SHARPENING THE TRIANGLE INEQUALITY: ENVELOPES BETWEEN L^2 AND L^p SPACES

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Abstract. Motivated by the inequality $\|f+g\|_2^2 \leq \|f\|_2^2 + 2\|fg\|_1 + \|g\|_2^2$, Carbery (2006) raised the question what is the "right" analogue of this estimate in L^p for $p \neq 2$. Carlen, Frank, Ivanisvili and Lieb (2018) recently obtained an L^p version of this inequality by providing upper bounds for $\|f+g\|_p^p$ in terms of the quantities $\|f\|_p^p, \|g\|_p^p$ and $\|fg\|_{p/2}^{p/2}$ when $p \in (0,1] \cup [2,\infty)$, and lower bounds when $p \in (-\infty, 0) \cup (1,2)$, thereby proving (and improving) the suggested possible inequalities of Carbery. We continue investigation in this direction by refining the estimates of Carlen, Frank, Ivanisvili and Lieb. We obtain upper bounds for $\|f+g\|_p^p$ also when $p \in (-\infty,0) \cup (1,2)$ and lower bounds when $p \in (0,1] \cup [2,\infty)$. For $p \in [1,2]$ we extend our upper bounds to any finite number of functions. In addition, we show that all our upper and lower bounds of $\|f+g\|_p^p$ for $p \in \mathbb{R}$, $p \neq 0$, are the best possible in terms of the quantities $\|f\|_p^p, \|g\|_p^p$ and $\|fg\|_{p/2}^{p/2}$, and we characterize the equality cases.

1. Introduction

For any real-valued functions $f, g \in L^p$ on an arbitrary measure space, and any $p \geq 1$, one has the inequality

$$\|f+g\|_p^p \leq 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right). \tag{1}$$

The estimate (1) follows from the fact that the map $x \mapsto |x|^p$ is convex. If $f = g$ in (1) then the constant 2^{p-1} is sharp and the inequality becomes equality. On the other hand, if f and g have disjoint supports then the constant 2^{p-1} is not needed. We remark that the estimate (1) reflects the convexity of the unit ball in L^p, which is equivalent to the usual L^p triangle (Minkowski) inequality (see e.g. [3]).

In [2], Carbery asked under what conditions on the sequence of functions $\{f_j\} \subset L^p$ the inequality $\sum \|f_j\|_p^p < \infty$ would imply $\sum f_j \in L^p$. If we try to adapt the inequality (1) to say n number of functions $f_1, f_2, \ldots, f_n$ instead of two, then the constant 2^{p-1} should be replaced by n^{p-1} which grows with n. To remove dependence on n Carbery suggested several extensions of inequality (1) which were motivated by the estimate $\|f+g\|_2^2 \leq \|f\|_2^2 + 2\|fg\|_1 + \|g\|_2^2$. All of them involve the extra parameter $\|fg\|_{p/2}^{p/2}$, which measures the "overlap" between the functions, and the strongest one in case of two functions he could prove only for indicator functions of sets. Recently a sharpened form of the triangle inequality was obtained [3] which implied the proposed estimates of Carbery's. Namely, take any $p \in \mathbb{R} \setminus \{0\}$, and

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put

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p} \quad \text{and} \quad \Gamma_p := \frac{2\|fg\|_{p/2}^{p/2}}{\|f\|_p^p + \|g\|_p^p}.$$

Then

$$\|f+g\|_p^p \leq \left(1+\Gamma_p^{2/p}\right)^{p-1} (\|f\|_p^p + \|g\|_p^p) \tag{2}$$

holds true if $p \in (0,1] \cup [2,\infty)$, and the inequality reverses if $p \in (-\infty,0) \cup (1,2)$, where in the latter case we assume that f, g are positive almost everywhere. Since by Cauchy–Schwarz $\Gamma_p \in [0,1]$ for all $p \in \mathbb{R} \setminus \{0\}$ we see that (2) improves on the trivial bound (1).

In this paper we continue investigation in this direction and we address the following questions:

1. Can one further sharpen the right hand side of the estimate (2) if we are allowed to use only the quantities $\|f\|_p, \|g\|_p, \|fg\|_{p/2}$?
2. What is the optimal upper bound on $\|f+g\|_p^p$ in terms of the quantities $\|f\|_p, \|g\|_p, \|fg\|_{p/2}$, also when $p \in (-\infty,0) \cup (1,2)$? The same question about lower bounds on $\|f+g\|_p^p$, also when $p \in (0,1] \cup [2,\infty)$.
3. Can one extend these estimates to many functions, more than 2?

We will give complete answers to Questions 1 and 2, and we will provide an answer to Question 3 when $p > 0$. In particular we show that for $p \in [1,2]$, if $\sum_j \|f_j\|_p^p < \infty$ and $\sum_{i<j} \|f_i f_j\|_{p/2}^{p/2} < \infty$, then $\sum_j f_j \in L^p$.

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2. Main results

Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space. In what follows we consider functions f, g on X that are measurable and nonnegative. Given $p \in \mathbb{R} \setminus \{0\}$ we will be always assuming that $\|f\|_p^p$, $\|g\|_p^p < \infty$. When $p < 0$ we allow f, g to take the value $+\infty$, where we understand f^p, $g^p = 0$.

Theorem 2.1. *For any $p \in (0,1] \cup [2,\infty)$, and any nonnegative f, g on any measure space we have*

$$\|f+g\|_p^p \leq \left(\left(\frac{1+\sqrt{1-\Gamma_p^2}}{2}\right)^{1/p} + \left(\frac{1-\sqrt{1-\Gamma_p^2}}{2}\right)^{1/p}\right)^p (\|f\|_p^p + \|g\|_p^p). \tag{3}$$

The inequality reverses if $p \in (-\infty,0) \cup [1,2]$. Equality holds if $(fg)^{p/2} = k(f^p + g^p)$ for some constant $k \in \left[0, \frac{1}{2}\right]$.

