Regularity in the Calculus of Variations

Connor Mooney

Contents

1 Introduction

In these notes we outline the regularity theory for minimizers in the calculus of variations. The first part of the notes deals with the scalar case, with emphasis on the minimal surface equation. The second deals with vector mappings, which have different regularity properties due to the loss of the maximum principle.

These notes are based on parts of courses given by Prof. Ovidiu Savin in Fall 2011 and Spring 2014.

2 Scalar Equations

Scalar divergence-form equations generally arise by taking minimizers of interesting energy functionals. For example, by minimizing

$$\int_{B_1} |\nabla u|^2$$

among all functions $u \in H^1(B_1)$ with $u|_{\partial B_1} = \phi$ we obtain the divergence form of Laplace's equation:

$$\text{div}(\nabla u) = 0.$$

More generally, let

$$F : \mathbb{R}^n \to \mathbb{R}$$

be strictly convex, and let

$$E(u) = \int_\Omega F(\nabla u).$$

Examples of interesting F include $F(p) = |p|^2$ (the Dirichlet energy) and $F(p) = \sqrt{1 + |p|^2}$, the area functional. If we make a compact perturbation of a C^2 minimizer u by $\epsilon\varphi$ we have

$$F(\nabla u + \epsilon\varphi) = F(\nabla u) + \epsilon\nabla F(\nabla u) + O(\epsilon^2)$$

so the first variation equation is

$$\int_\Omega \nabla F(\nabla u) \cdot \nabla\varphi = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Written in divergence form, this is

$$\text{div}(\nabla F(\nabla u)) = 0,$$

and in nondivergence form (passing the derivative) this is

$$F_{ij}(\nabla u)u_{ij} = 0.$$

By restricting ourselves to perturbing one way or another, we obtain important differential inequalities. If perturbing downwards increases energy, we have a subsolution: for positive φ,

$$0 \geq \int_\Omega \nabla F(\nabla u) \cdot \nabla\varphi$$

which after integration by parts gives

$$\text{div}(\nabla F(\nabla u)) \geq 0.$$

Similarly, supersolutions are those functions with the prescribed boundary data whose energy increases when we perturb upwards.

Later in these notes we will focus on minimal surfaces, mininimizers of the area functional. An important observation is that $\sqrt{1 + |p|^2}$ behaves like $|p|^2$ for small p, so we expect "flat" minimal surfaces to behave like harmonic functions. However, the area functional grows linearly at infinity so we have very different behavior for minimal surfaces with large gradient.

We next discuss why we would like F to be strictly convex. First, planes are minimizers. To see this, first assume that p is a minimum for F. Then it is clear that any compact perturbations of the planes $p \cdot x + c$ will increase the energy, so they are minimizers. Observe that if we change F by a linear function, i.e.

$$\tilde{F} = F - (p_0 \cdot p + c)$$

then the minimizers of $\tilde{F}$ are the same as for F. Indeed, the energy changes by

$$\int_{\Omega} (c + p_0 \cdot \nabla u) = c|\Omega| + \int_{\partial\Omega} \phi p_0 \cdot \nu$$

which is constant. Thus, even if p is not a minimum for F, by convexity it is a minimum for

$$\tilde{F} = F - (\nabla F(p) \cdot x + F(p)),$$

and by the preceding discussion $p \cdot x + c$ are minimizers for F.

The intuition is that for convex F, adding "wiggles" can only increase the energy, so we don't expect wild minimizers. To examine what goes wrong with non-convex F, imagine it has two wells of equal depth centered at $\pm e_1$. Then 0 would not be a minimizer for F, but it "should be". We could imagine taking a very crinkled and flat "accordion" made from pieces of planes with slopes $\pm e_1$ and cutting off towards $\partial\Omega$ to get the smallest energy possible. Since these functions are approximations to 0, minimizing with respect to F is like taking the convex envelope of F and then minimizing.

The strict convexity of F also implies uniqueness. As noted above, any C^2 minimizer solves the Euler Lagrange equation

$$\text{div}(\nabla F(\nabla u)) = 0.$$

Conversely, if we can find a C^2 function u solving this equation, then

$$\int_{\Omega} F(\nabla u + \nabla \varphi) \geq \int_{\Omega} [F(\nabla u) + \nabla F(\nabla u) \cdot \nabla \varphi] = \int_{\Omega} F(\nabla u),$$

with strict inequality if φ is not constant. Thus, if we can find a C^2 solution to the Euler-Lagrange equation then it is the unique minimizer for the energy $\int_{\Omega} F(\nabla u)$.

The problem of existence of such a minimizer is equivalent to the estimate

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|\phi\|_{C^{2,\alpha}(\partial\Omega)})$$

for smooth solutions to the Euler-Lagrange equation by the method of continuity, outlined in a later section. Such an estimate is known as an "apriori estimate." Heuristically, given a smooth solution, such we can solve the problem for all nearby boundary data, and with this estimate we can perturb the boundary data all the way to whatever we like without the solutions becoming singular. The geometry of the domain is important to solvability of the equation; we will discuss this below. For now we restrict to the simple situation $\Omega = B_1$. Our main goal is to prove:

Theorem 1 (Apriori Estimate). *If $u \in C^{2,\alpha}(\overline{B_1})$ is a solution to the Euler-Lagrange equation*

$$F_{ij}(\nabla u)u_{ij} = 0 \text{ in } B_1, \quad u|_{\partial B_1} = \phi$$

with $\phi \in C^{2,\alpha}(\partial B_1)$ then

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \leq C(\|\phi\|_{C^{2,\alpha}(\partial B_1)}.$$

Remark 1. *To get existence of minimizers there are, broadly speaking, two methods. One, mentioned above, is to find a smooth solution to the Euler-Lagrange equation. The strict convexity of F then guarantees that this solution is the unique minimizer.*

The other method, known as the "direct method," is to compactify our space of smooth functions and search for a minimizer in the compactified space by taking the limit of a minimizing sequence. Typically this is relatively easy, and requires some sort of lower semicontinuity for the functional which says that energy can only be lost in the limit. The interesting problem then is to show the minimizer is regular.

Method of Continuity: Before proving the apriori estimate, we use it to find C^2 minimizers. The idea is to make small perturbations of a model solution, and using estimates depending only on the structure we can perturb all the way to the actual solution.

Indeed, suppose we have solved the problem

$$F_{ij}(\nabla u)u_{ij} = 0, \quad u|_{\partial B_1} = t_0\phi$$

for some $t_0 \in [0, 1]$. (The solution is 0 for $t = 0$). If we perturb by a function ϵv_1 we get

$$F_{ij}(\nabla u + \epsilon \nabla v_1)(u + \epsilon v_1)_{ij} = F_{ij}(\nabla u)u_{ij} + \epsilon L_u(v_1) + \epsilon^2 G_1(u, v_1)$$

where

$$L_u(v_1) = F_{ij}(\nabla u)(v_1)_{ij} + F_{ijk}(\nabla u)u_{ij}(v_1)_k$$

is the linearization of the equation around u. If we solve

$$L_u(v_1) = 0, \quad v_1|_{\partial B_1} = \epsilon\phi$$

then we have perturbed u towards a solution, where the error in the equation has order ϵ^2 with a constant depending on $\|u\|_{C^{2,\alpha}}$ by Schauder estimates.

