)

3.2 Li-Yau Harnack Inequality

For caloric functions we obtained the maximum principle and Harnack inequality using integration by parts and special properties of the fundamental solution. (The correct choice of test function gave a positive weight in the mean value formula). Here we give an elementary proof by finding a differential inequality for a clever quantity. For clarity we first treat the elliptic case.

If u is harmonic and positive in B_1, we consider the equations for $v := \log u$ and its derivatives. Let $w = |\nabla v|^2$. We have

$$\Delta v = -|\nabla v|^2, \quad \Delta w = -\nabla v \cdot \nabla w + 2|D^2 v|^2 \geq -2\nabla v \cdot \nabla w + \frac{2}{n}w^2.$$

This is a powerful differential inequality for w; heuristically, if $w(0, 0)$ is very large, then the w^2 on the right side will force w to blow up before it reaches the boundary. One directly obtains an upper bound for w in $B_{1/2}$ depending on n by constructing a radial barrier that blows up on ∂B_1. The super-solution inequality becomes

$$f'' + \frac{n-1}{r}f' + \frac{4}{3}(f^{3/2})' < \frac{2}{n}f^2.$$

Taking $f = \frac{A}{(1-r^2)^2}$ (which near the boundary has the correct scaling for the equation $f'' = f^2$) this becomes

$$C(n)(Ar^2 + A(1 - r^2) + A^{3/2}r) < A^2$$

which is true in B_1 for A large.

The Li-Yau proof of the parabolic Harnack inequality follows similar lines. For u caloric in Q_1 let $v = \log u$. We compute

$$(\partial_t - \Delta)v = |\nabla v|^2, \quad (\partial_t - \Delta)(\Delta v) = 2\nabla v \cdot \nabla(\Delta v) + 2|D^2 v|^2.$$

The last term is larger than $\frac{2}{n}(\Delta v)^2$, and the next to last term is a gradient term. Since one solution to the ODE $f' = \frac{2}{n}f^2$ is $-\frac{n}{2(1+t)}$ always starts below Δv, we conclude that

$$v_t - |\nabla v|^2 \geq -\frac{n}{2(1+t)}.$$

This can be regarded as a differential form of the Harnack inequality; note that it is sharp on the fundamental solution starting at $(0, -1)$.

By integrating this along spacetime lines connecting points in $B_{1/2} \times \{-1/2\}$ to $(0, 0)$ we recover the Harnack inequality.

Remark 9. *The above computations illustrate a connection between the radial direction in elliptic problems, and the negative time direction in parabolic problems. This can be a useful heuristic; we will see the same idea appear when we discuss monotonicity formulae and unique continuation, and a way to view caloric functions as high-dimensional limits of harmonic functions.*

3.3   ABP Estimate

We proved the strong maximum principle e.g. for the equation $u_t - a_{ij}(x, t)u_{ij} = 0$ using only the uniform ellipticity (and not the regularity) of the coefficients with a barrier. Since the Harnack inequality is a quantitative version of the strong maximum principle, it is not surprising that it too holds for such equations. In view of this fact, it is natural to ask for a proof of the Harnack inequality for these equations that only uses cleverly constructed barriers. I am not aware of such a proof. Instead, it seems important to connect the local information of the equation to information in measure. This is accomplished by the ABP estimate.

Remark 10. *Estimates that don't use anything except ellipticity are very important for nonlinear problems. Consider for example the fully nonlinear flow $u_t - F(D^2u) = 0$. The derivatives of the equation solve*

$$(u_k)_t = F_{ij}(D^2u)(u_k)_{ij}.$$

If we don't have any a priori control of D^2u, then such estimates are all we can use.

To fix ideas we consider the elliptic case. If a positive harmonic function is 1 at the origin, a natural first step in the direction of the Harnack inequality is to show that it is bounded in a set of positive measure (it doesn't hover high and occasionally spike down). This is what ABP accomplishes:

Theorem 4. *There are universal constants C, $\delta > 0$ such that if $u \geq 0$, $\Delta u \leq 0$ and $u(0) = a$, then*

$$\frac{|\{u < Ca\} \cap B_r|}{|B_r|} > \delta.$$

Remark 11. *One might attempt a proof as follows: if u is large in most places, then the data on some reasonable closed curve containing 0 will be large on average (but possibly highly oscillatory). In view of the representation formula for harmonic functions, it seems reasonable to expect that the solution is large on the interior as well. Interestingly, this is false for oscillatory coefficients. (See e.g. the paper "Harmonic measures for elliptic operators of non-divergence form" by Wu.)*

To connect the local information of the equation to nonlocal information, the idea is to slide convex shapes from below until they touch u and look at the map from contact point to vertex. The Jacobian is controlled because the whole Hessian of u is controlled at the contact point.

Proof. By scaling and multiplying by constants we may assume that $r = 10$ and $u(0) = 1$. Slide the paraboloids $t - |y - x|^2/2$ with vertex $y \in B_1$ from below u until they touch for the first time. We have

$$y = x + \nabla u(x), \quad D_x y = I + D^2u(x).$$

Since the paraboloid touches from below at x we have $D_x y \geq 0$. Furthermore, by the equation we have $|D^2 u(x)| < C(n)$. Let A denote the set of contact points. We conclude that

$$|B_1| = \int_A \det D_x y \, dx \leq C(n)|A|.$$

$\square$

Remark 12. *We use paraboloids because they reflect the natural scaling of the problem. In fact, we could use any smooth convex function ψ such that the eigenvalues of $D^2\psi$ are comparable at each point. In this case we have*

$$y = x + (\nabla\psi)^{-1}(\nabla u(x)) = x + \nabla\psi^*(\nabla u(x))$$

$$D_x y = I + D^2\psi^*(\nabla u(x)) \cdot D^2 u(x) = D^2\psi^* \cdot (D^2\psi + D^2 u).$$

Here ψ^* is the Legendre transform of ψ. The geometric information at x says that $D_x y \geq 0$, and the equation plus pointwise comparability of eigenvalues of $D^2\psi$ says that $D^2 u \leq CD^2\psi$. Thus, $\det D_x y < C$.

To see why one needs the comparability of eigenvalues, consider sliding cones $|y - x|$ from below. In general, there will be one contact point for a family of vertices on a line. However, cones can be useful for degenerate problems (see below).