Remark 2.2. The right hand side of (3) is the best possible in the following sense: consider the measure space $([0,1], \mathcal{B}, dx)$. Pick any nonnegative numbers x, y and z such that $0 \leq z \leq \sqrt{xy}$. Then, for any $p \in (0,1] \cup [2,\infty)$ the supremum of the left hand side of (3) over all nonnegative f, g with fixed $\|f\|_p^p = x$, $\|g\|_p^p = y$, $\|fg\|_{p/2}^{p/2} = z$

coincides with the right hand side of (3). Similarly, for any $p \in (-\infty, 0) \cup [1, 2]$ the infimum of the left hand side of (3) over all such f, g coincides with the right hand side of (3). We justify this remark in Section 3.

Remark 2.2 implies in particular that Theorem 2.1 refines the estimate (2). As a consequence we have the following peculiar estimate:

Corollary 2.3. *For any $p \in (0, 1] \cup [2, \infty)$, and any number $\Gamma \in [0, 1]$ we have*

$$\left(\left(\frac{1+\sqrt{1-\Gamma^2}}{2}\right)^{1/p} + \left(\frac{1-\sqrt{1-\Gamma^2}}{2}\right)^{1/p}\right)^p \leq \left(1+\Gamma^{2/p}\right)^{p-1}. \tag{4}$$

The inequality reverses if $p \in (-\infty, 0) \cup [1, 2]$.

If we set $\Gamma := 2\frac{(ab)^{p/2}}{a^p+b^p}$ for nonnegative a, b, then after a short computation inequality (4) becomes

$$\frac{(a+b)^p}{a^p+b^p} \leq \left(1+\left(2\frac{(ab)^{p/2}}{a^p+b^p}\right)^{2/p}\right)^{p-1}. \tag{5}$$

This estimate was previously obtained in [3] (where it was also shown to be equivalent to the inequality (2)), and the arguments are quite involved.

Remark 2.4. If we let $q := 1/p$ and $x = \sqrt{1-\Gamma^2}$, then inequality (4) can also be written as the following two-point-type inequality:

$$\frac{(1+x)^q + (1-x)^q}{2} \leq \left(\frac{1+(1-x^2)^q}{2}\right)^{1-q} \tag{6}$$

for all $q \in \left(-\infty, \frac{1}{2}\right] \cup [1, \infty)$, $x \in [0, 1]$, and the inequality reverses if $q \in \left[\frac{1}{2}, 1\right)$. This inequality is reminiscent of Bonami's two-point inequality

$$\left(\frac{\left|y+\sqrt{\frac{p-1}{q-1}}u\right|^q + \left|y-\sqrt{\frac{p-1}{q-1}}u\right|^q}{2}\right)^{\frac{1}{q}} \leq \left(\frac{|y+u|^p + |y-u|^p}{2}\right)^{\frac{1}{p}}, \tag{7}$$

which holds true for all y, $u \in \mathbb{R}$ and $1 \leq p \leq q < \infty$ (see [1]). Indeed, if we take $y = 1$, $p = 2$, and $u = x\sqrt{q-1}$ then we get

$$\frac{|1+x|^q + |1-x|^q}{2} \leq (1+(q-1)x^2)^{q/2}. \tag{8}$$

The right sides of inequality (6) and (8) are not comparable. For example, when $x = 1$ the estimate (6) gives better upper bounds for $q > 2$, while near $x = 0$ it gives worse upper bounds.

Next, let $p \in \mathbb{R} \setminus \{0\}$, and set[1]

$$C_p := \frac{\min\{\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}\}}{\|fg\|_{p/2}^{p/2}}.$$

[1]If $\|fg\|_{p/2}^{p/2} = 0$ then we set $C_p = 1$.

Theorem 2.5. *For any $p \in (1,\, 2)$ and any nonnegative f, g on any measure space we have*

$$\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p + \Big((C_p^{-1/p} + C_p^{1/p})^p - C_p^{-1} - C_p\Big)\, \|fg\|_{p/2}^{p/2}. \tag{9}$$

The inequality reverses if $p \in (0,1] \cup [2,\infty)$. Equality holds in (9) if one of the following three conditions holds: $f = g$ on $\{fg > 0\}$, $g = \lambda f$ on $\{f > 0\}$ for some $\lambda \geq 1$, or $f = \lambda g$ on $\{g > 0\}$ for some $\lambda \geq 1$.

For $p \in (-\infty,\, 0)$ we have

$$\|f+g\|_p^p \leq \left(C_p^{-1/p} + C_p^{1/p}\right)^p \|fg\|_{p/2}^{p/2}. \tag{10}$$

Equality holds in (10) if one of the following three conditions holds: $f = g$ on $\{fg < \infty\}$, $g = \lambda f$ on $\{f < \infty\}$ for some $\lambda \leq 1$, or $f = \lambda g$ on $\{g < \infty\}$ for some $\lambda \leq 1$.

Exactly the same remark as before applies to Theorem 2.5; that is, the right hand sides of (9) and (10) are the best possible. Together, Theorems 2.1 and 2.5, along with the remarks about optimality, answer Questions 1 and 2.

Finally, we state a partial answer to Question 3 in the case $p > 0$.

