STRICT 2-CONVEXITY OF CONVEX SOLUTIONS TO THE QUADRATIC HESSIAN EQUATION

CONNOR MOONEY

Abstract. We prove that convex viscosity solutions to the quadratic Hessian inequality

$$\sigma_2(D^2u) \geq 1$$

are strictly 2-convex. As a consequence we obtain short proofs of smoothness and interior C^2 estimates for convex viscosity solutions to $\sigma_2(D^2u) = 1$, which were proven using different methods in recent works of Guan-Qiu [GQ], McGonagle-Song-Yuan [MSY] and Shankar-Yuan [SY2].

1. Introduction

In this note we consider convex viscosity solutions to the quadratic Hessian inequality

$$\sigma_2(D^2u) \geq 1. \tag{1}$$

Our main result is their strict two-convexity. That is:

Theorem 1.1. *Let u be a convex viscosity solution to (1) in $\Omega \subset \mathbb{R}^n$, and let L be a supporting linear function to u in Ω. Then*

$$dim\{u = L\} \leq n - 2.$$

Theorem 1.1 is sharp in view of the example $u = x_1^2 + x_2^2$, with $L = 0$.

Local smoothness of convex viscosity solutions to

$$\sigma_2(D^2u) = 1 \tag{2}$$

follows from Theorem 1.1, using the classical solvability of the Dirichlet problem [CNS] and the Pogorelov-type interior C^2 estimate from [CW] (see Section 2). With a compactness argument we can in fact prove a universal modulus of strict 2-convexity (see Proposition 4.1). As a result we obtain:

Theorem 1.2. *Let u be a convex viscosity solution of (2) in $B_1 \subset \mathbb{R}^n$. Then u is smooth, and*

$$|D^2u(0)| \leq C\left(n, \|u\|_{L^\infty(B_1)}\right). \tag{3}$$

Inequality (3) was recently proven for smooth convex solutions of (2) in [GQ] and [MSY], and Theorem 1.2 was proven in [SY2]. A subtle issue in passing to the viscosity case is that smooth approximations of convex viscosity solutions may not be convex. An advantage of our approach is that it avoids using a priori estimates for smooth convex solutions, which allows us to bypass this issue. The methods in the above-mentioned works are quite different from ours, based in [GQ] on the

2010 *Mathematics Subject Classification.* 35J60, 35B65.

Key words and phrases. Sigma-2 equation, regularity.

Bernstein technique, and in [MSY] and [SY2] on the properties of the equation for the Legendre-Lewy transform of u.

An interesting question is whether the conclusion of Theorem 1.2 holds without assuming that u is convex. It is true when $n = 2$ (in which case solutions are automatically convex and (2) is the Monge-Ampère equation, [H]) and when $n = 3$ (in which case (2) is equivalent to the special Lagrangian equation, [WY]). It is also known to be true if u is slightly non-convex [SY2]. Finally, an interior C^2 estimate of the form (3) was recently obtained in [SY1] for smooth solutions to (2) that satisfy the semi-convexity condition $D^2u \geq -KI$, with C depending also on K. The general case in dimension $n \geq 4$ remains open.

Remark 1.3. Local smoothness and interior C^2 estimates are false for convex viscosity solutions to the k-Hessian equation

$$\sigma_k(D^2u) = 1$$

when $k \geq 3$, in view of the well-known Pogorelov example ([P], [U]). The same example shows that convex viscosity solutions to $\sigma_k(D^2u) \geq 1$ are not always strictly k-convex when $k \geq 3$. In particular, Theorems 1.1 and 1.2 are both special to the quadratic Hessian equation.

The paper is organized as follows. In Section 2 we recall a few classical results about the k-Hessian equation, and we use them to show that Theorem 1.1 implies that convex viscosity solutions of (2) are smooth. In Section 3 we prove Theorem 1.1. Finally, in Section 4 we prove a quantitative version of Theorem 1.1 using a compactness argument, and we use it to complete the proof of Theorem 1.2.

Acknowledgments

The author is grateful to Ravi Shankar and Yu Yuan for comments. This research was supported by NSF grant DMS-1854788.

2. Preliminaries

In this section we recall a few classical facts about the k-Hessian equation. Below Ω denotes a bounded domain in $\mathbb{R}^n$, and $1 \leq k \leq n$.

We first recall some facts about the σ_k operator. The function σ_k on $Sym_{n\times n}$ denotes the k^{th} symmetric polynomial of the eigenvalues. It is elliptic on the cone

$$\Gamma_k := \{M \in Sym_{n\times n} : \sigma_l(M) > 0 \text{ for each } 1 \leq l \leq k\},$$

and has convex level sets in Γ_k. Furthermore, the function σ_k is uniformly elliptic on compact subsets of Γ_k.