Remark 13. *The technique adapts well to degenerate problems. For example, if an equation is uniformly elliptic only where the gradient is bounded (e.g. the minimal surface equation), then ABP works with shapes that have bounded gradient in B_1. If the equation is uniformly elliptic only at large gradient (e.g. p-Laplace equation) then one can slide cones and run a dichotomy argument: either there are many points touching at the vertex and we are done, or there is a good covering by disjoint balls in which a cone touches away from the vertex. In the latter case, one can slide paraboloids near the contact points to ensure that they touch at large gradient, and conclude.*

Another important example is the linearized Monge-Ampère equation $tr(D^2u^{-1}D^2\varphi) = 0$. Here $\det D^2 u \in [\lambda, \Lambda]$ is bounded away from 0 and ∞, and the domain is a level set of u that is comparable to a ball. In this case, slide "tiltings" $u(x) - \nabla u(y) \cdot (y - x)$ from below until they touch, giving the relation

$$\nabla u(y) = \nabla u(x) + \nabla\varphi(x), \quad D^2 u(y) D_x y = D^2 u(x) + D^2\varphi(x).$$

Since $\det D^2 u$ is comparable to 1 and $tr(D^2 u^{-1}(D^2 u + D^2\varphi)) = n$, the bound $\det D_x y < C$ follows from the arithmetic-geometric mean inequality.

The parabolic case is similar. It is convenient to work in the forward and backward spacetime parabolas

$$E_r^{\pm} = \{\pm t > |x|^2\} \cap \{|t| < r^2\}.$$

The analogue is: There are universal constants $C, \delta > 0$ such that if $u \geq 0$, $u_t - \Delta u \geq 0$ in E_1^+, and $u(0, 1) = 1$, then

$$|\{u < C\} \cap E_1^+| > \delta.$$

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To prove this result, instead slide the space-time paraboloids $P(x, t) = t - s - |y - x|^2/2$ until they cross for the first time at (x, t). This gives the relation

$$(y, s) = (x + \nabla u(x, t), t - u(x, t) - |\nabla u(x, t)|^2/2).$$

Then $D_{x,t}(y, s)$ is an $(n+1) \times (n+1)$ matrix, with the top left $n \times (n+1)$ block $(I + D^2 u, \nabla u_t)$, and bottom row $(0, 1 - u_t) + ((I + D^2 u)\nabla u, \nabla u_t \cdot \nabla u)$. The second vector is a linear combination of the top n rows, so we may ignore it when checking positivity and computing the Jacobian determinant. It follows from $D^2 u \geq -I$ and $u_t \leq 1$ at the contact point that $D_{x,t}(y, s)$ is positive, and from the equation that $D^2 u(x, t)$ and $(1 - u_t)$ are bounded, giving

$$\det D_{x,t}(y, s) = \det(I + D^2 u)(1 - u_t) < C.$$

One concludes with the same argument as above.

Remark 14. *In the case of equations that degenerate at large gradient, this technique is effective. However, when the equation degenerates at small spacetime gradient, attempting to use the parabolic version of cones (1-homogeneous in space and $1/2$-homogeneous in time) doesn't immediately work, since one can't use the information at contact points behind the vertex in time.*

3.4   Krylov-Safonov Harnack Inequality

For non-divergence equations, the information from the equation is local. Above we discussed a way to pass from this local information to information in measure. In this section we use barriers to localize this measure estimate. By applying it at all scales, we obtain the very important Krylov-Safonov inequality:

Theorem 5. *Assume that $u \geq 0$ in Q_1, and $u_t - a_{ij}(x, t)u_{ij} \geq 0$ for some uniformly elliptic coefficients a_{ij}. Then there exist ϵ, $C > 0$ universal such that*

$$\left(\int_{E_{1/2}^-} u^\epsilon \, dx \, dt \right)^{\frac{1}{\epsilon}} < Cu(0, 0).$$

This can be viewed as a generalization of the mean value inequality to equations with rough coefficients. It says that if a positive super-solution is small at some point, then it is small in most of the spacetime parabola behind this point.

One simple consequence is oscillation decay: if $u_t - a_{ij}(x, t)u_{ij} = 0$ in Q_1 then

$$\operatorname{osc}_{Q_{1/2}} u < (1 - \delta)\operatorname{osc}_{Q_1} u,$$

for some $\delta > 0$ universal. By iterating we obtain $C^{\alpha, \alpha/2}$ regularity for solutions. The idea is to apply the Krylov-Safonov inequality upside-up and upside-down. By adding and multiplying by constants we can assume that $0 \leq u \leq 1$. If u were very close to 0 at some point in $Q_{1/2}$ and very close to 1 at another, then by Theorem 5, u would be very close in measure to 0 and 1 in spacetime parabolas behind these points. The parabolas intersect in some region of nontrivial measure, giving a contradiction.

Proof of Krylov-Safonov Inequality. We may assume after multiplying by a constant that $u(0, 0) = 1$. It is enough to show that

$$|\{u > M^{k+1}\} \cap E_{1/2}^-| < (1 - \eta)|\{u > M^k\} \cap E_{1/2}^-|$$

for some $M, \eta > 0$ universal and all k. By the invariance of the equation under multiplication we may assume that $k = 0$.

For each $(x, t) \in E_{1/2}^- \cap \{u > 1\}$, take the largest r such that $u > 1$ in $E_r^+(x, t) \cap E_{1/2}^-$. (The sets $E_r^+(x, t)$ are guaranteed to touch $\{u \leq 1\}$ eventually since $u(0, 0) \leq 1$.) Call this $r(x, t)$.

There is a cylinder in $E_{r(x, t)}^+(x, t) \cap E_{1/2}^-$ with spatial radius order r and time radius order r^2. We claim that $u < M$ at some point in this cylinder. This is a simple barrier argument: If not, rescale to size 1 and start a large multiple of an appropriate translation of the useful barrier constructed above somewhere in this cylinder. By choosing the parameters for the barrier appropriately we contradict that $u \leq 1$ somewhere on the top of $E_{r(x, t)}^+(x, t)$. (The barrier stays large long enough and spreads quickly enough to cross u).

Applying the ABP estimate behind the point where $u < M$ in this cylinder, we conclude (up to redefining M) that

$$|\{u < M\} \cap E_{r(x, t)}^+(x, t) \cap E_{1/2}^-| > c|E_{r(x, t)}^+(x, t)|$$

for some c small universal.

Finally, we use a covering argument. A version of the Vitali covering lemma allows us to choose a disjoint collection $\{E_{r_i}^+\}$ such that their C-times enlargements cover $\{u > 1\} \cap E_{1/2}^-$, and furthermore

$$|\{u < M\} \cap E_{r_i}^+ \cap E_{1/2}^-| > c|E_{r_i}^+|.$$

We conclude that

$$
\begin{aligned}
|\{u > M\} \cap E_{1/2}^-| &< |\{u > 1\} \cap E_{1/2}^-| - \sum_i |\{u \leq M\} \cap E_{r_i}^+ \cap E_{1/2}^-| \\
&< |\{u > 1\} \cap E_{1/2}^-| - c \sum_i |E_{Cr_i}^+| \\
&< (1 - c)|\{u > 1\} \cap E_{1/2}^-|,
\end{aligned}
$$

completing the proof. $\qquad\square$

Remark 15. *Good references for a more detailed treatment of this this topic include: Savin, Small perturbation solutions to elliptic equations, and Yu Wang's parabolic analogue: Small perturbation solutions to parabolic equations.*

In these works the authors treat degenerate situations in which the equation only holds when the derivatives of u are close to the derivatives of a nearby smooth solution. Interestingly, the methods give enough compactness to do perturbation theory: blow up to a model situation (harmonic/ caloric) from which regularity is inherited. Roughly, if u is close to a smooth solution then the iteration outlined above can be carried out enough times to linearize.