Corollary 2.6. *For any $p \in [1,2]$, and any sequence of nonnegative functions $\{f_j\}_{j\geq 1}$ we have*

$$\|\sum_j f_j\|_p^p \leq \sum_j \|f_j\|_p^p + (2^p - 2)\sum_{i<j} \|f_i f_j\|_{p/2}^{p/2}.$$

If $p \in (0,1] \cup [2,\infty)$ the inequality reverses. Equality holds if and only if

$$\left(\sum_j f_j\right)^p = \sum_j f_j^p + (2^p - 2)\sum_{i<j} (f_j f_j)^{p/2}$$

almost everywhere.

In particular, when $p \in [1,\, 2]$ we have that $\sum_j f_j \in L^p$ provided $\sum_j \|f_j\|_p^p < \infty$ and $\displaystyle\sum_{i<j} \|f_j f_j\|_{p/2}^{p/2} < \infty$.

Remark 2.7. After we finished writing this paper we received the preprint [4], in which the authors obtain an upper bound for the L^p norm of a sum of N functions in the case $p \geq 2$, in terms of a certain analogue for N functions of the quantity Γ_p. Their estimate complements our result Corollary 2.6, which holds for $p \in (1,\, 2)$, and is obtained using different techniques.

The rest of the paper is organized as follows. In Section 3 we reduce the proofs of Theorems 2.1 and 2.5, as well as the remarks about their optimality, to computing the concave and convex envelopes of a certain function defined on the boundary of a convex cone in $\mathbb{R}^3$. In Section 4 we compute these envelopes. Finally, in Section 5 we prove Corollary 2.6 using an observation about the proof of Theorem 2.5.

3. Reductions

In this section we reduce Theorems 2.1 and 2.5 to computing explicitly the convex and concave envelopes of a certain function defined on the boundary of a convex cone in $\mathbb{R}^3$. Let

$$\Omega := \{x,\, y \geq 0,\, 0 \leq z \leq \sqrt{xy}\}$$

be the convex cone in $\mathbb{R}^3$ whose vertical cross-sections $\Omega \cap \{x + y = c > 0\}$ are half-ellipses. For $p \in \mathbb{R}\backslash\{0\}$ define φ_p on $\partial\Omega$ by

$$\varphi_p(x,\, y,\, \sqrt{xy}) = (x^{1/p} + y^{1/p})^p,\; x,\, y > 0, \quad \varphi_p(x,\, y,\, 0) = \begin{cases} x + y, & p > 0 \\ 0, & p < 0. \end{cases}$$

Let f and g be nonnegative functions on an arbitrary measure space $(X,\, \mathcal{A},\, \mu)$ with $\|f\|_p^p,\, \|g\|_p^p < \infty$. Note that the triple $(\|f\|_p^p,\, \|g\|_p^p,\, \|fg\|_{p/2}^{p/2}) \in \Omega$ by the Cauchy-Schwarz inequality. By the equality case, if the triple is in $\partial\Omega$ we have $\|f+g\|_p^p = \varphi_p(\|f\|_p^p,\, \|g\|_p^p,\, \|fg\|_{p/2}^{p/2})$. Our approach is based on the following lemma:

Lemma 3.1. *Let $p \in \mathbb{R}\backslash\{0\}$, and assume that $H \in C(\Omega)$ is a concave, one-homogeneous function on Ω with $H|_{\partial\Omega} = \varphi_p$. Then*

$$\|f + g\|_p^p \leq H\left(\|f\|_p^p,\, \|g\|_p^p,\, \|fg\|_{p/2}^{p/2}\right).$$

If H is convex, the inequality reverses.

Proof. By the boundary conditions, we have

$$1 = H\left(\frac{f^p}{(f+g)^p},\, \frac{g^p}{(f+g)^p},\, \frac{(fg)^{p/2}}{(f+g)^p}\right)$$

on the set $X' = \{f + g > 0\}$ when $p > 0$, or $\{f + g < \infty\}$ when $p < 0$. Integrating this identity with respect to the probability measure $\frac{(f+g)^p\, d\mu}{\|f+g\|_p^p}$ on X' and applying Jensen's inequality gives

$$1 \leq H\left(\frac{\|f\|_p^p}{\|f+g\|_p^p},\, \frac{\|g\|_p^p}{\|f+g\|_p^p},\, \frac{\|fg\|_{p/2}^{p/2}}{\|f+g\|_p^p}\right)$$

when H is concave, and the other inequality for H convex. The result follows from the one-homogeneity of H. $\square$

Lemma 3.1 reduces our problem to computing the concave and convex envelopes of φ_p on Ω. By concave envelope we mean the infimum of linear functions on Ω that are greater than φ_p on $\partial\Omega$, and by convex envelope the supremum of linear functions on Ω that are smaller than φ_p on $\partial\Omega$. Let $\overline{H}_p$ denote the concave envelope, and $\underline{H}_p$ the convex envelope. For $(x,\, y,\, z) \in \Omega$, define

$$w(x,\, y,\, z) := \frac{2z}{x + y}, \quad v(x,\, y,\, z) := \min\left\{\frac{x}{z},\, \frac{y}{z},\, 1\right\},$$

where we take $w = 0$ at the origin and $v = 1$ on $\Omega \cap \{z = 0\}$. Define the one-homogeneous functions F_p, G_p on Ω by

$$F_p(x,\, y,\, z) := \frac{x+y}{2}\left((1+\sqrt{1-w^2})^{1/p} + (1-\sqrt{1-w^2})^{1/p}\right)^p, \tag{11}$$

$$G_p(x,\, y,\, z) := \begin{cases} x + y + \left((v^{1/p} + v^{-1/p})^p - (v + v^{-1})\right) z, & p > 0 \\ (v^{1/p} + v^{-1/p})^p\, z, & p < 0. \end{cases} \tag{12}$$