We can continue making higher-order perturbations; indeed, expanding the operator on $u + \epsilon v_1 + \epsilon^2 v_2$ we get

$$\epsilon^2(L_u(v_2) + G_2(u, v_1)) + \epsilon^3 H(u, v_1, v_2)$$

so by solving

$$L_u(v_2) = -G_2(u, v_1), v_2|_{\partial B_1} = 0$$

we obtain a function solving the equation with error of order ϵ^3 depending again on $\|u\|_{C^{2,\alpha}}$. (To get this use that when we expand F to first order around ∇u the error looks like

$$\epsilon^2(\nabla v_1 + \epsilon \nabla v_2)^T D^2 F_{ij}(\nabla v_1 + \epsilon \nabla v_2) = \epsilon^2 H_1(u, v_1) + \epsilon^3 H_2(u, v_1, v_2).)$$

In general, we can continue by solving

$$L_u(v_k) = G_k(u, v_1, ..., v_{k-1}), v_k|_{\partial B_1} = 0$$

to get

$$\left| F_{ij}\left(\nabla\left(u+\sum_{i=1}^{k}\epsilon^i v_i\right)\right)\left(u+\sum_{i=1}^{k}\epsilon^i v_i\right)_{ij}\right| \leq C_k\epsilon^{k+1},$$

where C_k depends only on k and $\|u\|_{C^{2,\alpha}(\overline{B_1})}$ by Schauder theory. Thus, by choosing ϵ small depending on $\|u\|_{C^{2,\alpha}}$, we obtain the the limit a $C^{2,\alpha}$ solution to

$$F_{ij}(\nabla u_\epsilon)(u_\epsilon)_{ij} = 0, u_\epsilon|_{\partial B_1} = (t_0+\epsilon)\phi.$$

The key point now is that u_ϵ has the same estimate as u by the apriori estimate, so we may iterate and perturb our boundary data all the way to ϕ without the solutions degenerating.

2.1 Gradient Bounds Inherited From the Boundary

The first step in the apriori estimate is the following gradient bound:

Proposition 1 (Gradient Estimate). *If*

$$div(\nabla F(\nabla u)) = 0 \text{ in } B_1, \quad u|_{\partial B_1} = \phi$$

then

$$|\nabla u| \leq C\|\phi\|_{C^{1,1}(\partial B_1)}.$$

The philosophy is that if we have a translation-invariant equation with a maximum principle, gradient bounds at the boundary can be made global:

Proof. The uniform convexity of ∂B_1 means that there is a cone of opening depending only on $\|\phi\|_{C^2}$ such that this cone centered at any point on the graph of the boundary data avoids the boundary data otherwise. If we translate and lift u so that the boundary data remain within the cones then the resulting translations

$$u_h = u(x-he) + Ch$$

lie above u on the boundary of the common domain, so these cones avoid u when centered at any point on the graph, i.e. u is Lipschitz. □

Practically, this must means if we differentiate the equation we see the derivative satisfies a linear equation satisfying the maximum principle:

$$\partial_i(F_{ij}(\nabla u)(u_k)_j) = 0.$$

Remark 2. *By the discussion above, to prove the above estimate we use that the gradients have their maxima at the boundary. Since the boundary is uniformly convex, planes grow quadratically along the boundary and thus trap the boundary data when they have slope large depending on* $\|\phi\|_{C^{1,1}}$*. Since planes are minimizers and they trap* u *on the interior as well, giving the gradient estimate at the boundary, hence globally.*

For more general boundary geometry (not uniformly convex) we must use the specific form of F *to find alternative barriers, or existence may fail (see later sections).*

Since our equation is

$$F_{ij}(\nabla u)u_{ij} = 0$$

this gives us uniformly elliptic, bounded coefficients. If we could estimate u in $C^{1,\alpha}$, then the coefficients would become C^α (a general property of quasilinear equations) and then Schauder theory would imply the desired $C^{2,\alpha}$ estimate. Thus, the problem is to fill the gap between Lipschitz and $C^{1,\alpha}$.

If we differentiate $\text{div}(\nabla F(\nabla u)) = 0$ we see that the derivatives solve

$$\partial_i(F_{ij}(\nabla u)(u_k)_j) = 0,$$

where F_{ij} are uniformly elliptic and bounded measurable by strict convexity of F and the gradient bound. De Giorgi and Nash independently proved that this gives $u_k \in C^\alpha$, the fundamental contribution that fills the gap. We describe this in the next section.

2.2 De Giorgi-Nash-Moser

The De Giorgi-Nash theorem is a non-perturbation result that exploits the difference in scaling class between the function and its gradient. To illustrate the idea we consider the following example. Let E be a set with minimal perimeter in B_1, let $V(r) = |E \cap B_r|$ and let $S(r) = |E \cap \partial B_r|_{\mathcal{H}^{n-1}}$. By the minimality of ∂E we have that

$$|\partial(E \cap B_r)|_{\mathcal{H}^{n-1}} \leq CS(r).$$

By the isoperimetric inequality we have

$$V(r)^{\frac{n-1}{n}} \leq C|\partial(E \cap B_r)|.$$

Putting these together and integrating from 0 to t we get

$$\int_0^t V(r)^{\frac{n-1}{n}}\, dr \leq C \int_0^t S(r)\, dr = CV(t),$$

and in particular if $\tilde{r} < r$ we get

$$V(\tilde{r})^{\frac{n-1}{n}} \leq \frac{C}{r - \tilde{r}} V(r).$$

We claim that if $V(1)$ is sufficiently small, then $V(1/2) = 0$, i.e. we don't expect "spikes" into the the center of B_1. To see this divide B_1 into rings of radius $r_k = \frac{1}{2} + 2^{-k}$ and let $v_k = V(r_k)$. The previous inequality gives

$$v_{k+1} \leq C^k v_k^{1+\frac{1}{n}},$$

i.e. each time we chop we lose some mass proportional to the previous mass to a *larger* power. Letting $\gamma = 1 + 1/n > 1$ we see by iteration that

$$v_k \leq C^{\gamma^k(\sum_i \frac{i}{\gamma^i})} v_1^{\gamma^k}$$

so if v_1 is sufficiently small we get $v_k \to 0$ as claimed.

The following interior estimate of De Giorgi-Nash gives the analagous result for subsolutions to divergence equations, which says that u cannot have spikes on the interior, or that if u is small in measure then it is small pointwise as well.

Theorem 2 (De Giorgi-Nash). *If $\partial_i(a^{ij}(x)u_j) \geq 0$ in B_1 with $\lambda I \leq a^{ij} \leq \Lambda I$ and a^{ij} bounded measurable, then*

$$\|u^+\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^2(B_1)}.$$

Taking the positive part is necessary; take for example the fundamental solution $-|x|^{2-n}$ and cap it off with a paraboloid at radius r. Then the function has size r^{2-n}, but L^2-norm going like $r^{2-n/2}$.