4 Variational Approach

We give another approach to study the heat equation based on its gradient flow structure. This perspective gives new methods to study uniqueness and regularity issues, based mostly on integral estimates. Good references include the works of Caffarelli-Vasseur, and the lecture notes of Šverák on the Theory of PDE.

4.1 Gradient Flows

In the finite-dimensional setting, a gradient flow is a solution to the ODE

$$x'(t) = -\nabla F(x(t)),$$

where x is a curve in $\mathbb{R}^n$ and F is a smooth function (an "energy") on $\mathbb{R}^n$. It is clear that the stationary solutions are the critical points of F, and solutions travel normal to the level sets of F and generally tend towards its stable critical points ($D^2 F > 0$). (Some trajectories tend towards saddle points).

The basic "energy estimate" for solutions is that F decreases along the flow:

$$\frac{d}{dt} F(x(t)) = -|x'(t)|^2, \quad F(x)|_0^T + \int_0^T |x'(t)|^2 \, dt = 0. \tag{6}$$

In particular, if F is bounded below then we have an a priori H^1 bound for x.

Remark 16. *Assume that $F \to \infty$ as $x \to \infty$. In one dimension it is not hard to show that solutions converge to critical points. Indeed, by the energy estimate they remain bounded, and by ODE uniqueness they are monotonic.*

However, in two dimensions this is false for smooth F. One can imagine taking constructing F as a "spiraling staircase" over a curve spiraling from the outside towards S^1, with values $\sim e^{-\frac{1}{r-1}}$ and gradient tangent to the spiral, and making a smooth extension. Then some trajectories spiral for infinite time without converging.

(If F is analytic, the result may be true).

Remark 17. *The notion of gradient flow depends on the choice of scalar product. Indeed, if $\langle u, v \rangle = g(u) \cdot v$ for some positive definite symmetric g, then for x' to move in the direction that decreases F the fastest with respect to g, we need*

$$x' = -g^{-1}(\nabla F).$$

In particular, if A is a constant matrix then the equation $x' = Ax$ is a gradient flow if gA is symmetric for some positive definite g, or equivalently, if A is diagonalizable over the reals.

(Indeed, if A is diagonalizable then $A = B^{-1}DB$ for some invertible B and diagonal D. We may take $g = B^T B$ in this case. In the other direction, we have $A = GS$ for some symmetric S and positive definite G. The point is that we can take the square root of G. We have that $\sqrt{G}^{-1} A \sqrt{G}$ is symmetric, so it can be written ODO^T for some diagonal D and orthogonal O. Choosing $B = O^T \sqrt{G}^{-1}$ yields $A = B^{-1}DB$.)

The equation $u_t = \Delta u$ can be interpreted in several ways as a gradient flow (in infinite dimensions now). Recall that the choice of scalar product is important.

To see this consider the energy

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^n} u^2 \, dx.$$

Then $J'(u)v = (u \cdot v)_{L^2}$. In analogy with the finite dimensional case, we can write this as $J'(u)v = -(g(\Delta u) \cdot v)_{L^2}$, where $g = (-\Delta)^{-1}$. Thus, the heat equation is the gradient flow of J with respect to the scalar product $\langle w, v \rangle = ((-\Delta)^{-1}w \cdot v)_{L^2}$.

The energy estimate for this gradient flow is:

$$\frac{1}{2} \int_{\mathbb{R}^n} u^2 \, dx \Big|_0^T + \int_0^T \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \, dt = 0. \tag{7}$$

The heat equation is also the L^2 gradient flow of the Dirichlet energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx.$$

Remark 18. *More generally, one can consider gradient flows of functionals of the form $\int_\Omega F(D\mathbf{u}) \, dx$ for maps $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^m$ and functions F on $M^{m \times n}$. The gradient flow $\mathbf{v}$ satisfies*

$$\mathbf{v}_t - \mathrm{div}(\nabla F(D\mathbf{v})) = 0, \tag{8}$$

and when $\mathbf{v}_t|_{\partial\Omega} = 0$ the energy estimate is

$$\int_\Omega F(D\mathbf{v}) \, dx \Big|_0^T + \int_0^T \int_\Omega |\mathbf{v}_t|^2 \, dx \, dt = 0. \tag{9}$$

Remark 19. *The energy estimates also follow directly from differentiating the energy, using the equation and integrating by parts. However, the gradient flow perspective yields the meaning of the energy estimates: the energy lost is an average in time of the gradient squared of the functional, which gives a useful bound when the energy is positive.*

4.2 Consequences of the Energy Estimate

The basic energy estimate for a gradient flow gives a useful and general approach to uniqueness and regularity questions.

4.2.1 Uniqueness

It is easy to see that the energy estimate (7) also holds for caloric functions that vanish on the boundary of a bounded domain. An immediate consequence (by linearity) is uniqueness for the Dirichlet problem.

This argument works more generally for parabolic systems of the form

$$\partial_t \mathbf{u} = \mathrm{div}(A(x, t)D\mathbf{u}) + D\mathbf{u} \cdot b(x, t) + c(x, t)\mathbf{u},$$

where $\mathbf{u} : \Omega \times [0, T] \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is an evolving map, $A(x, t)$ are (possibly rough) coefficients that are a positive definite quadratic form on $M^{m \times n}$ for each (x, t) satisfying the ellipticity condition

$$\nu I \leq A \leq \nu^{-1} I,$$

and b, c are bounded. In this case, the modified energy $e^{-\lambda t} \int_\Omega |\mathbf{u}|^2 \, dx$ decreases when $\mathbf{u}$ has zero boundary data and λ is large.