Proposition 3.2. *The concave and convex envelopes $\overline{H}_p$, $\underline{H}_p$ of φ_p in Ω are in $C(\Omega)$ and are given explicitly by the formulae*

$$\overline{H}_p = \begin{cases} F_p, & p \in (0,\, 1] \cup [2,\, \infty), \\ G_p, & p \in (-\infty,\, 0) \cup (1,\, 2) \end{cases}$$

and

$$\underline{H}_p = \begin{cases} F_p, & p \in (-\infty,\, 0) \cup (1,\, 2), \\ G_p, & p \in (0,\, 1] \cup [2,\, \infty). \end{cases}$$

We delay the proof of Proposition 3.2 to Section 4, and immediately note that Theorems 2.1 and 2.5 follow quickly:

Proof of Theorems 2.1 and 2.5: To prove the inequalities, just apply Lemma 3.1 to the functions $\overline{H}_p$ and $\underline{H}_p$. To check the equality cases, observe that in the proof of Lemma 3.1, we have equality in Jensen provided $\{(f^p,\, g^p,\, (fg)^{p/2})\}$ lie in a set where H is linear.

Since F_p is linear when restricted to the hyperplanes $\{z = k(x+y)\} \cap \Omega$ (which are nontrivial when $k \in [0,\, 1/2]$) we obtain the equality case in Theorem 2.1.

We note that G_p is linear on the triangular cone $\{z \leq \min\{x,\, y\}\} \cap \Omega$, and on the hyperplanes $\{z = \gamma x\} \cap \Omega$ and $\{z = \gamma y\} \cap \Omega$ for each $\gamma \geq 1$. The first condition gives $(fg)^{p/2} \leq \min\{f^p,\, g^p\}$, so $f = g$ on $\{fg > 0\}$ in the case $p > 0$ and on $\{fg < \infty\}$ in the case $p < 0$. The second condition gives $(fg)^{p/2} = \gamma f^p$, and the third $(fg)^{p/2} = \gamma g^p$. When $p > 0$ the second condition gives that $g = \lambda f$ on $\{f > 0\}$ for some $\lambda \geq 1$, and the third gives that $f = \lambda g$ on $\{g > 0\}$ for some $\lambda \geq 1$; when $p < 0$ the second condition gives $g = \lambda f$ on $\{f < \infty\}$ for some $\lambda \leq 1$, and the third gives that $f = \lambda g$ on $\{g < \infty\}$ for some $\lambda \leq 1$. □

To conclude the section we address the optimality of Theorems 2.1 and 2.5 in the measure space $(X,\, \mathcal{A},\, \mu) = ([0,\, 1],\, \mathcal{B},\, dx)$. We define

$$\overline{B}_p(x,\, y,\, z) = \sup\left\{\|f+g\|_p^p : (\|f\|_p^p,\, \|g\|_p^p,\, \|fg\|_{p/2}^{p/2}) = (x,\, y,\, z)\right\},$$

$$\underline{B}_p(x,\, y,\, z) = \inf\left\{\|f+g\|_p^p : (\|f\|_p^p,\, \|g\|_p^p,\, \|fg\|_{p/2}^{p/2}) = (x,\, y,\, z)\right\}.$$

It is easy to see that $\overline{B}_p$, $\underline{B}_p$ are defined on a cone $\Omega_p \subset \Omega$, are locally bounded by the inequalities $(f+g)^p \leq 2^{p-1}(f^p + g^p)$ for $p \in (-\infty,\, 0) \cup [1,\, \infty)$ and $(f+g)^p \leq f^p + g^p$ for $p \in (0,\, 1]$, are one-homogeneous, and equal φ_p on $\partial\Omega$ (by the equality case of Cauchy-Schwarz). Furthermore, by Lemma 3.1 we have

$$\underline{H}_p \leq \underline{B}_p \leq \overline{B}_p \leq \overline{H}_p$$

on the common domain of definition.

Lemma 3.3. *If $\overline{B}_p$ ($\underline{B}_p$) is defined on all of Ω and is concave (convex), then*

$$\overline{H}_p = \overline{B}_p \quad (\underline{B}_p = \underline{H}_p).$$

Proof. Local boundedness and concavity of $\overline{B}_p$ implies continuity in the interior of Ω, and since $\overline{B}_p$ is trapped between envelopes that attain the data continuously, we have $\overline{B}_p \in C(\Omega)$. Since $\overline{H}_p$ is the smallest such concave function, we conclude that $\overline{B}_p \geq \overline{H}_p$. The argument is similar for $\underline{B}_p$. □

Thus, it just remains to show that when $(X, \mathcal{A}, \mu) = ([0, 1], \mathcal{B}, dx)$, the domain of definition for $\overline{B}_p$ and $\underline{B}_p$ is all of Ω, and that $\overline{B}_p$ is concave and $\underline{B}_p$ is convex.

Lemma 3.4. *For $(X, \mathcal{A}, \mu) = ([0, 1], \mathcal{B}, dx)$ we have $\Omega_p = \Omega$ for all $p \neq 0$, that $\overline{B}_p$ is concave in Ω, and that $\underline{B}_p$ is convex in Ω.*

The optimality of the inequalities in Theorems 2.1 and 2.5 follows:

Proof of Optimality Statements: For either inequality, given

$$(x, y, z) = (\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}),$$

the functions $\overline{B}_p(x, y, z)$ and $\underline{B}_p(x, y, z)$ are by definition the best we can do. These are equal to the envelopes $\overline{H}_p$, $\underline{H}_p$ by Lemmas 3.3 and 3.4. □

Remark 3.5. For given $(x, y, z) \in \Omega$ and $p \in \mathbb{R}\backslash\{0\}$, the supremum (infimum) in the definition of $\overline{B}_p$ ($\underline{B}_p$) is in fact attained.

For equality in (3) consider pairs of the form $(f, g) = (a, b)\chi_{[0, c]} + (b, a)\chi_{[c, 1]}$ for a, b, c chosen appropriately.