Proof. The analogue of minimality comes from using the test function $\phi = \eta^2 u^+$ for some positive test function η which gives the Caccioppoli inequality (in fact, what we say below works for general divergence equations as long as they have the Caccioppoli inequality):

$$\int_{B_1} |\nabla u^+|^2 \eta^2 \leq C \int_{B_1} u^2 |\nabla \eta|^2.$$

Heuristically, the energy is controlled by u itself. The Sobolev inequality plays the role of the isoperimetric inequality, capturing the change in scaling class between the energy of u and the mass of u. Indeed, let $r_k = \frac{1}{2} + 2^{-k}$,

$$u_k = [u - (1 - 2^{-k})]^+$$

and let $a_k = \int_{B_{r_k}} u_k^2$. Let η_k be a smooth bump function which is 1 on B_{r_k} and 0 outside $B_{r_{k-1}}$. The Sobolev inequality gives that

$$\left(\int_{B_{r_k}} u_k^{2^*} \right)^{2/2^*} \leq C \int_{B_{r_{k-1}}} |\nabla(u_k \eta_k)|^2.$$

Combining with the Caccioppoli inequality we get

$$\left(\int_{B_{r_k}} u_k^{2^*} \right)^{2/2^*} \leq C^k a_{k-1}.$$

Finally, Hölder's inequality gives

$$a_k \leq \left(\int_{B_{r_k}} u_k^{2^*} \right)^{2/2^*} |\{u_k \geq 0\} \cap B_{r_k}|^{\delta(n)},$$

and the last term by Chebyshev is at most

$$C^k a_{k-1}^{\delta(n)}$$

giving

$$a_k \leq C^k a_{k-1}^{1+\delta(n)}$$

and proving the claim. □

Remark 3. *The key step of De Giorgi-Nash-Moser, and the only place we use the equation, is to get the Caccioppoli inequality. Any equations giving control of energy by mass will have a similar estimate.*

Remark 4. *For Laplace we have caccioppoli inequality for all derivatives, since they satisfy the same equation:*

$$\int |D^k u|^2 \leq C \int |D^{k-1} u|^2$$

(up to stepping in a bit). This is another approach to regularity for harmonic functions.

Remark 5. *Moser did the same as above avoiding taking slices by using the test function $\phi = (u^+)^\beta \eta^2$ instead of $u^+\eta^2$ for some $\beta \geq 1$, to get the Caccioppoli-type inequality*

$$\int_{B_1} |\nabla \left(u^{\frac{\beta+1}{2}}\right)|^2 \eta^2 \leq C \int_{B_1} u^{\beta+1} |\nabla \eta|^2.$$

(Of course, this test function is not a priori H^1 so to do this rigorously one performs cutoffs of u and takes a limit). Applying Sobolev one obtains

$$\left(\int_{B_{r_1}} u^{p\gamma}\right)^{\frac{1}{p\gamma}} \leq \left(\frac{C}{(r_2 - r_1)^2} \int_{B_{r_2}} u^p\right)^{1/p}$$

where $\gamma = \frac{2^*}{2} > 1$, $p = \beta + 1$ and the constant comes from the cutoff function. Taking r_k as above and iterating one obtains

$$\|u\|_{L^{2\gamma^k}(B_{r_k})} \leq C^{\sum_i \frac{i}{\gamma^i}} \|u\|_{L^2(B_1)}$$

and takes $k \to \infty$ to obtain the same estimate.

2.3 Density Estimates and Hölder Regularity

De Giorgi-Nash says that if solutions are small in large measure then they have no spikes on the interior. The density estimates sharpen this to say if we know that solutions are small on some set, we can control how far they spike:

Proposition 2 (**Density Estimates**). *If $\partial_i(a^{ij}(x)u_j) \geq 0$ in B_1 and*

$$|\{u \leq 0\} \cap B_1| \geq \delta |B_1|$$

then

$$osc_{B_{1/2}}(u^+) \leq (1-\epsilon) osc_{B_1} u^+$$

for some $\epsilon(\delta, n, \lambda, \Lambda)$.

We may assume $\sup_{B_1} u = 1$ my multiplying by a constant. If $\delta > 1 - \eta$ for some η small universal then oscillation decay comes directly from De Diorgi-Nash. The density estimates thus follow from the following lemma by subtracting, multiplying by constants and iterating:

Lemma 1. *Assume that*

$$\delta |B_1| \leq |\{u \leq 0\} \cap B_1| \leq (1-\eta)|B_1|.$$

Then

$$|u \leq 1/2 \cap B_1| \geq (\delta + \delta_1)|B_1|$$

for some $\delta_1(\delta, n, \lambda, \Lambda, \eta)$.

Proof. The idea is that if the derivative is in L^p for $p > 1$ we must pay in measure to jump from 0 to 1. Let $\tilde{u} = min\{u^+, 1/2\}$. Let $A = \{\tilde{u} = 0\}$, $B = \{\tilde{u} = 1/2\}$ and $D = \{0 < \tilde{u} < 1/2\}$. Taking $x \in A$ and integrating over lines into B one easily obtains

$$|B| \leq C \int_D \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy.$$

Integrating over $x \in A$ and using rearrangement one obtains the scale-invariant estimate

$$|B||A|^{1-1/n} \leq C \int_D |\nabla u| \leq C \left(\int_D |\nabla u|^2 \right)^{1/2} |D|^{1/2}.$$

By hypothesis A is not too small, and doesn't fill B_1. If $|B|$ is very small we are done, so assume not. Finally, from Caccioppoli we have

$$\|\nabla u\|_{L^2(B_{1-\mu})} \leq C(\mu)$$

which proves the claim up to stepping in a little bit (taking μ a small multiple of δ). □

For a solution we can apply the density estimates from both sides to get oscillation decay:

Theorem 3 (Hölder Regularity). *If* $\partial_i(a^{ij}(x)u_j) = 0$ *then*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^2(B_1)}.$$

In particular, this gives the interior $C^{1,\alpha}$ estimate for smooth minimizers in the calculus of variations.

Proof. By stepping in, subtracting and multiplying by a constant we can assume that u oscillates between -1 and 1 in B_1. If u is within ϵ of 1 at some point in $B_{1/2}$ then by uniform density

$$|\{u \geq 0\} \cap B_1| \geq 3/4|B_1|.$$

On the other hand, if u is also ϵ-close to -1 in $B_{1/2}$ as well then

$$|\{u < 0\} \cap B_1| \geq 3/4|B_1|,$$

a contradiction. Rescaling and iterating completes the proof. □

2.4 Harnack Inequalities

The Harnack inequalities turn pointwise information into information in measure. Usually, the statement says that if we have two solutions, one on top of the other, and they are close at a point, then they are comparably close in a neighborhood. We remark that the density estimate from the previous section is true for both divergence and nondivergence equations. In the divergence case we used the Caccioppoli inequality (an energy estimate) and change in scaling class. In the nondivergence case one uses the ABP estimate and comparison principle to obtain the same inequality. The theory in this section relies only on the density estimate, and thus holds for both divergence and divergence equations.

We begin with an analogue to the mean value inequality for harmonic functions:

Proposition 3 (Weak Harnack Inequality I). *Assume $\partial_i(a^{ij}(x)u_j) \leq 0$ in B_1 and that $u \geq 0$. Then*

$$\|u\|_{L^p(B_1)} \leq Cu(0)$$

for some small universal p.

This clearly implies the density estimate for supersolutions. By applying the density estimate at all scales and tying information together by a covering argument we obtain the weak Harnack inequality.