Remark 20. *Finally, this argument works for gradient flows of functionals of the form $\int_\Omega F(D\mathbf{u}) \, dx$, where $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^m$ and F is uniformly convex. The point is that the difference of solutions solves a linear uniformly parabolic problem: if $\mathbf{u}$ and $\mathbf{v}$ are solutions, then*

$$
\begin{aligned}
(\mathbf{u}^\alpha - \mathbf{v}^\alpha)_t &= \partial_i(F_{p_i^\alpha}(D\mathbf{u}) - F_{p_i^\alpha}(D\mathbf{v})) \\
&= \partial_i \left(\int_0^1 \frac{d}{ds} F_{p_i^\alpha}(D\mathbf{v} + s(D\mathbf{u} - D\mathbf{v})) \, ds \right) \\
&= \partial_i \left(\left(\int_0^1 F_{p_i^\alpha p_j^\beta}(D\mathbf{v} + s(D\mathbf{u} - D\mathbf{v})) \, ds \right) (\mathbf{u}^\beta - \mathbf{v}^\beta)_j \right).
\end{aligned}
$$

4.2.2 Regularity

By localizing the energy estimate for the heat equation one recovers the parabolic analogue of the Caccioppoli inequality from elliptic theory. Take a cutoff φ that is one in $Q_{1/2}$ and zero outside Q_1. We compute

$$\partial_t \int_{B_1} u^2 \, dx + 2 \int_{B_1} |\nabla u|^2 \varphi \, dx = \int_{B_1} u^2(\varphi_t + \Delta \varphi).$$

Integrating in time, we conclude that

$$\sup_{t \geq -1/4} \int_{B_{1/2}} u^2 \, dx + \int \int_{Q_{1/2}} |\nabla u|^2 \, dx \, dt \leq C \int \int_{Q_1} u^2 \, dx \, dt. \tag{10}$$

By applying this to all the derivatives of u we obtain local smoothness of solutions (and also space analyticity, but not time, just as above, using the scaling of the estimates).

The same local energy estimate holds for linear systems of the type considered above (with possibly rough coefficients), yielding

$$\sup_{t > -1/4} \int_{B_{1/2}} |\mathbf{u}|^2 \, dx + \int \int_{Q_{1/2}} |D\mathbf{u}|^2 \, dx \, dt < C(n, \nu) \int \int_{Q_1} |\mathbf{u}|^2 \, dx \, dt.$$

(In this case, multiply by the square of the cutoff and use Cauchy-Schwarz). Thus, solutions are uniformly L^2 in time and the average H^1 norm in time is bounded.

A Historical Aside

The importance of considering linear systems with rough coefficients is that the space and time derivatives of solutions to nonlinear systems of the form (8) solve this type of equation (as we saw above). If one shows that solutions to the linear system are continuous, then

the coefficients of the linearized equation are continuous, and smoothness for the nonlinear system follows from perturbation theory and bootstrapping.

In the elliptic two-dimensional case (both scalar and systems), this was shown by Morrey ('30s). The point is that the energy estimate is invariant under the scaling $u \to u(Rx)$ in two dimensions. One can in fact show that the energy decays by a fixed fraction in dyadic balls using the energy estimate and a Poincaré inequality, giving $r^2 \mathrm{avg}_{B_r}(|\nabla u|^2) < Cr^{2\alpha}$. The C^α regularity follows from standard results in Morrey-Campanato spaces. (In the scalar case there is an even more elementary proof using the maximum principle combined with H^1 control.)

In the scalar case, for $n \geq 3$ this was shown by De Giorgi and Nash in the late '50s. This breakthrough result crucially uses a version of the maximum principle: the energy estimate holds for positive sub-solutions to the equation $u_t - \mathrm{div}(A\nabla u) = 0$, hence for truncations of the solution. We discuss this result in the next section.

In the systems case, these techniques fail. In fact, continuity is false in the elliptic case when $n = m \geq 3$ by examples of De Giorgi and Giusti-Miranda (late '60s). The question of continuity for linear uniformly parabolic systems in the plane remained open until recently. It is in fact possible to construct examples of finite-time discontinuity formation (and even L^∞ blowup) from smooth data in this case.

Nonetheless, it is a beautiful result of Nečas and Šverák from the early '90s that gradient flows of smooth uniformly convex functionals are smooth when $n = 2$. We briefly discuss the idea. Instead of differentiating in space and using a continuity result for the linearized equation (which works in the scalar/ 2D elliptic case, but is now known to be false in the parabolic case by the aforementioned examples), use that the time derivative solves the linearized equation. The energy estimate gives uniform L^2 integrability in time of $\mathbf{v}_t$. The key observation of the paper is that this result holds true for small perturbations of the power: ignoring boundary terms, we compute for solutions $\mathbf{u}$ of the linear problem that

$$\partial_t \int |\mathbf{u}|^\gamma = \gamma \int |\mathbf{u}|^{\gamma-2}\mathbf{u} \cdot \mathrm{div}(AD\mathbf{u})$$

$$\leq -c\left(\int |\mathbf{u}|^{\gamma-2}|D\mathbf{u}|^2\right) - \gamma(\gamma-2)\int |\mathbf{u}|^{\gamma-2}A\left(\frac{\mathbf{u}\otimes\mathbf{u}}{|\mathbf{u}|^2}\cdot D\mathbf{u},\, D\mathbf{u}\right)$$

for a positive universal c. In the scalar case, the operator $(\mathbf{u}\otimes\mathbf{u})/|\mathbf{u}|^2 = 1$, indicating that we obtain some control on L^γ norms for any $\gamma \geq 2$. On the other hand, in the systems case, the components of the map interact, so the second term might cancel out the good contribution from the first.

However, this operator has one eigenvalue 1 and the rest zero, so it doesn't change the size of $D\mathbf{u}$ in the second term. Thus, for γ barely larger than 2, the coefficient of the second integral is small enough that the first term dominates, giving uniform $L^{2+\epsilon}$ integrability in time. This estimate for $\mathbf{v}_t$ is good enough that the nonlinear system can be treated as an elliptic problem for each fixed time. (Roughly, the mass of the right side "spreads out" well enough that Morrey's arguments work).

4.3 De Giorgi-Nash-Moser

Here we discuss the De Giorgi proof of the Hölder estimate for divergence-form equations with rough coefficients. The importance of this estimate is discussed above.

We will discuss two key steps: a "no spikes" estimate, and an oscillation decay estimate. We discuss them in contexts of increasing difficulty (minimal surfaces, elliptic, and finally parabolic cases).

4.3.1 "No Spikes Estimate"

Assume that $E \subset \mathbb{R}^n$ is a set of minimal perimeter (that is, if we make any compact perturbation of E, the perimeter increases). In this context, the no spikes estimate says: there exists $\delta(n) > 0$ such that if $|E \cap B_1| < \delta$, then $|E \cap B_{1/2}| = 0$. The interpretation is exactly that E doesn't have any thin spikes, at any scale. It implies in particular that if a sequence of sets of minimal perimeter converges in L^1 to a half-space, then the sets converge uniformly.

The idea of the inequality is that the perimeter minimality and isoperimetric inequalities compete, at every scale. Let $V(r) = |E \cap B_r|$. By perimeter minimality we have

$$Per(E, B_r) \leq Per(\partial B_r, E),$$

which implies

$$Per(E \cap B_r) \leq CPer(\partial B_r, E) = CV'(r).$$

On the other hand, by the isoperimetric inequality we have

$$V(r)^{1-\frac{1}{n}} < CPer(E \cap B_r).$$

Taking a decreasing sequence of radii r_k, we conclude that

$$V(r_k) = V(r_{k+1}) + \int_{r_{k+1}}^{r_k} V'(r)\, dr \geq c(r_k - r_{k+1})V(r_{k+1})^{1-\frac{1}{n}}.$$

Taking $r_k = 1 + 2^{-k}$, we thus have

$$V(r_{k+1}) < C^k V(r_k)^\gamma$$

for some $\gamma(n) > 1$. By iterating this inequality we conclude that

$$V(r_{k+1}) < C^{\gamma^k} \delta^{\gamma^k}$$

(for a possibly larger $C(n)$), and the conclusion follows for $\delta(n)$ small.

