SOME REMARKS ON "FINITE TIME BLOWUP FOR PARABOLIC SYSTEMS IN TWO DIMENSIONS"

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Abstract. In this note we show how to improve the target dimension from 4 to 3 for examples of singularity from smooth data for quasilinear parabolic systems in the plane. We also remark on the obstructions to finding such examples with target dimension 2.

1. Introduction

In [M] we considered regularity for weak solutions to the quasilinear parabolic system

$$\mathbf{u}_t = \operatorname{div}(a(\mathbf{u})D\mathbf{u}). \tag{1}$$

Here $\mathbf{u} : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}^m$, and $a = [a_{\alpha\beta}^{ij}(q)]_{\alpha,\,\beta\leq m}^{i,\,j\leq n}$ are smooth bounded coefficients on $\mathbb{R}^m$ satisfying the uniform ellipticity condition

$$\lambda|p|^2 \leq a_{\alpha\beta}^{ij}(q)p_i^\alpha p_j^\beta \leq \Lambda|p|^2 \tag{2}$$

for some positive constants λ, Λ, and for all $p \in M^{m\times n}$ and all $q \in \mathbb{R}^m$. We constructed examples of finite-time discontinuity from smooth data for (1) in the case $n = 2$, $m = 4$. The purpose of this note is to improve the target dimension to $m = 3$. It remains an interesting open problem whether such examples exist in the case $n = m = 2$. (If $a = a(x, t)$ then such examples exist, see [M]). We conclude the note with a short discussion of this problem.

2. The Example

By searching for self-similar solutions, we reduce the problem to finding a global bounded solution $\mathbf{U} : \mathbb{R}^2 \to \mathbb{R}^3$ to the elliptic system

$$\operatorname{div}(A(\mathbf{U})D\mathbf{U}) = \frac{1}{2}D\mathbf{U}\cdot x, \tag{3}$$

where $A = A_{\alpha\beta}^{ij}$ are smooth, uniformly elliptic coefficients on $\mathbb{R}^3$ and $\mathbf{U}$ satisfies

$$|D\mathbf{U}(x)| = O(|x|^{-1}), \quad |D\mathbf{U}\cdot x| = O(|x|^{-2})$$

(see [M], Section 2).

Let φ, f, h, R_0 and $E(r)$ be as in ([M], Section 3). Let ψ be a smooth increasing function that is 0 to the left of 0 and 1 to the right of 1, and let

$$\gamma(r) = \mu\psi\left(\frac{r - R_0}{R_0}\right)$$

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for $\mu > 0$ small to be chosen. Define

$$\mathbf{U} = \varphi(r)(\nu,\, 0) + \gamma(r)e_3,$$

where ν is the unit radial vector in $\mathbb{R}^2$. Finally, define coefficients

$$A_0 = \left(\begin{array}{cc|cc|cc} f & 0 & 0 & \eta & 0 & \alpha\sin(\theta) \\ 0 & h & -\eta & 0 & -\alpha\sin(\theta) & 0 \\ \hline 0 & -\eta & f & 0 & 0 & -\alpha\cos(\theta) \\ \eta & 0 & 0 & h & \alpha\cos(\theta) & 0 \\ \hline 0 & -\alpha\sin(\theta) & 0 & \alpha\cos(\theta) & 1 & 0 \\ \alpha\sin(\theta) & 0 & -\alpha\cos(\theta) & 0 & 0 & 1 \end{array}\right)$$

with respect to the coordinate system

$$(\nu, 0, 0),\ (\tau, 0, 0),\ (0, \nu, 0), ...,\ (0, 0, \tau),$$

where $\tau = \nu^{\perp}$ is the counter-clockwise rotation of ν by $\pi/2$, (v, w, z) denotes the 3×2 matrix with rows v, w and z, and $\eta(r)$, $\alpha(r)$ are to be chosen.

The system (3) (with coefficients $A = A_0(x)$) then becomes

$$\frac{\eta'\varphi - \alpha\gamma'}{r} = E(r), \quad \frac{(\alpha\varphi)'}{r} + \gamma'' + \frac{\gamma'}{r} = \frac{1}{2}r\gamma'.$$

Solving the second equation for α gives

$$\alpha = O(\mu R_0^2)\chi_{\{r \geq R_0\}}.$$

Solving the first equation for η and recalling the estimate (14) for E from ([M], Section 3) gives

$$\eta = O(\log R_0 + \mu^2 R_0^2)\chi_{\{r \geq R_0\}}.$$

The uniform ellipticity of A_0 when $\mu = 0$ follows from ([M], Section 4), so for $\mu > 0$ small, A_0 is smooth and uniformly elliptic.