For equality in (9), consider pairs of the form

$$(f, g) = (a, a)\chi_{[0, 1/2]} + (b, 0)\chi_{[1/2, 3/4]} + (0, c)\chi_{[3/4, 1]}$$

for a, b, c appropriately chosen when $z \leq \min\{x, y\}$, and $(f, g) = (a, b)\chi_{[0, 1/2]} + (c, d)\chi_{[1/2, 1]}$ when $z > \min\{x, y\}$ for appropriate a, b, c, d, with one of c, d equal to 0.

For equality in (10), consider pairs of the form

$$(f, g) = (a, a)\chi_{[0, 1/2]} + (b, \infty)\chi_{[1/2, 3/4]} + (\infty, c)\chi_{[3/4, 1]}$$

for a, b, c appropriately chosen when $z \leq \min\{x, y\}$, and $(f, g) = (a, b)\chi_{[0, 1/2]} + (c, d)\chi_{[1/2, 1]}$ when $z > \min\{x, y\}$ for appropriate a, b, c, d, with one of c, d equal to ∞.

Proof of Lemma 3.4: For the first part, if $p > 0$ take $f_s = (2x)^{1/p}\chi_{[s, s+1/2]}$ for $s \in [0, 1/2]$ and let $g = (2y)^{1/p}\chi_{[1/2, 1]}$. Then $\|f_s\|_p^p = x$ and $\|g\|_p^p = y$. Furthermore, we have $h(s) := \|f_s g\|_{p/2}^{p/2}$ is continuous, increasing, and $h(0) = 0$, $h(1/2) = \sqrt{xy}$. When $p < 0$, use the same example but set f_s, $g = \infty$ where they were previously zero.

For the second part, let $(x_i, y_i, z_i) \in \Omega$ with $i = 1, 2$, and for $\epsilon > 0$ choose f_i, g_i such that $(x_i, y_i, z_i) = (\|f_i\|_p^p, \|g_i\|_p^p, \|f_i g_i\|_{p/2}^{p/2})$ and

$$\|f_i + g_i\|_p^p \geq \overline{B}_p(x_i, y_i, z_i) - \epsilon, \quad i = 1, 2.$$

Extend f_i, g_i to be zero outside of $[0, 1]$, and define the rescalings

$$\tilde{f}_1(s) = 2^{1/p} f_1(2s),\ \tilde{g}_1(s) = 2^{1/p} g_1(2s),\ \tilde{f}_2(s) = 2^{1/p} f_2(2s-1),\ \tilde{g}_2(s) = 2^{1/p} g_2(2s-1),$$

so that $\tilde{f}_i$, $\tilde{g}_i$ are supported in $[0,\, 1/2]$ for $i = 1$ and in $[1/2,\, 1]$ for $i = 2$. We then have

$$\begin{aligned}\frac{\overline{B}_p(x_1,\, y_1,\, z_1) + \overline{B}_p(x_2,\, y_2,\, z_2)}{2} - \epsilon &\le \frac{1}{2}\left(\|\tilde{f}_1 + \tilde{g}_1\|^p_{L^p([0,\,1/2])} + \|\tilde{f}_2 + \tilde{g}_2\|^p_{L^p([1/2,\,1])}\right)\\ &= \frac{1}{2}\|\tilde{f}_1 + \tilde{g}_1 + \tilde{f}_2 + \tilde{g}_2\|^p_p\\ &= \left\|\frac{\tilde{f}_1 + \tilde{f}_2}{2^{1/p}} + \frac{\tilde{g}_1 + \tilde{g}_2}{2^{1/p}}\right\|^p_p\\ &\le \overline{B}_p\left(\frac{1}{2}(x_1 + x_2,\, y_1 + y_2,\, z_1 + z_2)\right).\end{aligned}$$

For the last inequality, we used that for $f_0 := 2^{-1/p}(\tilde{f}_1 + \tilde{f}_2)$, $g_0 := 2^{-1/p}(\tilde{g}_1 + \tilde{g}_2)$ we have

$$\|f_0\|^p_p = \frac{1}{2}(x_1 + x_2),\ \|g_0\|^p_p = \frac{1}{2}(y_1 + y_2),\ \|f_0 g_0\|^{p/2}_{p/2} = \frac{1}{2}(z_1 + z_2).$$

Taking $\epsilon \to 0$, we conclude that $\overline{B}_p$ is concave. The convex direction is similar. □

Remark 3.6. Lemma 3.4 holds for any measure space with translation and scaling properties similar to $([0,\, 1],\, \mathcal{B},\, dx)$, e.g. $(B_1 \subset \mathbb{R}^n,\, \mathcal{B},\, dx)$.

Remark 3.7. The fact that $\overline{B}_p$ is concave also follows from Theorem 1 in [5]. Since the argument is simple, we decided to include it for the reader's convenience.

4. Envelopes

In this section we prove Proposition 3.2. We begin with some simple observations.

First, to check concavity (convexity) in Ω and continuity up to $\partial\Omega$ of $\overline{H}_p$ ($\underline{H}_p$), by one-homogeneity it suffices to check these properties on the half-ellipse

$$D := \Omega \cap \{x + y = 2\}.$$

More generally, any one-homogeneous function B in a convex cone in $\mathbb{R}^n$ (say contained in $\{x_n > 0\}$) is concave (convex) if it is concave (convex) when restricted to a cross-section of the cone (say $\{x_n = 1\}$). Indeed, by one-homogeneity we have

$$B\left(\frac{x+y}{2}\right) = \frac{x_n + y_n}{2} B\left(\lambda \frac{x}{x_n} + (1-\lambda)\frac{y}{y_n}\right)$$

where $\lambda = \frac{x_n}{x_n + y_n}$, and the statement follows by applying concavity / convexity of B on the cross-section and then using one-homogeneity once more.