In the elliptic case, the no spikes estimate says: if $\operatorname{div}(A(x)\nabla u) \geq 0$ in B_1, with ellipticity ν, and $\|u\|_{L^2(B_1)} < \delta(n, \nu)$, then $\|u^+\|_{L^\infty(B_{1/2})} < 1$. It is the analogue of the mean-value inequality for subharmonic functions.

In two dimensions, the estimate is easy. The energy estimate says that $\int_{B_{3/4}} |\nabla u|\, dx < C\delta^{1/2}$. By the co-area formula, the integral of $|\nabla u|$ is the integral of perimeters of level sets of u. If u were larger than 1 at some point in $B_{1/2}$, then by the maximum principle it is

larger than 1 at some point on each circle with radius larger than $1/2$. Thus, the perimeters of the level sets of height t are order 1 for all $t < 1$, giving a contradiction.

In the higher dimensional case, the perimeters of level sets can still get very small, so we need to be more careful. We repeat the proof for minimal surfaces, with the energy estimate playing the role of area minimality, and the Sobolev inequality playing the role of the isoperimetric inequality. Let h_k be an increasing sequence of heights, and r_k a decreasing sequence of radii. The analogue of $V(r_k)$ is $\int_{B_{r_k}} [(u - h_k)^+]^2 \, dx := a_k$. The energy estimate gives:

$$\int_{B_{r_k}} [\nabla((u - h_k)^+ \varphi)]^2 \, dx < C(r_k - r_{k+1})^{-2} a_k,$$

where φ is a cutoff between $B_{r_{k+1}}$ and B_{r_k}. The Sobolev inequality gives:

$$\left(\int_{B_{r_{k+1}}} [(u - h_{k+1})^+]^{2*} \, dx \right)^{\frac{2}{2*}} < C \int_{B_{r_k}} [\nabla((u - h_k)^+ \varphi)]^2 \, dx.$$

Finally, we can take advantage of the change in scaling between these two inequalities using Hölder's inequality:

$$a_{k+1} \leq \left(\int_{B_{r_{k+1}}} [(u - h_{k+1})^+]^{2*} \, dx \right)^{\frac{2}{2*}} |\{(u - h_k)^+ > h_{k+1} - h_k\} \cap B_{r_k}|^{\delta(n)}$$

$$< C(r_k - r_{k+1})^{-2} (h_{k+1} - h_k)^{-2\delta} a_k^{2+\delta}.$$

Taking $r_k = 1/2 + 2^{-k}$ and $h_k = 1 - 2^{-k}$, we have an inequality of the same type as above in the minimal surface case, completing the proof.

Finally, we treat the parabolic case. The statement is that if $u_t - \operatorname{div}(A(x, t)\nabla u) \leq 0$ in Q_1, and $\|u\|_{L^2(Q_1)} < \delta(n)$, then $\|u^+\|_{L^\infty(Q_{1/2})} < 1$.

The analogue of a_k is $\int_{Q_{r_k}} [(u - h_k)^+]^2 \, dx := b_k$. Let $u_k = (u - h_k)^+$. The energy estimate and Sobolev inequality give

$$\sup_{t > -r_{k+1}^2} \int_{B_{r_{k+1}}} u_{k+1}^2 \, dx + \int_{-r_{k+1}^2}^0 \left(\int_{B_{r_{k+1}}} u_{k+1}^{2*} \, dx \right)^{\frac{2}{2*}} dt < C^k b_k.$$

Since the L^2 norm is uniformly bounded in time, and an L^p norm is bounded on average in time, we can interpolate using

$$\int v^{2+2/q} \leq \left(\int v^{2*} \right)^{2/2*} \left(\int v^2 \right)^{1/q}$$

(here q is the Hölder conjugate of $2*/2$). This gives (for $\gamma = 2 + 2/q$) that

$$\left(\int_{Q_{r_{k+1}}} u_{k+1}^\gamma \, dx \, dt \right)^{\frac{2}{\gamma}} < C^k b_k.$$

The rest of the proof goes exactly as before.

4.3.2 Decay Estimate

The "no spikes" estimate is the heart of the De Giorgi proof. The decay estimate combines the no spikes estimate with the observation that H^1 functions must "pay in measure" to go from one height to another.

To capture "paying in measure" precisely, we use a version of the Poincaré inequality. In the elliptic case, it says the following: Assume that $0 \leq v \leq 1$ in B_1, and that $\int_{B_1} |\nabla v|^2 \, dx \leq 1$. Then if $\delta |B_1| < |\{v = 0\}| < (1 - \delta)|B_1|$, there exists $\eta(\delta, n)$ such that

$$|\{0 < v < 1/2\}| > \eta.$$

In one dimension the proof is by Cauchy-Schwarz, which gives a $C^{1/2}$ estimate. In the higher-dimensional case, let $\bar{v} = \min\{v, 1/2\}$. We have by Cauchy-Schwarz that

$$\int_{B_1} |\nabla \bar{v}| \, dx < |\{0 < v < 1/2\}|^{1/2}.$$

By the Poincaré inequality, the left side controls $c(n) \int_{B_1} |\bar{v} - \mathrm{avg.}_{B_1} \bar{v}| \, dx$. If $|\{0 < v < 1/2\}| < \eta$ is extremely small, then this quantity is bounded away from zero, contradicting the above inequality.

Now assume that $-1 < u < 1$ and that $\mathrm{div}(A\nabla u) = 0$. By combining this result with the no spikes estimate we get the C^α estimate $\|u\|_{C^\alpha(B_{1/2})} < C$.

By the scaling invariance of the problem, it suffices to show oscillation decay from B_1 to $B_{1/2}$. Up to changing a sign, we may assume that $|\{u < 0\} \cap B_1|/|B_1| > 1/2$. If $u^+ = 0$ in a sufficiently large fraction of B_1, then by no spikes we obtain $u|_{B_{1/2}} < 1/2$. If not, we can apply the above result to say that $|\{2(u - 1/2)^+ = 0\}| > |\{u^+ = 0\}| + \eta$ small. Iterating, we eventually reach the situation that $2^{k_0}(u - (1 - 2^{-k_0}))^+$ is zero on a large enough fraction of the ball that no spikes kicks in (for universal k_0). In particular, $u < 1 - 2^{-k_0-1}$ in $B_{1/2}$, completing the proof.