Proof. Assume by multiplying by a constant that $u(0) = 1$. By the density estimate,

$$|\{u \leq M\} \cap B_1| \geq \frac{1}{2}|B_1|$$

for some large M universal. Let $D_k = \{u \leq M^k\}$. For each point $x \in B_1 - D_1$ take the largest ball around x in $B_1 - D_1$ (it's an open set by Hölder regularity) that is in $B_1 - D_1$, say $B_r(x)$, so that the boundary touches D_1. If

$$|\{u > M^2\} \cap B_r(x) \cap B_1| > cr^n$$

then the uniform density estimates would give $u > M$ at the touching points (up to increasing M if necessary), a contradiction. Thus, if we take all such balls and take a Vitali subcover, we must swallow up a universal fraction of $B_1 - D_1$ when we remove $\{u \leq M^2\}$, i.e.

$$|B_1 - D_k| \leq \gamma |B_1 - D_{k-1}|$$

for some $\gamma < 1$ universal. Iterating we obtain geometric decay of the level sets, proving the proposition. □

Remark 6. *The Moser iteration technique in fact shows that the first weak Harnack inequality holds for $0 < p < \frac{n}{n-2}$. Furthermore, this is sharp, even for Δ. Take for example the fundamental solution v shifted so that $v(0) = 1$.*

We can say an analagous thing for subsolutions by applying this estimate upside-down, for any p:

Proposition 4 (Weak Harnack Inequality II). *Assume $\partial_i(a^{ij}u_j) \geq 0$ in B_1 and that $u \geq 0$. Then*

$$\sup_{B_{1/2}} u \leq C(p)\|u\|_{L^p(B_1)}$$

for any $p > 0$.

Proof. By multiplying by a constant assume that $\|u\|_{L^p(B_1)} = 1$. Let $\gamma > 1$ be chosen such that if v satisfies the hypotheses of the weak Harnack inequality I and $v(0) = \gamma - 1$ then

$$|\{v > 1/2\} \cap B_1| < \frac{1}{2}|B_1|.$$

Assume that $u \geq M$ at $x_0 \in B_{1/2}$ for some M large. If $u \leq \gamma M$ in $B_r(x_0)$ for some r, then by applying the weak Harnack inequality I upside-down in $B_r(x_0)$ we obtain

$$|\{u \geq M/2\} \cap B_r(x_0)| \geq cr^n.$$

We thus have

$$1 = \int_{B_1} u^p \geq cM^p r^n,$$

giving

$$r < cM^{-p/n}.$$

Thus, we can find a point x_1 within $M^{-p/n}$ of x_0 such that $u(x_1) > \gamma M$. Iterating, if M is sufficiently large (depending on p) we get that u blows up in $B_{3/4}$, a contradiction. □

The weak Harnack inequalities combine to give the full Harnack inequality for divergence equations:

Theorem 4 (Harnack Inequality). *If $\partial_i(a^{ij}(x)u_j) = 0$ in B_1 and $u \geq 0$ then*

$$\sup_{B_{1/2}} u \leq Cu(0).$$

Remark 7. *In two dimensions the Harnack inequality has an easy proof based on the fact that the oscillation on rings is controlled by the energy (integrate long the circle, then outwards and use that oscillation is increasing by maximum principle). One proceeds by looking at the equation satisfied by $v = \log(u)$.*

2.5 Boundary Estimates

With a mild condition on the boundary geometry, De Giorgi-Nash ideas extend easily to the boundary.

Theorem 5 (Boundary Estimates). *Assume that Ω^c has uniform density in balls centered on $\partial\Omega$ (for example, exterior cone condition). If*

$$\partial_i(a^{ij}(x)u_j) = 0 \text{ in } \Omega, \quad u \in C(\overline{\Omega}) \text{ with } u|_{\partial\Omega} = \phi$$

and $\phi \in C^\alpha(\partial\Omega)$ for some small $\alpha > 0$, then

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C\|\phi\|_{C^\alpha(\partial\Omega)}.$$

Proof. Let $0 \in \partial\Omega$ and assume by adding a constant and multiplying that u has oscillation 1 in B_1 and $0 \leq u \leq 1$. Let k, K denote the smallest and largest values of ϕ on $\partial\Omega \cap B_1$. Note that $(u-K)^+$ and $(k-u)^+$ (extended to be 0 outside Ω) are subsolutions in B_1 and are 0 on sets of density bounded away from 0. Applying De Giorgi-Nash and density estimates we obtain that

$$\epsilon k \leq u|_{B_{1/2}} \leq 1 - \epsilon(1-K),$$

giving the estimate

$$osc_{B_{1/2}} u \leq (1-\epsilon) osc_{B_1} u + \epsilon osc_{B_1} \phi.$$

We conclude by rescaling and iterating this (using uniform density of Ω^c) and using that ϕ is Hölder continuous. □

Remark 8. *The exterior cone condition is sufficient but not necessary; see the Wiener criterion. For example, on the slit disc in* $\mathbb{R}^2$ *we can prescribe values on either side and everything is ok, but on the punctured disc we cannot (there is no harmonic function* 0 *on* ∂B_1 *and* 1 *at* 0 *by comparison with small multiples of* $\log(r)$*). Heuristically, surfaces around codimension* 2 *sets have arbitrarily small perimeter so they cannot influence the mass of* Δ *(interpreted as a boundary integral of* u_ν *by integration by parts).*

The key point is that divergence equations are invariant under bilipschitz maps (for divergence equations, only the gradient appears in the equation, unlike the nondivergence case). Indeed, let T *be a bilipschitz map of* Ω *and* u *be a solution to*

$$div((\nabla u)^T A) = 0$$

for some uniformly elliptic A*. Then for* $\tilde{u}(Tx) = u(x)$ *we have*

$$\begin{aligned} 0 &= \int_\Omega (\nabla u)^T A(\nabla \phi)\, dx \\ &= \int_\Omega (\nabla \tilde{u})^T (DT \cdot A \cdot (DT)^T)(\nabla \tilde{\phi}) dx \\ &= \int_{T\Omega} (\nabla \tilde{u})^T \tilde{A} (\nabla \tilde{\phi}) dy \end{aligned}$$

where

$$\tilde{A} = \frac{DT \cdot A \cdot (DT)^T}{|\det(DT)|}.$$

The slit disc is equivalent to a half-disc by a locally bilipschitz transformation T *(i.e.* $cI \leq DT \leq CI$*) obtained by "opening the mouth." Note that we cannot do the same for the sphere minus a line segment (touching the north pole, say) in* $\mathbb{R}^3$*, since the analagous map would send points near the north pole far from each other.*

Equipped with the boundary estimate we can complete the proof of the apriori estimate.

Last Step in Apriori Estimate: Recall that the derivatives u_k solve linear divergence-form uniformly elliptic PDE, so the interior estimates above give an interior $C^{1,\alpha}$ estimate for u.

Assume for simplicity that the boundary is flat and lies on $\{x_n = 0\}$ locally (otherwise, use rotational derivatives). Then $u_i = \phi_i$ for $i < n$ so the by the boundary estimates

$$\|u_i\|_{C^\alpha(\overline{B_{1/2}^+})} \leq C\|\nabla \phi\|_{C^\alpha(\partial B_1^+)}.$$

The only thing left is to get a modulus of continuity on u_n near the boundary.