Second, to prove that $\overline{H}_p$ ($\underline{H}_p$) is the concave (convex) envelope of φ_p, it suffices to check that each point in the interior of D lies on a segment that connects boundary points of D, on which $\overline{H}_p$ ($\underline{H}_p$) is linear. Indeed, then any linear function larger (smaller) than φ_p on $\partial\Omega$ will then be larger than $\overline{H}_p$ (smaller than $\underline{H}_p$) in the interior of Ω.

Proof of Proposition 3.2. We first examine F_p, and then G_p.

The Function F_p. On D we can write $F_p(1 + s,\, 1 - s,\, t) = u(t)$, where

$$u(t) := \left[(1 + \sqrt{1 - t^2})^{1/p} + (1 - \sqrt{1 - t^2})^{1/p}\right]^p, \quad t \in [0, 1].$$

It is clear that F_p is continuous up to ∂D for each $p \in \mathbb{R}\backslash\{0\}$, and $u(0) = \varphi_p$ (that is, 2 if $p > 0$ and 0 if $p < 0$) on the bottom of D and $F_p(1-s,\, 1+s,\, \sqrt{1-s^2}) = ((1+s)^{1/p} + (1-s)^{1/p})^p = \varphi_p$ on the top of D. Since F_p is constant along the horizontal segments in D, it suffices to check that u is concave when $p \in (0,\, 1] \cup [2,\, \infty)$, and convex otherwise. To that end, we let $t = \sin(x)$, with $x \in [0, \pi/2]$. Then

$$u(\sin(x)) = \left[(1+\cos(x))^{1/p} + (1-\cos(x))^{1/p}\right]^p .$$

Let us rewrite the last equality as

$$\frac{1}{2}u(\sin(2s)) = \left[\sin^{2/p}(s) + \cos^{2/p}(s)\right]^p ,$$

where $s = x/2 \in [0,\, \pi/4]$. Differentiating both sides of the equality in s, we obtain

$$\begin{aligned}
&u'(\sin(2s))\cos(2s)\\
&= p\left[\sin^{2/p}(s) + \cos^{2/p}(s)\right]^{p-1} \frac{2}{p}\left(\sin^{2/p-1}(s)\cos(s) - \cos^{2/p-1}(s)\sin(s)\right)\\
&= p\left[\sin^{2/p}(s) + \cos^{2/p}(s)\right]^{p-1} \frac{2\cos^{2/p}(s)}{p}\left(\tan^{2/p-1}(s) - \tan(s)\right).
\end{aligned}$$

Taking the derivative a second time we obtain

$$\begin{aligned}
&2u''(\sin(2s))\cos^2(2s) - 2u'(\sin(2s))\sin(2s) =\\
&p(p-1)\left[\sin^{2/p}(s) + \cos^{2/p}(s)\right]^{p-2}\left[\frac{2\cos^{2/p}(s)}{p}\left(\tan^{2/p-1}(s) - \tan(s)\right)\right]^2 +\\
&p\left[\sin^{2/p}(s) + \cos^{2/p}(s)\right]^{p-1} \times \left(-\frac{4\cos^{2/p}(s)\tan(s)}{p^2}\left(\tan^{2/p-1}(s) - \tan(s)\right) +\right.\\
&\left.\frac{2\cos^{2/p}(s)}{p}\left(\left(\frac{2}{p} - 1\right)\tan^{2/p-2}(s) - 1\right)\left(1 + \tan^2(s)\right)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&2u''(\sin(2s))\cos^2(2s) = \left[\sin^{2/p}(s) + \cos^{2/p}(s)\right]^{p-2} \times \frac{4}{p} \times \cos^{4/p}(s) \times\\
&\left[(p-1)\left[\left(\tan^{2/p-1}(s) - \tan(s)\right)\right]^2 + \left[1 + \tan^{2/p}(s)\right] \times\right.\\
&\left(-\tan(s)\left(\tan^{2/p-1}(s) - \tan(s)\right) + \left(\left(1 - \frac{p}{2}\right)\tan^{2/p-2}(s) - \frac{p}{2}\right)\left(1 + \tan^2(s)\right)\right) +\\
&\left. p\tan(2s)\left[1 + \tan^{2/p}(s)\right]\left(\tan^{2/p-1}(s) - \tan(s)\right)\right].
\end{aligned}$$

Since $\tan(2s) = \frac{2\tan(s)}{1-\tan^2(s)}$, after denoting $\tan(s) = w \in [0, 1]$ we obtain

$$\begin{aligned}
&\frac{2u''(\sin(2s))\cos^2(2s)}{\left[\sin^{2/p}(s)+\cos^{2/p}(s)\right]^{p-2}\cos^{4/p}(s)} \\
&= \frac{4(p-1)}{p}\left(w^{2/p-1}-w\right)^2 \\
&+ \frac{4(1+w^{2/p})}{p}\times\left(-w^{2/p}+w^2+\left(\left(1-\frac{p}{2}\right)w^{2/p-2}-\frac{p}{2}\right)(1+w^2)\right) \\
&+ \frac{8w}{1-w^2}(1+w^{2/p})(w^{2/p-1}-w) \\
&= \frac{2(1+w^2)^2}{1-w^2}\left(w^{\frac{4}{p}-2}+\left(\frac{2}{p}-1\right)w^{\frac{2}{p}-2}(1-w^2)-1\right).
\end{aligned}$$