In the parabolic case, note that sub-solutions can drop from 1 to 0 immediately in time without paying in measure. Thus, the statement of the pay in measure estimate requires some mass of the zero set to be near the bottom of a parabolic cylinder, and some mass of $\{v > 1/2\}$ to be near the top. Let v satisfy $0 \leq v \leq 1$ in Q_1, and assume the energy estimate

$$\int_{B_1} v^2 \, dx \big|_s^t + \int_s^t \int_{B_1} |\nabla v|^2 \, dx \, dt < (t - s)$$

holds for all $-1 \leq s < t \leq 0$. (This is the energy estimate satisfied by positive bounded sub-solutions to the parabolic problem). Thus, the H^1 norm is bounded on average, and the L^2 norm is "upwards Lipschitz" in time. Then if $|\{v = 0\} \cap \{t < -1/2\}| > \delta$ and $|\{v \geq 1/2\} \cap \{t > -1/2\}| > \delta$, then

$$|\{0 < v < 1/2\} \cap Q_1| > \eta(\delta, n).$$

Indeed, assume that $|\{0 < v < 1/2\}| < \eta$ very small. Let $\mu(t) = |\{v(., t) = 0\}|$ and $\eta(t) = |\{0 < v(., t) < 1/2\}|$. Then at all but $\sim \eta^{1/2}$ measure of times, $\eta(t) < \eta^{1/2}$.

Furthermore, at such times, whenever $\delta^2 < \mu(t) < |B_1| - \delta^2$, we have by the stationary case above that $\int_{B_1} |\nabla v|^2 \, dx > C(\eta, \delta) \to \infty$ as $\eta \to 0$. By the energy estimate, as $\eta \to 0$, the measure of the set of times when $\delta^2 < \mu(t) < |B_1| - \delta^2$ goes to zero.

Thus, for η small, at the vast majority of times, we have that either $v = 0$ or $v > 1/2$ in all but $|B_1| - \delta^2$ measure of the ball. Since the zero set has some mass in $\{t < -1/2\}$, the first case happens at some $t_0 \in \{t < -1/2\}$. Using the first term in the energy estimate, we see that for some universal small s_0, there exists $t_1 > t_0 + s_0$ such that the first case also holds at t_1, and the second case can never happen for $t_0 \le t \le t_1$. Iterating, we contradict that $\{v > 1/2\}$ has some mass in $\{t > -1/2\}$, completing the proof.

The proof of oscillation decay proceeds as above. Assume that $u_t - \operatorname{div}(A\nabla u) = 0$ in Q_1, with $-1 \le u \le 1$. We may assume without loss of generality that $\{u \le 0\} \cap \{t < -1/2\}$ fills at least half of $Q_1 \cap \{t < -1/2\}$. If $\{u > 1/2\}$ has tiny mass in the top half of Q_1, we get oscillation decay by the no spikes case. If not, we gain at least η measure going from $\{u \le 0\}$ to $\{u - 1/2 \le 0\}$. Iterating, we reach the threshold required to use the no spikes estimate after a finite universal number of steps.

5 Homogeneous Solutions to the Stationary Navier-Stokes System

Here we discuss Šverák's striking rigidity results for homogeneous solutions to the stationary Navier-Stokes system, from the paper "On Landau's solutions of the Navier-Stokes equations." An interesting insight in the paper is that in three dimensions, there is a connection between the structure of the equations for homogeneous solutions and the conformal geometry of S^2.

5.1 Preliminaries

We briefly introduce the Navier-Stokes system, and discuss some important scaling properties. The Navier-Stokes system is

$$u_t + Du \cdot u - \Delta u + \nabla p = 0, \quad \operatorname{div}(u) = 0,$$

where $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is a vector field representing the velocity of fluid flow, and $p : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a function representing the pressure. The first two terms in the equation measure how u changes as we move with the fluid. The last two represent the forces in the fluid. (The equation is Newton's law mass times acceration = force). The Laplace term represents particle interactions that tend to average the velocity, e.g. if the fluid moves very quickly at some point, but slowly in regions around this point, then when diffusion dominates the velocity tends toward the average. Finally, the second equation says that the flow preserves volume (it is incompressible).

Observe that by the second equation, the pressure is recovered by a nonlocal operation on the derivatives of u:
$$p = (\Delta)^{-1}((Du)^2).$$

The Navier-Stokes equations are invariant under parabolic rescalings that fix -1-homogeneous vector fields, and -2-homogeneous pressures:

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

also solves the system.

As a first step in understanding possible singularities of Navier-Stokes, it is thus natural to investigate examples of -1-homogeneous stationary solutions.

Remark 21. *Since $|\partial B_r| \sim r^{n-1}$, the equation $div(u) = 0$ holds across the origin (in the distributional sense) for -1-homogeneous u when $n \geq 3$. In the case $n = 2$, the divergence of u can have a Dirac mass at the origin.*

Furthermore, the first equation is $div(Du - u \otimes u) - \nabla p = 0$. Since the quantities differentiated are -2-homogeneous, in dimension $n \geq 4$ the first equation holds across the equation, but when $n = 3$ the expression can have a Dirac mass. In dimension two, it is not clear how to interpret the equation in the distributional sense.

Finally, note that in dimension $n \geq 5$ the energy $\int |Du|^2$ is locally bounded.

We will restrict our attention to the case $n \geq 3$ here. In the case $n = 2$, Šverák classified the -1 homogeneous solutions with zero flux around the origin (see the paper). Interestingly,

they are either purely rotational, or one of a countable family of purely radial solutions expressible in terms of elliptic functions of θ.

We conclude this introductory section by mentioning useful symmetric situations in the study of the $3D$ Navier-Stokes system, and stating the main theorem. We say that u is axisymmetric if it is invariant under rotations around the x_3 axis, i.e. $u(Rx) = Ru(x)$ for all R that are pure rotations in the $x_1 - x_2$ variables. This reduces the problem two variables r, x_3 in cylindrical coordinates (one, angle from x_3 axis, if homogeneity is imposed), still three components.

We say that it is axisymmetric with no swirl if, in addition, u is tangent to each plane through the x_3 axis (it has no components in the directions generated by rotations around the x_3 axis). This reduces the number of components to two.

Landau computed interesting solutions to this problem in three dimensions by imposing axisymmetry and no swirl, which reduces the problem to an ODE system in angle from x_3 axis, for two components of the vector field. As we will see, the solutions resemble the velocity field one might expect from a jet that propels air upwards. These are called Landau solutions, and it is known for these solutions that the first equation has a Dirac mass on the right side in the direction of some vector.

The main theorem is:

Theorem 6. *(Šverák): If u is a -1-homogeneous, smooth solution to the Navier-Stokes system in $\mathbb{R}^n \backslash \{0\}$, then in the case $n = 3$, u is a Landau solution (up to a rotation), and in the case $n \geq 4$, $u = 0$.*

5.2   Quantity with Maximum Principle

Since the Laplace is the leading-order term in Navier-Stokes equations, it is natural to search for quantities that satisfy the maximum principle, possibly with useful remaining terms. There is one very useful quantity which we compute here.