It remains to show that $A_0(x)$ can be written as $A(\mathbf{U})$ for a uniformly elliptic, smooth matrix field A on $\mathbb{R}^3$. Let $\Gamma \subset \mathbb{R}^2$ be the image $(\varphi, \gamma)((0, \infty))$. Then Γ is a smooth embedded curve that looks like a "hook" provided $\gamma' > 0$ where φ takes its maximum, which is easily arranged (see Figure 1). The coefficients $A_{\alpha\beta}$ for $\alpha,\, \beta \leq 2$ can be defined as in ([M], Section 6), and A_{33} is constant. Note that the coefficients of the rotations defining $(A_0)_{13}$ and $(A_0)_{23}$ can be written $(\alpha/\varphi)U^2$ and $(\alpha/\varphi)U^1$, respectively. Since α vanishes for $r < R_0$ it is easy to extend the function G on Γ defined by $G(\varphi,\, \gamma) = \alpha/\varphi$ to a compactly supported smooth function on $\mathbb{R}^2$ that vanishes in a strip around the vertical axis, and then write the coefficients A_{13} and A_{23} as $G(|q'|,\, q_3)q^2$, resp. $G(|q'|,\, q_3)q^1$ times rotations. This completes the construction.

3. The case $m = 2$

An important feature of the above example is that $|\mathbf{U}|$ is not radially increasing. This feature is not just an artifact of the construction; in [M] we also show that if $\mathbf{U}$ is bounded, solves a uniformly elliptic system of the form $\text{div}(A_0(x)D\mathbf{U}) = \frac{1}{2}D\mathbf{U} \cdot x$ in $\mathbb{R}^2$, and $|\mathbf{U}|$ is radially increasing, then $\mathbf{U}$ is constant (see [M], Section 7). We briefly discuss how this is an obstruction to constructing solutions to (3) with target dimension $m = 2$.

To construct solutions to (3) our strategy is to find a solution $\mathbf{U}$ to a linear uniformly elliptic system with coefficients $A_0(x)$, which is a diffeomorphism onto its image $\mathbf{U}(\mathbb{R}^2)$. Then we can write the coefficients as functions of $\mathbf{U}$. The advantage

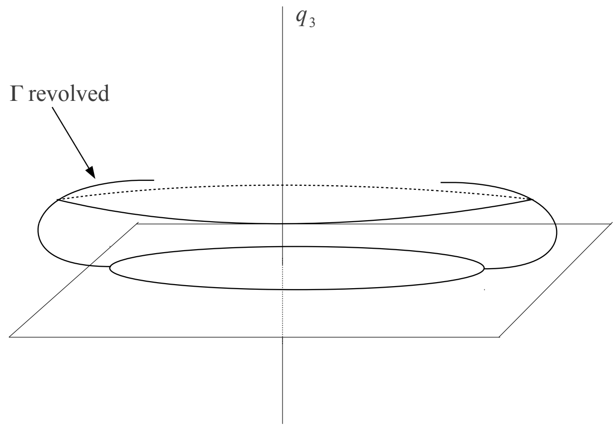

FIGURE 1. $\mathbf{U}$ is a diffeomorphism onto $\mathbf{U}(\mathbb{R}^2)$, which is the hook Γ revolved around the vertical axis

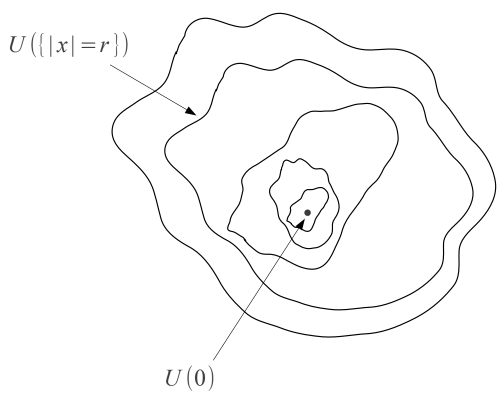

FIGURE 2. If $\mathbf{U}$ is a diffeomorphism, then the curves $\mathbf{U}(\partial B_r)$ form an "increasing" foliation of $\mathbf{U}(\mathbb{R}^2)$

of using target dimension $m = 3$ or higher is that we have room to "disperse" the values of $\mathbf{U}$, while allowing $|\mathbf{U}|$ to attain a radial maximum. In contrast, when $m = 2$, the images of the circles ∂B_r under a diffeomorphism form an "increasing" foliation of $\mathbb{R}^2$ (see Figure 2). In view of the Liouville theorem mentioned above, this gives hope that there is a positive result when the target is $\mathbb{R}^2$.

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References

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