(The last equality is a tedious computation, but can be checked by hand). Since $\frac{2(1+w^2)^2}{1-w^2}>0$ we see after denoting $x:=w^2\in[0,1]$ that $\mathrm{sgn}(u'')=\mathrm{sgn}(v(x))$, where

$$v(x)=x^{\frac{2}{p}-1}+\left(\frac{2}{p}-1\right)x^{\frac{1}{p}-1}(1-x)-1,\quad x\in[0,1].$$

Let us study the sign of $v(x)$. Without loss of generality assume that $p\neq 1,2$, otherwise the claims about concavity/convexity of u are trivial. First notice that $v(1)=0$, and

$$v'(x)=x^{\frac{1}{p}-2}\left(\frac{2}{p}-1\right)\left(x^{\frac{1}{p}}-\left(1+\frac{1}{p}(x-1)\right)\right).$$

Therefore, if $p\in(2,\infty)$ it follows from concavity of $x\mapsto x^{1/p}$ that $v'\geq 0$, and hence $v\leq 0$, i.e., u is concave. Similarly, if $p\in(1,2)$, then $v\geq 0$, i.e., u is convex. Next, if $p\in(0,1)$ then $x\mapsto x^{1/p}$ is convex, and hence $v'\geq 0$, i.e., u is concave. Finally, if $p\in(-\infty,0)$ then $x\mapsto x^{1/p}$ is convex, and therefore $v'\leq 0$, i.e., u is convex.

The Function G_p. Let $b_p(s,\,z)=G_p(1+s,\,1-s,\,z)$, with $(s,\,z)$ in the upper half-disc. For $p>0$ we can write b_p explicitly as

$$b_p(s,\,z)=2+\begin{cases} w(1-|s|,\,z), & z\geq 1-|s| \\ (2^p-2)z, & z<1-|s|, \end{cases}$$

where w is the one-homogeneous function given by

$$w(t,\,z):=\left(t^{1/p}+(z^2/t)^{1/p}\right)^p-(t+(z^2/t))$$

with $(t,\,z)\in(0,\,1)^2$. It is easy to check that b_p continuously takes the boundary values $b_p(s,\,0)=2=\varphi_p$ and $b_p(s,\,\sqrt{1-s^2})=((1+s)^{1/p}+(1-s)^{1/p})^p=\varphi_p$. Let

$$h(t):=w(t,\,1)=\left(t^{1/p}+t^{-1/p}\right)^p-(t+t^{-1}),\quad t\in(0,\,1).$$

By the one-homogeneity of w and the fact that b_p is linear on the triangle $\{z<1-|s|\}$ with vertical gradient, if we show that $h'(1)=0$ and that h is concave / convex on $[0,\,1]$, then b_p is C^1 away from $(s,\,z)=(\pm1,\,0)$ and concave / convex. Furthermore, b_p is linear when restricted to the segments through $(s,\,z)=(\pm1,\,0)$ that lie outside of the triangle $\{z\leq 1-|s|\}$, so G_p is the concave / convex envelope provided the above conditions on h are confirmed. To that end we compute the first two derivatives of h. The first derivative is

$$h'(t) = (t^{1/p} + t^{-1/p})^{p-1}(t^{1/p-1} - t^{-1/p-1}) - (1 - t^{-2}).$$

This confirms that $h'(1) = 0$. The second derivative is

$$\begin{aligned} h''(t) &= \frac{p-1}{p}(t^{1/p} + t^{-1/p})^{p-2}(t^{1/p-1} - t^{-1/p-1})^2 \\ &+ \frac{1}{p}(t^{1/p} + t^{-1/p})^{p-1}((1-p)t^{1/p-2} + (1+p)t^{-1/p-2}) - 2t^{-3} \\ &= \frac{1}{p}(t^{1/p} + t^{-1/p})^{p-2}[(p-1)(t^{1/p-1} - t^{-1/p-1})^2 \\ &+ (t^{1/p} + t^{-1/p})((1-p)t^{1/p-2} + (1+p)t^{-1/p-2})] - 2t^{-3} \\ &= \frac{2}{p}(t^{1/p} + t^{-1/p})^{p-2}[pt^{-2/p-2} + (2-p)t^{-2}] - 2t^{-3} \\ &= 2t^{-3}[(t^{1/p} + t^{-1/p})^{p-2}(t^{1-2/p} + (2/p-1)t) - 1] \\ &= 2t^{-3}[(1 + t^{2/p})^{p-2}(1 + (2/p-1)t^{2/p}) - 1]. \end{aligned}$$

Let $x := t^{2/p} \in [0, 1]$. It suffices to show that

$$g_p(x) := (1 + (2/p - 1)x) - (1 + x)^{2-p}$$

satisfies $g_p \leq 0$ on $[0, 1]$ for $p \in (1, 2)$ and $g_p \geq 0$ on $[0, 1]$ for $p \in (0, 1] \cup [2, \infty)$. Note that $g_p(0) = 0$. The desired inequality for $g_p(1)$ is equivalent to the fact that the linear function p crosses the convex function 2^{p-1} at $p = 1$ and $p = 2$. Finally, we observe that the first term in g_p is linear, and the second term is convex for $p \in (1, 2)$ and concave for $p \in (0, 1) \cup (2, \infty)$. The desired inequality for $g_p(x)$ with $x \in (0, 1)$ follows immediately from this observation and the inequalities at the endpoints $x = 0$ and $x = 1$.