Next we recall the notion of viscosity solution. We say that a function $u \in C^2(\Omega)$ is k-convex if $D^2u \in \overline{\Gamma_k}$. Given a nonnegative function $f \in C(\Omega)$, we say that a function $u \in C(\Omega)$ is a viscosity solution of

$$\sigma_k(D^2u) \geq (\leq)\, f$$

if, whenever a k-convex function $\varphi \in C^2(\Omega)$ touches u from above (below) at a point $x_0 \in \Omega$, we have

$$\sigma_k(D^2\varphi(x_0)) \geq (\leq)\, f(x_0).$$

We say that $u \in C(\Omega)$ is a viscosity solution of

$$\sigma_k(D^2u) = f$$

if it is a viscosity solution of both $\sigma_k(D^2u) \geq f$ and $\sigma_k(D^2u) \leq f$. Viscosity solutions are closed under uniform convergence, and the notions of classical and viscosity solution coincide on C^2 functions that are k-convex.

Third we recall the classical solvability of the Dirichlet problem for the k-Hessian equation, proven in [CNS]:

Theorem 2.1. *Let $g \in C^\infty(\partial B_R)$. Then there exists a unique k-convex solution $u \in C^\infty\left(\overline{B_R}\right)$ to the Dirichlet problem*

$$\sigma_k(D^2u) = 1 \text{ in } B_R, \quad u|_{\partial B_R} = g.$$

The result in fact holds for smooth bounded $k-1$-convex domains.

Finally we recall the Pogorelov-type estimate Theorem 4.1 from [CW]:

Theorem 2.2. *Assume that $u \in C^\infty\left(\overline{\Omega}\right)$ is a k-convex solution to*

$$\sigma_k(D^2u) = 1 \text{ in } \Omega,$$

and that there exists a k-convex function $w \in C\left(\overline{\Omega}\right)$ such that $u < w$ in Ω and $u = w$ on $\partial\Omega$. Then

$$\sup_{\Omega}\left((w-u)^4|D^2u|\right) \leq C\left(n,\, k,\, \|u\|_{C^1(\Omega)}\right). \tag{4}$$

Inequality (4) implies in particular that the equation for u is uniformly elliptic on compact subdomains of Ω. By the Evans-Krylov theorem (see [CC]), interior derivative estimates of all higher orders follow.

To conclude the section we show local smoothness of convex viscosity solutions to (2). We assume u is defined in $B_1 \subset \mathbb{R}^n$, and it suffices to prove smoothness in a neighborhood of the origin. After subtracting a supporting linear function we may assume that $u(0) = 0$ and that $u \geq 0$. By Theorem 1.1 we have after a rotation that $\{u = 0\}$ is contained in the subspace spanned by $\{e_3, ..., e_n\}$. Let

$$w_\delta(x) := \delta[2(n-2)(x_1^2 + x_2^2) - (x_3^2 + ... + x_n^2)],$$

and notice that w_δ is 2-convex for all $\delta > 0$. Furthermore, we can choose δ, η, $\mu > 0$ small (depending on u) such that

$$u > w_\delta + \eta \text{ on } \partial B_{1/2} \quad \text{and} \quad \overline{B_\mu} \subset \{u < w_\delta + \eta\}.$$

Let $\{v_j\}$ be a sequence of smooth 2-convex (but not necessarily convex) solutions to (2) that converge uniformly to u in $B_{1/2}$. (One obtains the functions v_j e.g. by taking smooth approximations to u on $\partial B_{1/2}$ and applying Theorem 2.1 with $R = 1/2$ and $k = 2$.) Applying Theorem 2.2 to v_j with $w = w_\delta + \eta$ and $k = 2$, we see that the solutions v_j enjoy uniform derivative estimates of all orders in B_μ as $j \to \infty$. We conclude that u is smooth in B_μ.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Proof of Theorem 1.1: Assume by way of contradiction that there exists a supporting linear function L to u such that $\dim\{u = L\} \geq n - 1$. After subtracting L, translating, rotating, and quadratically rescaling, we may assume that u is defined in B_2, that $u \geq 0$, and that $u = 0$ on $\{x_n = 0\} \cap B_2$. After subtracting another supporting linear function of the form ax_n with $a \geq 0$, we may also assume that

$$u(te_n) = o(t) \text{ as } t \to 0^+.$$

Letting $x = (x', x_n)$, it follows that $\{u < h\}$ contains a cylinder of the form