By taking the dot product of the stationary equations with u we obtain

$$u \cdot \Delta u - u \cdot \nabla(|u|^2/2 + p) = 0.$$

Let $H = |u|^2/2 + p$ be the quantity of interest. We then have

$$\Delta H - u \cdot \nabla H = |Du|^2 - \Delta p = |Du|^2 - tr((Du)^2).$$

The quantity on the right is twice the square of the asymmetric part of Du, $(Du - (Du)^T)/2$. We denote Du_{asymm} by ω. (In three dimensions, it is essentially $\nabla \times u$). We thus have

$$\Delta H - u \cdot \nabla H = 2|\omega|^2.$$

In particular, if H has a local maximum, then it is constant, so $\omega = 0$. It would follow that u is a gradient, hence harmonic by the incompressibility equation. If in addition u is -1-homogeneous, then it vanishes in dimension $n \geq 4$ since such singularities are removable in this case. (In dimension 3 the fundamental solution is r^{-1}.)

5.3 System on the Sphere

Since we impose -1-homogeneity, the system can be written on the sphere. It is a useful exercise to compute it. Let ν be the unit radial vector. We write

$$u = r^{-1}(f\nu + v)$$

where f is a zero-homogeneous function, and v is a zero-homogeneous vector field tangent to the sphere. By mild abuse of notation we write the pressure as $r^{-2}p$ for p a zero-homogeneous function.

We have

$$\nabla(\text{pressure}) = -2p\nu + \nabla p.$$

The divergence of $r^{-1}\nu$ is the Laplace of $\log r$, which is $n-2$ on the sphere. Using this we compute

$$\text{div}(u) = \text{div}(v) + (n-2)f = 0.$$

Since v is zero-homogeneous, the full divergence is the same as the divergence on the sphere.

We split $Du \cdot u$ into two pieces. By homogeneity, the radial derivative of u on the sphere is $-u$, so the derivative in the direction of the radial component of u is $-fu = -f^2\nu - fv$. The tangential derivative of $r^{-1}f\nu$ is $\nu \otimes \nabla f + fD\nu$. Since the second fundamental form $D\nu$ is the projection onto the sphere, this part of Du contributes $(v \cdot \nabla f)\nu + fv$. Finally, the remaining part is $Dv(v)$. We can split this into its tangential component $\nabla_v v$ (here ∇ is covariant differentiation on the sphere) and normal component $Dv(v) \cdot \nu = v \cdot (Dv)^T(\nu) = -v \cdot (D\nu)^T(v) = -|v|^2$. Putting this together we obtain

$$Du \cdot u = (v \cdot \nabla f - f^2 - |v|^2)\nu + \nabla_v v.$$

Finally, we compute the Laplace. We have $\Delta(f\nu/r) = (\Delta f)\nu + 2\nabla f - 2f\Delta(\nabla \log r) = (\Delta f)\nu + 2\nabla f + 2\text{div}(v)f$, using that $D\nu$ is the projection to tangent, and also the zero-divergence equation. Finally, we have that $\Delta(r^{-1}v) \cdot \nu = -v \cdot \Delta u - 2tr(D\nu^T Dv) = -2\text{div}(v)$. We conclude that

$$\Delta u = (\Delta f)\nu + 2\nabla f + [\Delta(v/r)]_T.$$

Here, the subscript T denotes tangential part.

The full system then becomes:

$$[\Delta(v/r)]_T - \nabla_v v + \nabla(2f - p) = 0,$$
$$\Delta f - v \cdot \nabla f + (f^2 + |v|^2) + 2p = 0,$$
$$\text{div} v + (n-2)f = 0.$$

Here, all operators are taken on the sphere. The quantity $[\Delta(v/r)]_T$ is related to the Hodge Laplacian on the sphere.

5.4 The Case $n \geq 4$

Recall from the above discussion that in dimension $n \geq 4$, if the quantity $H = |u|^2/2 + p$ is constant and u is -1-homogeneous, then u vanishes.

In view of this observaiton, we compute the equation for $H = r^{-2}H(\nu)$ on the sphere. Using the homogeneity of H and the fact that the mean curvature of the sphere is $n - 1$ we compute

$$\Delta H = \Delta_{S^{n-1}} H + 2(4 - n)H.$$

Observe that for $n \geq 4$, the second term has the correct sign to apply the maximum principle at positive maxima of H; this is the key point. If we manage to show H is non-positive, then it vanishes. Indeed, the equation for f is

$$\Delta f - v \cdot \nabla f = -2H,$$

using the above formulae and the definition of H. It would follow from the maximum principle that f is constant, hence vanishes by the incompressibility equation.

The second term in the equation for H is

$$-(f\nu + v) \cdot (\nabla H - 2H\nu) = -v \cdot \nabla H + 2fH.$$

Using the incompressibility equation we arrive at

$$\Delta_{S^{n-1}} H + 2(4 - n)H - v \cdot \nabla H - \frac{2}{n-2} H \operatorname{div}(v) = 2|\omega|^2.$$

In the case $n = 4$, the last three terms are $-\operatorname{div}(Hv)$, so by integrating we conclude that $\omega = 0$ directly.

In the case $n > 4$, we can remove the last two terms by multiplying by the correct power α of $H_+ = \max\{H, 0\}$ and integrating the last term by parts (corresponding to downward variations of H in the positivity region). Doing this, the last two terms become

$$-\int_{S^{n-1}} \left(1 - \frac{2(\alpha + 1)}{n - 2}\right) H_+^\alpha \nabla H_+ \cdot v \, ds = 0$$

when we take $\alpha = \frac{n-4}{2}$. This completes the proof in the case $n \geq 4$.

5.5   The Case $n = 3$

In the case $n = 3$ there are the nontrivial Landau solutions to the Navier-Stokes system away from 0. We derive them here.

5.5.1   Conformal Geometry and the Landau Solutions

In the case that u is axisymmetric with no swirl, the vector field v/r is the gradient of a zero-homogeneous function depending only on x_3. Here we consider more generally the consequences of the relation $v/r = \nabla\varphi$.

First, the incompressibility equation becomes

$$f = -\Delta_{S^2}\varphi.$$

We also have

$$[\Delta(v/r)]_T = (\nabla(r^-2\Delta_{S^2}\varphi))_T = -\nabla f.$$

Finally, observe that

$$\nabla_v v = (D(v/r) \cdot (v/r))_T = (D^2\varphi \cdot \nabla\varphi)_T = \nabla_{S^2}(|v|^2/2).$$

We conclude from the first equation in the system that

$$f - p - |v|^2/2 = c$$

for some constant c. Using this in the second equation, along with the incompressibility, we obtain

$$\Delta f + 2f - \operatorname{div}(fv) = const.$$

By integrating and using that f is a divergence we conclude that the constant is zero. Replacing f with $\tilde{f} = f + 2$ we obtain

$$\Delta\tilde{f} - \operatorname{div}(\nabla\varphi\tilde{f}) = 0.$$

Making the ansatz $\tilde{f} = a(x)e^\varphi$ we see that a solves an elliptic equation with no zeroth-order term, and is thus is a constant. We have $a > 0$ using that $-\Delta\varphi + 2 = ae^\varphi$, and we may assume that $a = 2$ after multiplying φ by a constant.