We use the equation

$$F_{ij}(\nabla u) u_{ij} = 0$$

to get

$$|u_{nn}| \leq C \sum_{(i,j)\neq(n,n)} |u_{ij}|.$$

This gives us the estimate

$$\int_{B_r^+} |\nabla u_n|^2 \leq C \sum_{i<n} \int_{B_r^+} |\nabla u_i|^2.$$

Letting $v = u_i$ we have by the Caccioppoli inequalty that

$$\int_{B_1} |\nabla v|^2 \leq C \int_{B_2} v^2 \leq Cosc_{B_2} v^2.$$

Rescaling this inequality we get

$$r^{2-n} \int_{B_r} |\nabla v|^2 \leq Cosc_{B_{2r}} v^2 \leq Cr^{2\alpha}$$

since v are continuous by the boundary estimate. We conclude that

$$\int_{B_r} |\nabla u_n|^2 \leq Cr^{n-2+2\alpha}.$$

Finally, we use integral characterization of Hölder continuity. The claim is that

$$\int_{B_r} |\nabla w|^2 \leq Cr^{n-2+2\alpha}$$

implies Hölder continuity (Morrey spaces). Poincarè gives

$$\int_{B_r} |w - w_{B_r}|^2 \leq Cr^{n+2\alpha}.$$

By comparing averages on dyadic balls one obtains

$$|w(x) - w_{B_r(x)}| \leq Cr^{\alpha}$$

and for $r = |x-y|$ by integrating over the overlap of the balls $B_r(x), B_r(y)$ one obtains

$$|w_{B_r(x)} - w_{B_r(y)}| \leq Cr^{\alpha}.$$

The preceding discussion gives the result

$$\|u\|_{C^{1,\alpha}(\overline{B_1})} \leq C(\|\phi\|_{C^{1,1}(\partial B_1)}).$$

Since the coefficients $F_{ij}(\nabla u)$ are then C^{α} with uniform estimates we can apply Schauder theory to get the desired apriori estimate. □

We will conclude our discussion of scalar equations by treating some properties special to minimal surfaces that don't hold for general F.

2.6 Dirichlet Problem for Minimal Surfaces

We would like to see when we can solve the Dirichlet problem

$$H(u) = f(x) \text{ in } \Omega, \quad u|_{\partial\Omega} = \phi$$

for ϕ continuous and f smooth. Roughly, two things can go wrong.

The first has to do with the relative "sizes" of f and Ω. For an example of non-solvability take $\Omega = B_1$ and $f > 2n$. Then by lowering the ball of radius $\frac{1}{2}$ into the domain, by the maximum principle it cannot touch the solution, so any solution must go to $-\infty$.

The second has to do with the existence of barriers at the boundary. The only step above that relied on the particular geometry of ∂B_1 (in particular, its uniform convexity) above is the gradient estimate, coming from using planes as barriers at the boundary (which will do for a minimizer for any reasonable F). For special F, however, the boundary geometry can be generalized, but without conditions guaranteeing the gradient estimate, classical solvability may fail.

Indeed, take for example an annulus with radii $1, r$ and the graph v of a catenoid with height 0 on the outer sphere and C on the inner sphere, with r chosen so that the normal derivative of v blows up on the inner sphere. Then there cannot be any classical solution with boundary data 0 on the outer sphere and data $\geq C$ on the inner sphere by the Hopf maximum principle. Sliding u from below it can't touch on the interior, so it must touch on the inner annulus, giving that u_ν is infinite on the inner sphere, and in fact tangent to the catenoid. Tilting the picture we obtain two minimal surfaces that touch and have the same gradient at a point, which violates the Hopf lemma (which would give a positive angle of separation).

To take care of the first issue (and guarantee that u is bounded) one approach is to find a subsolution lying below ϕ on the boundary. A sharper condition can be found using a mean-curvature form of the ABP estimate:

Proposition 5. *If $u \geq 0$ on $\partial\Omega$ and*

$$\int_\Omega \left(\frac{f^+}{n}\right)^n dx < \omega_n$$

then u is bounded below.

Proof. Assume that the graph Σ of u is touched below by a plane at x, and we have coordinates y on this tangent plane. Let ν be the normal to Σ. This gives a one-sided bound on the second fundamental form: $D_y\nu = II \geq 0$. Since its trace is $f(x)$ by AGM we have

$$\det(D_y\nu) \leq \left(\frac{f^+}{n}\right)^n.$$

This quantity $\det D_y\nu$ measures the infinitesimal ratio of volumes between the image of a patch around $(x, u(x))$ on Σ under ν on S^n and the patch on Σ. Note that the tangent planes to Σ at x and to S^n at $\nu(x)$ are parallel, so the projection of these volumes onto horizontal planes keeps the same ratio, giving

$$\det D_x\nu_x = \det D_y\nu \leq \left(\frac{f^+}{n}\right)^n,$$

where ν_x are the first n components of ν. The hypothesis thus says that the image of the Gauss map covers strictly less than the lower half-sphere.

If $|u|$ was very large, then the image of the convex envelope under the gradient map would cover most of the lower half-sphere, a contradiction. $\square$

To get boundary barriers, the trick is to use the boundary geometry itself.

Proposition 6. *If $H(\partial\Omega)(x_0) > f(x_0)$ then there is a barrier (class of nonpositive subsolutions touching 0 at x_0 that go to $-\infty$ as fast as we like).*

(The definition of barrier in the proposition is enough to ensure we can find function getting close to continuous boundary data at a point and lying below, giving continuity of solutions up to the boundary).

Proof. The idea of the proof is to view $\partial\Omega$ as a graph above x_0 in the normal direction at x_0 and make a perturbation of $\partial\Omega \times \mathbb{R}$ to obtain the barrier. Call this function $g(x)$. Then for small perturbations we have

$$H(g + \epsilon w) = H(g) + \epsilon L_g w + O(\epsilon^2),$$

where L_g is the linearized mean curvature operator at g. Since ∇g is so small nearby, the linearized operator is basically Δ, so to find the correct perturbation we just look for some $w > 0$ which is subharmonic and is 0 on a parabola so that it is a perturbation of the boundary cylinder (and thus lies below 0 when we tilt back as desired). This is easy to arrange with polynomial like

$$\left(\frac{1}{\epsilon}|x'|^2 - x_n\right)^2$$

where x_n is now the direction where the cylinder is constant. $\square$

Remark 9. *We have seen that solutions u to the minimal surface equation in a domain Ω minimize area among graphs with the same boundary data. It is natural to ask whether other surfaces lying in the same cylinder above Ω (maybe non-graphical) have smaller area. There are two easy ways to see why this is not the case. One is to slide the graph of u until it touches a point of the competitor by above or below, where the competitor is locally a graph, and get a violation of the strong maximum principle.*

Another related method, known as "calibration," is to extend the normal vector field ν to be constant in the e_{n+1} direction. Let V be the region between the competitor Σ for u, with exterior normal ν_V. We obtain via the divergence theorem that

$$Area(u) = \int_{\Sigma} \nu \cdot \nu_{\Sigma} \leq Area(\Sigma).$$

An important remark is that if Σ escapes the cylinder over Ω and Ω is not mean-convex, then the competitor may have less area.