$$Q_h := \{|x'| < 1\} \times (0, H),$$

with $h/H \to 0$ as $h \to 0^+$. For h small, the convex paraboloid

$$P_h := h|x'|^2 + 4\frac{h}{H^2}(x_n - H/2)^2$$

thus satisfies that $P_h \geq h \geq u$ on ∂Q_h, that $P_h(He_n/2) = 0 \leq u$, and that

$$\sigma_2(D^2 P_h) = c_1(n)h^2 + c_2(n)\frac{h^2}{H^2} < 1,$$

which contradicts (1). □

4. Proof of Theorem 1.2

In this section we prove a quantitative version of Theorem 1.1, and we use it to complete the proof of Theorem 1.2. For a set $S \subset \mathbb{R}^n$ and $r > 0$ we let S_r denote the r-neighborhood of S.

Proposition 4.1. *For $K > 0$, $r > 0$ and $n \geq 2$, there exists $\delta(n, K, r) > 0$ such that if u is a convex viscosity solution to (1) in $B_1 \subset \mathbb{R}^n$ with $\|u\|_{L^\infty(B_1)} \leq K$ and L is a supporting linear function to u at 0, then*

$$\{u < L + \delta\} \subset\subset T_r$$

for some $n-2$-dimensional subspace T of $\mathbb{R}^n$.

Proof. Assume not. Then there exist convex viscosity solutions u_j to (1) on B_1 with $\|u_j\|_{L^\infty(B_1)} \leq K$ and supporting linear functions L_j at 0 such that the conclusion fails with $\delta = 1/j$. Up to taking a subsequence, the functions u_j converge locally uniformly to a convex viscosity solution v of (1) in B_1, and L_j converge to a supporting linear L to v at 0 such that $\{v = L\}$ is not compactly contained in T_r for any $n-2$-dimensional subspace T. This contradicts Theorem 1.1. □

Proof of Theorem 1.2: We proved that u is smooth at the end of Section 2. The proof of the estimate (3) follows the same lines. We call a constant universal if it depends only on n and $\|u\|_{L^\infty(B_1)}$. We may assume after subtracting a supporting linear function with universal C^1 norm that $u(0) = 0$ and that $u \geq 0$. Write $x = (y, z)$ with $y \in \mathbb{R}^2$ and $z \in \mathbb{R}^{n-2}$. By Proposition 4.1 there exists $\delta > 0$ universal such that, after a rotation, $u > \delta$ on $\{|y| = 1/(2n)\} \cap B_1$. It follows that

$$u > w := \delta\left(2(n-2)|y|^2 - |z|^2 + \frac{1}{8}\right)$$

on the boundary of $B_{3/4} \cap \{|y| < 1/(2n)\}$. Notice also that w is 2-convex. The estimate (3) follows by applying Theorem 2.2 in the connected component of the set $\{u < w\}$ that contains the origin. □

References

[CC] Caffarelli, L.; Cabré, X. *Fully Nonlinear Elliptic Equations.* Colloquium Publications 43. Providence, RI: American Mathematical Society, 1995.

[CNS] Caffarelli, L.; Nirenberg, L.; J. Spruck, J. The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian. *Acta Math.* **155** (1985), 261-301.

[CW] Chou, K.-S.; Wang, X.-J. A variational theory of the Hessian equation. *Comm. Pure Appl. Math.* **54** (2001), 1029-1064.

[GQ] Guan, P.; Qiu, G. Interior C^2 regularity of convex solutions to prescribing scalar curvature equations. *Duke Math. J.* **168** (2019), no. 9, 1641-1663.

[H] Heinz, E. On elliptic Monge-Ampère equations and Weyl's embedding problem. *J. Analyse Math.* **7** (1959),1-52.

[MSY] McGonagle, M.; Song, C.; Yuan, Y. Hessian estimates for convex solutions to quadratic Hessian equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36** (2019), no. 2, 451-454.

[P] Pogorelov, A. V. *The multidimensional Minkowski problem.* Izdat. "Nauka," Moscow, 1975.

[SY1] Shankar, R.; Yuan, Y. Hessian estimate for semiconvex solutions to the sigma-2 equation. *Calc. Var. Partial Differential Equations*, to appear.

[SY2] Shankar, R.; Yuan, Y. Regularity for almost convex viscosity solutions of the sigma-2 equation. *J. Math. Study*, to appear.

[U] Urbas, J. On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations. *Indiana Univ. Math. J.* **39** (1990), no. 2, 355-382.

[WY] Warren, M.; Yuan, Y. Hessian estimates for the sigma-2 equation in dimension three. *Comm. Pure Appl. Math.* **62** (2009), no. 3, 305-321.

Department of Mathematics, UC Irvine
E-mail address: mooneycr@math.uci.edu