When $p < 0$ we can write b_p explicitly as

$$b_p(s, z) = \begin{cases} \tilde{w}(1 - |s|, z), & z \geq 1 - |s| \\ 2^p z, & z < 1 - |s|, \end{cases}$$

where $\tilde{w}$ is the one-homogeneous function given by

$$\tilde{w}(t, z) := \left(t^{1/p} + (z^2/t)^{1/p}\right)^p$$

with $(t, z) \in (0, 1)^2$. The same considerations as above reduce the problem to showing that

$$\tilde{h}(t) := \tilde{w}(t, 1) = \left(t^{1/p} + t^{-1/p}\right)^p$$

satisfies $\tilde{h}'(1) = 0$ and $\tilde{h}$ is concave on $[0, 1]$. We have

$$\tilde{h}' = (t^{1/p} + t^{-1/p})^{p-1}(t^{1/p-1} - t^{-1/p-1}) \Rightarrow \tilde{h}'(1) = 0,$$

$$\tilde{h}'' = 2t^{-2}(t^{1/p} + t^{-1/p})^{p-2}[t^{-2/p} + (2/p - 1)],$$

and the conclusion follows quickly using $p < 0$. □

Remark 4.1. It follows from the concavity / convexity properties of G_p that

$$G_p(x, y, z) \leq x + y + (2^p - 2)z$$

when $p \in [1,2]$, and the inequality reverses for $p \in (0,\,1] \cup [2,\,\infty)$. Indeed, G_p agrees with the linear function on the right hand side on an open set. We conclude from Theorem 2.5 that for any nonnegative numbers a, b, and any $p \in [1,2]$, we have

$$(a+b)^p \leq a^p + b^p + (2^p-2)(ab)^{p/2},$$

and the inequality reverses if $p \in (0,1] \cup [2,\infty)$.

5. Proof of Corollary 2.6

In this final section we prove Corollary 2.6.

Proof of Corollary 2.6: Recall from Remark 4.1 that for any nonnegative numbers a, b, and any $p \in [1,2]$, we have

$$(a+b)^p \leq a^p + b^p + (2^p-2)(ab)^{p/2},$$

and the inequality reverses for $p \in (0,\,1] \cup [2,\,\infty)$. Since for $p \in [0,2]$ we have $(a+b)^{p/2} \leq a^{p/2} + b^{p/2}$, and the reverse inequality if $p \geq 2$, it follows by induction that for any nonnegative numbers $a_j \geq 0$ we have

$$(\sum a_j)^p \leq \sum_j a_j^p + (2^p-2)\sum_{i<j}(a_i a_j)^{p/2} \tag{13}$$

holds true for $p \in [1,2]$, and the reverse inequality if $p \in (0,1] \cup [2,\infty)$. Finally it remains to put $a_j = f_j(x)$ and integrate the inequality. □

Remark 5.1. When $p < 0$, inequality (13) does not hold with three or more a_j. Take e.g. $a_j = 1$ for $j \leq 3$.

6. Concluding Remarks on Envelopes

An important challenge in this work was to compute the envelopes (11) and (12). In this section we briefly explain how we found them.

We recall from Section 3 that for the measure space $([0,1], \mathcal{B}, dx)$ we have $\overline{B}_p = \overline{H}_p$ is defined on Ω, one-homogeneous, and equals φ_p on $\partial\Omega$; that is, $\overline{H}_p(x,\,y,\,\sqrt{xy}) = (x^{1/p}+y^{1/p})^p$. We also recall from the discussion at the beginning of Section 4 that by one-homogeneity, to compute $\overline{H}_p$ it is enough to restrict our attention to the cross-section $D = \Omega \cap \{x+y=2\}$. Writing $D = \{(1+s,\,1-s,\,z)\}$ with $(s,\,z)$ in the upper half-disc, this reduces the problem understanding how the upper boundary of the convex envelope of the space curve

$$\gamma(s) = (s, \sqrt{1-s^2}, ((1-s)^{1/p} + (1+s)^{1/p})^p), \quad s \in [-1,1]$$

looks. One can show that the torsion τ_γ of the space curve γ changes sign only once from $-$ to $+$, at $s = 0$, when $p \in (0,1) \cup (2,\infty)$, and from $+$ to $-$ when $p \in (-\infty,0) \cup (1,2)$. Consider the case $p \in (0,1) \cup (2,\infty)$. Then it follows from Lemma 29 of Section 3.2 in [6] that locally, say for some $\delta \in (0,1]$, there exists a function $a(s) : [0,\delta] \to [-1,0]$ such that $a(0) = 0$, $a(s)$ is strictly decreasing, and the function $B(u,w)$ defined parametrically by

$$B(\lambda(a(s), \sqrt{1-a(s)^2}) + (1-\lambda)(s, \sqrt{1-s^2})) =$$
$$\lambda((1-a(s)^{1/p} + (1+a(s))^{1/p})^p + (1-\lambda)((1-s)^{1/p} + (1+s)^{1/p})^p$$

for $\lambda \in [0,1], s \in [0,\delta]$ is concave. In other words B has the prescribed boundary condition, i.e., $B(s, \sqrt{1-s^2}) = ((1-s)^{1/p} + (1+s)^{1/p})^p$, it is linear along the line segments $\ell(s) := [(a(s), \sqrt{1-a(s)^2}), (s, \sqrt{1-s^2})]$, and B is concave. It follows that "locally" B is a concave envelope. Because of the symmetry in x and y of the boundary data φ_p, one can show that the line segments $\ell(s)$ must be horizontal, i.e., $a(s) = -s$, and in fact $\delta = 1$. This means that B is a global concave envelope

$$B(u, w) = ((1 - \sqrt{1-w^2})^{1/p} + (1 + \sqrt{1-w^2})^{1/p})^p$$

for all $|u| \leq 1$ and $0 \leq w \leq \sqrt{1-u^2}$. Now it remains to change variables back to recover the envelope (11).

The case $p \in (-\infty, 0) \cup (1, 2)$ is different because τ_γ changes sign from $+$ to $-$, and in this case an "angle" arises with vertex sitting around the point $s = 0$ (see Section 3 in [6]).

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