Remark 10. *To find the shape of the catenoid, look for a radial function satisfying the MSE, $g(r)$. The curvature in the radial direction is easy:*

$$\frac{g''(r)}{(1+g'(r))^{3/2}}.$$

The tangential curvatures we compute with a geometric trick. Take the sphere beneath the graph of g centered on the $-x_{n+1}$ axis that touches g by below (and agrees with g to second order at the touching points in the tangential directions) with radius R. By similar triangles we have

$$\frac{r}{R} = \frac{|g'|}{\sqrt{1+g'^2}}.$$

Since the curvature is $\frac{1}{R}$ in the remaining $n-1$ directions, we obtain the following ODE for g:

$$g''(r) = \frac{n-1}{r}(1+g'(r)^2)|g'(r)|.$$

We obtain any other catenoid as a dilation of the one obtained this way. (To get the vertical pieces write as a graph another direction and solve, then piece together).

Observe that for r large and $g'(r)$ small, we obtain behavior like the equation

$$h''(r) + \frac{n-1}{r}h'(r) = 0,$$

which is exactly the equation for the fundamental solution of Δ. In particular, for 3-dimensional catenoids in $\mathbb{R}^4$ (or higher dimension) we get a surface trapped by two planes, whereas the 2-dimensional catenoid in $\mathbb{R}^3$ goes logarithmically out in both directions. An interesting consequence of this is that embedded minimal surfaces in $\mathbb{R}^3$ that lie in a half-space are planes, the analogue of the fact that subharmonic functions bounded above on $\mathbb{R}^2$ are constants by comparison with log.

2.7 Interior Gradient Estimate for Minimal Surfaces

For special equations the boundary data calms down as we step in and we can get interior estimates independent of boundary data. For example, if

$$\Delta u = 0$$

then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

This follows, for example, by the mean value property (the average in balls can only change as fast as the oscillation of u). We briefly outline a more general technique: Roughly, identify a special quantity related to $|\nabla u|$ that is a subsolution, so it cannot have interior maxima, and do some sort of cutoff and modification. This is known as the Bernstein technique. Take for example $w = |\nabla u|^2$. Then

$$\Delta w = \sum_{i,j} u_{ij}^2 \geq 0.$$

If we multiply by a cutoff η^2 we get

$$\Delta(\eta^2 w) = \eta^2 \Delta w + 2(\nabla \eta^2) \cdot \nabla w + (\Delta \eta^2) w \geq -C|\nabla u|^2$$

by the previous and Cauchy-Schwarz. Noting that $\Delta u^2 = 2|\nabla u|^2$ we have

$$\Delta(\eta^2 w + Au^2) \geq 0$$

for A large universal, and thus $\eta^2 w + Au^2$ must take its maxima on the boundary. The gradient estimate follows easily.

In this section, we use similar techniques to show an interior gradient bound for the minimal surface equation.

Theorem 6. *If $Mu = 0$ in B_1 and $u \in C^3(B_1)$ then*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)}).$$

Note that the dependence of the constant on the right side is not linear. Heuristically, this theorem says that no matter how wiggly and terrible we make the boundary data, it always calms down when we move in.

Before we begin the proof we make some pictures that clarify our choice of cutoff function. Imagine that the graph of u is a minimal surface, and we make a small perturbation

$$(x, u) \to (x, u) + \epsilon \phi \nu$$

where ν is the upward normal to the graph and ϕ is a positive function on $\mathbb{R}^{n+1}$. Choose coordinates so that at 0 the surface has 0 gradient. If it was a curve C with curvature κ, then if ϕ was constant the new osculating circle has radius

$$\frac{1}{\kappa} + \epsilon \phi$$

giving a new curvature of

$$\kappa + \epsilon(\phi \kappa^2) + O(\epsilon^2).$$

If ϕ is not constant its second derivative tangential to the curve further affects the curvature at order ϵ (since gradient is 0 the derivative of ϕ can only give higher-order change) giving

$$\kappa + \epsilon \phi \kappa^2 + \epsilon \phi''_C$$

as the new curvature to first order in ϵ, where to compute ϕ''_C we project the values of ϕ from the curve C to the tangent line and compute there. In higher dimensions, call the graph of u a surface Σ. The analagous computation gives the second fundamental form of the perturbed surface $\tilde{\Sigma}$:

$$A_{\tilde{\Sigma}}(0) = A_\Sigma(0) + \epsilon \left(D^2 \phi_\Sigma + A_\Sigma^2 \phi \right) + O(\epsilon^2),$$

where D^2_Σ is obtained by projecting ϕ to the tangent plane T and taking the usual derivatives. To lowest order, the difference between ϕ on T and on Σ a distance ϵ from the tangent point is $\frac{\kappa}{2}\epsilon^2 \nabla \phi \cdot \nu$ where κ is the curvature in that direction, giving

$$D^2_\Sigma \phi = D^2_T \phi + A \nabla \phi \cdot \nu.$$

In particular, for a minimal surface, the Laplace-Beltrami operator and tangential Laplace coincide. We conclude that

$$H(0) + \epsilon(\Delta_T \phi + \phi c^2)$$

where c^2 is the sum of squares of principal curvatures and $\Delta_T \phi$ is the Laplacian on the tangent hyperplane.

The key observation now is that the functions ν^k (coordinate functions of the unit normal) generate translations of the surface, which are minimal themselves. We conclude that

$$\Delta_T \nu^k + c^2 \nu^k = 0.$$

Since ν^{n+1} is positive, we have that ν^{n+1} is a supersolution to Laplace on tangent hyperplanes. Heuristically then, ν^{n+1} should not get too small on the interior (corresponding to a large gradient).

Slide u up so that it is positive and let $u(0) = A$. From here the idea is to construct a very small test function ϕ such that $\Delta_T \phi$ is positive on almost-vertical hyperplanes and vanishes outside the parabola $|x|^2 + x_n = 0$. We can then slide ϕ in the x_{n+1} direction and its graph cannot touch ν^{n+1} by below (by the maximum principle, we use support on paraboloids so that the slides are compactly supported on Σ), which will give us a lower bound for ν^{n+1}.

Let $v = -(2A|x|^2 + x_{n+1})$ and take $\phi = e^{Cv} - 1$ for large C. First, since it is positive at $(0, A)$ we know $\phi(x, u) > 0$ at $(0, u(0))$, and since $u \geq 0$ and ϕ disappears outside the paraboloid, it is a compact perturbation. Second, along vertical planes the Laplacian of ϕ looks like

$$\phi(C^2 - AC|x|)$$

which is positive for large C depending on A (in fact we can take C like A), giving the geometric proof. We now do the Bernstein technique.