The interpretation of the equation

$$-\Delta\varphi + 2 = 2e^\varphi$$

is that $e^\varphi g_0$ has Gauss curvature 1 (here g_0 is the standard metric on S^2), and is thus isometric to g_0. We conclude that for some conformal diffeomorphism h of S^2, $h^*g_0 = e^\varphi g_0$, hence

$$\varphi = \log|Dh|^2.$$

The conformal diffeomorphisms of S^2 are well-known. If h fixes a pair of antipodal points, then by composing with stereographic projections we get an automorphism of $\mathbb{C}$ that fixes the origin, corresponding to a dilation and a rotation λz. These are in fact the building blocks of all conformal diffeomorphisms, using the fact that h must in general take some pair of antipodal points to another pair.

To see this, observe that after composing with a rotation we can assume h fixes the north pole. Composing with stereographic projections, we get an automorphism of $\mathbb{C}$, which after another rotation fixing the north pole is $\lambda z + b$ for some $\lambda \in \mathbb{R}$, $\lambda > 0$. Consider the restriction to the circle the plane containing the x_3 axis and b, let θ be the angle from the north pole, and let $h(\theta)$ be the angle of the image. Then $h(0) = 0$ and $h(2\pi) = 2\pi$, and furthermore, $h(\pi) < \pi$. We conclude by continuity that $h(\theta + \pi) - h(\theta) = 0$ for some $\theta \in (0, \pi)$. Thus, after composing with an isometry we have that h fixes a pair of antipodal points.

In particular, if h_λ are the conformal diffeomorphisms fixing the north and south poles and planes through the x_3 axis, corresponding to the automorphisms λz of $\mathbb{C}$ (that is, if P is the stereographic projection from S^2 to $\mathbb{C}$, then $h_\lambda = P^{-1} \circ \lambda z \circ P$), then

$$\varphi(x) = \log|Dh_\lambda|^2(Rx)$$

for some rotation R. We conclude that φ is axisymmetric with respect to some axis, hence u is a rotation of a Landau solution.

We conclude this subsection by explicitly computing and describing the Landau solutions corresponding to h_λ. Let $\lambda = e^{-\kappa}$ be the dilation factor, and let $h_\kappa(\theta)$ be the angle of the image of a point at angle θ, where the angle is taken from the x_3 axis. Since h_λ is conformal, we only need to compute one component of its derivative (the other is the same size). The easy one to compute is the derivative in the horizontal direction, given by the ratio of radii at $h_\kappa(\theta)$ and θ:

$$|Dh|^2 = \frac{\sin^2(h_\kappa)}{\sin^2(\theta)}.$$

The stereographic projection relation is

$$\frac{\sin(h_\kappa)}{1 - \cos(h_\kappa)} = e^{-\kappa}\frac{\sin(\theta)}{1 - \cos(\theta)}.$$

We can simplify this relation to

$$\frac{\sin(h_\kappa)}{\sin(\theta)} = \frac{2}{e^\kappa(1 - \cos\theta) + e^{-\kappa}(1 - \sin(\theta))} = \frac{1}{\cosh(\kappa) - \sinh(\kappa)\cos(\theta)}.$$

Thus,

$$e^\varphi = \frac{1}{(\cosh\kappa - \sinh\kappa\cos(\theta))^2}.$$

From this it is easy to derive f and v via $v = \nabla\varphi$ and $f = 2e^\varphi - 2$:

$$v = \frac{-2\sin\theta}{\coth\kappa - \cos\theta}e_\theta, \quad f = \frac{2}{(\cosh\kappa - \sinh\kappa\cos\theta)^2} - 2.$$

The flow lines of these solutions focus upwards towards the x_3 axis like a jet stream.

Finally, it is useful to compute the pressure. Using $p = f - |v|^2/2$ we have up to a constant that

$$p = 2\frac{1 - \sinh^2\kappa\sin^2\theta}{(\cosh\kappa - \sinh\kappa\cos\theta)^2}.$$

Observe that the pressure has a local maximum at $\theta = 0$, so the points along the x_3 axis should be unstable with respect to perturbations. On the other hand, experiments show that a ping-pong ball can be suspended in the stream from a hair dryer in a stable way; Thus, the Landau solutions are not likely to produce this scenario. On the other hand, if we don't ask that the solution is smooth on the whole sphere, one can solve the ODE locally and get examples where the pressure has a local minimum on the x_3 axis (see the work of Yanyan Li et. al.).

5.5.2 Rigidity in Three Dimensions

To prove the main theorem, Šverák shows that v/r is in fact a gradient. The idea is that in two-dimensional problems, the vorticity $\nabla \times u$ is a useful scalar quantity. Since we impose -1-homogeneity, our problem is two-dimensional, and it is natural to compute the equation

for the vorticity $\omega = (\nabla \times u) \cdot \nu$ on S^2. Roughly, by taking the curl of the first equation in the system one can derive (as in the flat case) that

$$\Delta \omega - \operatorname{div}(v\omega) = 0.$$

Roughly, the gradient terms disappear, and the second term above coming from taking the curl of $Dv \cdot v$ can be handled by direct computation. (See Šverák's paper for a more natural geometric derivation).

In view of the previous discussion, it suffices to show that $\omega = 0$. The equation for ω doesn't have the maximum principle, but the adjoint operator $L^* = \Delta + v \cdot \nabla$ does. Thus, the kernel of L^* is one-dimensional, and L^* is surjective into the space normal to ω. Furthermore, by the maximum principle, the image of L^* contains no functions with a sign. If ω changes sign, then there is some positive function ψ perpendicular to ω, giving a contradiction. Thus, ω has a sign. We conclude using Stokes' theorem that $\omega = 0$, completing the proof.

6 Topics for Next Semester

Interesting topics for future talks include:

- Backwards uniqueness and applications to Navier-Stokes (Escauriaza-Seregin-Sverak)

- Liouville theorems for Navier-Stokes (Necas-Ruzicka-Sverak, Koch-Nadirashvili-Seregin-Sverak)

- Evans' Minimization Principle (based on comparison with an optimal space-time surface rather than just time slices)

- Equations with divergence-free drifts (Seregin-Silvestre-Sverak-Zlatos)

- Probabilistic approach (caloric functions as high-dimensional limits of harmonic functions, Terry Tao's blog)

- Tensor maximum principle, two-point maximum principle, applications to curvature flows (works of Hamilton, Brendle, and others)