Proof. The function we cook up to see if it is a subsolution is

$$v = \sqrt{1 + |\nabla u|^2}.$$

Let ν be the upward unit normal to the graph and define

$$\nu^i = \frac{u_i}{v}, \quad g^{ij} = \delta^{ij} - \nu^i \nu^j.$$

We compute

$$v_i = u_{ik}\nu^k, v_{ij} = (u_k)_{ij}\nu^k + \frac{u_{ik}u_{jk} - u_{ik}u_{jl}\nu_k\nu_l}{v}.$$

Note that the second term, when written with respect to coordinates where D^2u is diagonal, is

$$u_{ii}u_{jj}(\delta_{ij} - \nu_i\nu_j)$$

which is a positive matrix since $|\nu| \leq 1$. Thus, we have

$$g^{ij}v_{ij} \geq g^{ij}(u_k)_{ij}\nu^k.$$

We differentiate the equation to reduce the third order terms:

$$g^{ij}u_{ij} = 0, \quad g^{ij}u_{kij}\nu^k = \frac{2}{v}(u_{ij}u_{jk}\nu^i\nu^k - \nu^i\nu^j u_{ij}u_{kl}\nu^k\nu^l) = \frac{2}{v}g^{ij}v_iv_j,$$

giving

$$g^{ij}v_{ij} \geq \frac{2}{v}g^{ij}v_iv_j > 0$$

as desired.

To obtain the estimate, we must cut off v with a function $\eta(x, u)$. Let $w(x) = \eta(x, u)v(x)$. Then w has an interior maximum, and at this point

$$0 = w_i = (\eta)_i v + \eta v_i.$$

Furthermore, the Hessian of w

$$w_{ij} = (\eta)_{ij}v + (\eta)_i v_j + (\eta)_j v_i + \eta v_{ij} = (\eta)_{ij}v + \eta\left(v_{ij} - 2\frac{v_iv_j}{v}\right)$$

(using the vanishing gradient identity) is nonpositive at this maximum point, giving

$$0 \geq g^{ij}w_{ij} = g^{ij}(\eta)_{ij}v + \eta\left(g^{ij}v_{ij} - \frac{2}{v}g^{ij}v_iv_j\right) \geq g^{ij}(\eta)_{ij}.$$

Adding a constant so that $u \geq 0$ and taking

$$\eta(x, x_{n+1}) = e^{C(2A(1-|x|^2)-x_{n+1})^+} - 1$$

(as above) we see that η is nontrivial at 0 and is a compact perturbation (as above). Furthermore, we compute (dropping constants)

$$g^{ij}(\eta)_{ij} = (\eta + 1)C^2 g^{ij}(u_iu_j + Au_ix_j + A^2x_ix_j - A/C\delta_{ij}),$$

coming from

$$(\eta)_i = \eta_i + \eta_{n+1}u_i, \quad (\eta)_{ij} = \eta_{ij} + \eta_{i,n+1}u_j + \eta_{j,n+1}u_i + \eta_{n+1,n+1}u_iu_j.$$

Computing, we obtain the following bound by below:

$$g^{ij}w_{ij} \geq \frac{1}{1 + |\nabla u|^2}(|\nabla u|^2 - A|\nabla u \cdot x| - A/C(1 + |\nabla u|)^2).$$

Thus, if $|\nabla u|^2$ is very large then w is a subsolution in a neighborhood of the maximum, a contradiction.

We conclude that

$$\eta(0, A)v(0) \leq \max(\eta v) \leq C(A)$$

since η is bounded depending on A and $|\nabla u|$ is bounded at the maximum, completing the proof. □

One application of the interior gradient estimate is an existence result with lower regularity assumptions on the boundary data:

Theorem 7. *The equation*

$$Mu = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = \phi$$

with $\phi \in C(\partial\Omega)$ *and* $\Omega \in C^2$ *with* $H(x) \geq 0$ *for all* $x \in \partial\Omega$ *is uniquely solvable for* $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$.

The proof is to approximate ϕ with $C^{2,\alpha}$ functions and obtain approximate solutions converging uniformly by the maximum principle. The interior gradient estimate is independent of the boundary data, giving interior C^k estimates for any k (De Giorgi-Nash-Moser plus Schauder), so the limit is smooth on the interior. The condition $H(x) \geq 0$ cannot be relaxed, as seen in a previous section.

2.8 Optimal Gradient Bound

The key point of the interior gradient estimate for graphs is that ν^{n+1} is a positive supersolution to

$$\Delta_\Sigma \nu^{n+1} = 0,$$

where Σ is the graph in $\mathbb{R}^{n+1}$ and Δ_Σ is computed by projecting to the tangent plane and computing the Laplace there. If $0 \leq u$ and $u(0) = M$ we constructed an η of the form

$$\eta(x, x_{n+1}) = e^{-CM(M|x|^2 + x_n - (M+1))},$$

so convex in the x_{n+1} direction that $\Delta_T \eta$ is positive on almost-vertical planes, and 0 outside of a paraboloid so that η lies below ν^{n+1} on the boundary of Σ. Note that $\eta(0, M) = e^{CM^2}$. Taking $\delta\eta$ a family of barriers from below starting with $\delta = 0$, we see that $\delta\eta$ cannot touch ν^{n+1} by below for δ small on the scale of e^{-CM^2}, because if it did then it would touch where T is almost vertical. This gives a lower bound

$$|\nabla u(0)| \leq e^{CM^2}.$$

However, heuristics suggest that this is not optimal. Indeed, one could imagine that the configuration giving the largest gradient at 0 would be to make $u = 0$ on B_1^- and $u = M$ on B_1^+. Viewing this picture as a graph in the x_n direction, the gradient looks very small at 0, so we expect behavior like a harmonic function. The model solution which is 0 on a long strip and ± 1 at the end edges is something like

$$\frac{1}{e^M} \cos(y)(e^x - e^{-x}),$$

so we expect just exponential growth of the gradient bound with M. The following is an alternate, more involved proof (using integration by parts) which gives the optimal bound.

We develop the ideas in several steps. First, recast our definition of $\Delta_\Sigma v$ for a function v on the surface as follows. Extend v to be constant in the normal direction, and call this extension v_{ext}. Then we claim

$$\Delta_\Sigma v = \Delta v_{ext}.$$

Indeed, v is constant in the normal direction so we can ignore that direction, and at a point on the tangent plane T a distance ϵ from the tangent point, the projection onto Σ normal to T and normal to Σ differ by order ϵ^3, so this doesn't affect the second derivatives. The reason this definition is useful is that it allows us to prove the integration by parts formula

$$\int_\Sigma \Delta v \varphi \, d\mathcal{H}^n = -\int_\Sigma \nabla v \cdot \nabla \varphi \, d\mathcal{H}^n$$

for any $\phi \in C_0^\infty(\Sigma)$ easily. (Indeed, just fatten the surface by ϵ and use the usual IBP, and the boundary term goes away since the extensions are constant in the normal direction).

Next, we claim the mean value inequality

$$\Delta_\Sigma v \geq 0 \quad \to \quad v(0) \leq C(n) \int_{B_1 \cap \Sigma} v \, d\mathcal{H}^n.$$

To see this, take the usual radial test function φ from $\mathbb{R}^n$ obtained by capping off the fundamental solution at levels 1 and δ. For δ small, the tantential Laplacian is like the usual Laplacian because tangent planes almost contain the radial direction. However, further out, the tangent planes have some non-radial component, which can only decrease the Laplace of the test function, which is radially convex but concave in the other directions. Thus, we have the inequality

$$\Delta\phi \leq -\delta_0 + c\chi_{B_1 \cap \Sigma}.$$

Integrating this against v we get the desired inequality.

Finally, the last trick is to build a subsolution on the surface with a helpful term: let

$$w = -\log(\nu^{n+1}).$$

It is easy to verify that $\Delta_\Sigma w = |\nabla_\Sigma w|^2$ (where ∇_Σ is the projection of the gradient of w_{ext}, and in general for a function on $\mathbb{R}^{n+1}$ the projection onto the tangent plane) and $w > 0$. Let $x_0 = (0, u(0))$. By the MVI we only need to control w in average, i.e. show that

$$\int_{B_{1/2} \cap \Sigma} w \, d\mathcal{H}^n < C.$$

To that end we construct a test function and use integration by parts. Let $\eta(x_{n+1})$ be a function which is linear off of $(M-1, M+1)$, quadratic in $(M-1, M+1)$ and 0 for $x_{n+1} \geq M+1$, and let ϕ be a cutoff function in the x directions. First, we claim we only need to bound

$$\int_{B_1 \cap \Sigma} \Delta_\Sigma \eta (w\phi) \, d\mathcal{H}^n.$$

Indeed, $\Delta_\Sigma \eta$ is positive when the tantent plane is non-horizontal, i.e. when w is bigger than $\frac{1}{2}$, say. Integrating by parts on all of Σ (using that ϕ cuts things off at the boundary) we obtain

$$w(x_0) \leq C + C \int_\Sigma \eta'(x_{n+1})(|\nabla_\Sigma x_{n+1} \cdot \nabla_\Sigma \phi| w + |\nabla_\Sigma x_{n+1} \cdot \nabla_\Sigma w|\phi) \, d\mathcal{H}^n.$$

We focus on the first term. Since $\nabla\phi$ is in the x directions, the largest component it can have in the direction of e_n projected onto the tangent plane is $C\nu^{n+1}$, so the first term is bounded, and when we integrate we get it is less that CM.

For the second term we use the Cacciopoli inequality coming from the equation for w:

$$\int_\Sigma |\nabla_\Sigma w|^2 \phi^2 \, d\mathcal{H}^n = \int_\Sigma \Delta_\Sigma w \phi^2 \, d\mathcal{H}^n,$$

which upon integration by parts and an application of Cauchy-Schwarz gives

$$\int_\Sigma |\nabla_\Sigma w|^2 \phi^2 \leq CM.$$

The second term can be bounded by $C|\nabla w|\phi$, so by using Cauchy-Schwarz we bound the second term also by CM.

We conclude that

$$|\nabla u(0)| \leq e^{w(x_0)} \leq e^{CM},$$

giving the optimal scaling of the gradient bound with M.

Remark 11. *For a subsolution $\Delta v \geq 0$ with $v \geq 0$, we always have the Cacciopoli inequality*

$$\int |\nabla v|^2 \eta^2 \leq \int v^2 |\nabla \eta|^2.$$

The trick of taking $w = -\log(f)$ for a positive supersolution f is so useful because it allows us to swallow the dependence on the solution on the right hand side; indeed, one computes

$$\Delta w \geq |\nabla w|^2$$

and uses integration by parts as in the proof above.

Another lesson from the above proof is that for minimal surfaces, in many situations we can work as if the surface is itself $\mathbb{R}^n$ (doing IBP, etc.).

2.9 Concluding Remarks on Minimal Surfaces

We have seen that the classical solvability of the minimal surface equation breaks down depending on the boundary geometry and boundary data, for example if we set $\phi = 0$ on ∂B_1 and M on $\partial B_{1/2}$ for M large, the solution will have a discontinuity at $\partial B_{1/2}$ (by the maximum principle, comparison with the catenoid).

One could generalize the problem by looking for BV solutions to the problem

$$\min_{u \in BV} \int_\Omega \sqrt{1 + |\nabla u|^2} + \int_{\partial\Omega} |u - \phi| d\mathcal{H}^{n-1}.$$

One can conclude that in Ω, u is in fact smooth and if ϕ is continuous at a point where $\partial\Omega$ has positive mean curvature we have uniqueness.

The key point is that we already know that the singular set has $\mathcal{H}^{n-1}$ measure zero (in fact, much better, it has Hausdorff dimension $n-7$). This gives that ∇u is in fact in L^1 (as a measure it cannot concentrate on sets of $\mathcal{H}^{n-1}$ measure 0). Since the functional is strictly convex on $W^{1,1}$ we obtain a unique minimizer, up to constants. The condition on ϕ gives barriers that ensure uniqueness.

The solution is smooth on the interior, using approximation and the fact that on the boundary of a little ball the singular set is still very small. On $\partial\Omega$, u solves an obstacle problem (when we tilt our heads), which is well-studied. The basic result is that solutions with a nice obstacle are $C^{1,1}$ with quadratic separation, so u is $C^{1/2}$ at the boundary, and the free boundary is continuous and smooth away from a small set.

For an example of non-uniqueness, take for example a region made of four circular arcs bending away from the origin, with large data on one pair of opposing sides and small data on the other pair. Like the catenoid, the solution will eventually hit vertical (at 0 angle) and travel upwards or downwards, so by translating up and down we get a family of solutions. We need the circles to bend away so that the boundary has negative mean curvature, otherwise we know that it will be a graph and continuous up to these parts by barriers.

2.10 Stability in the Calculus of Variations

If we have a critical point of some variational problem (the weakest way to guarantee it solves some elliptic equation) it is natural to ask whether tiny perturbations increase energy. If this is the case, we say the solution is stable. Consider the general functional

$$J(u) = \int_\Omega F(\nabla u, u, x)\, dx$$

where $F = F(p, z, x)$. Then we have the expansion

$$J(u+\epsilon v) = J(u)+\epsilon \int_\Omega (\nabla_p F \cdot \nabla v + F_z v)\ dx + \frac{\epsilon^2}{2} \int_\Omega \left(\nabla v^T \cdot D_p^2 F \cdot \nabla v + F_{p_i z} v v_i + F_{zz} v^2\right)\ dx.$$

Integrating by parts, the stability condition says that

$$\int_\Omega \left(\nabla v^T \cdot A \cdot \nabla v - B v^2\right)\ dx \geq 0$$

for all $v \in H_0^1(\Omega)$, where A and B depend on $\nabla u, u, x$. If we minimize this value subject to the constraint $\int_\Omega v^2\, dx = 1$ we solve the eigenvalue problem

$$L_u(v) = \lambda v$$

where

$$L_u(v) = \text{div}(A \nabla v) + Bv.$$

Using v as a perturbation, we see that stability is equivalent to the statement that $\lambda \leq 0$, so this is a useful nonvariational characterization of the stability condition.

A geometrically compelling way to establish stability, then, is to find a positive supersolution w to the linearized problem:

$$L_u(w) < 0, \quad w > 0 \text{ in } \Omega.$$

If we manage to do this then we have stability since if λ were positive then v would be a strict subsolution to the problem where v is positive and we could take a multiple of w touching v from above, a contradiction of the maximum principle. (The same argument works from below).

Philosophically, in a variational problem if one finds a foliation of the regions on either sides of the solution with other solutions, then we have stability by this maximum principle argument. This is especially powerful when one can use the symmetries of the problem and the original solution to get the foliation. For example, minimal graphs are stable since translations are solutions and foliate. Dilations are also solutions so one can use these to get foliations in other situations. If we look at the catenoid then translations and dilations cross the original surface so we don't get global stability (as expected) but stability on smaller regions is easy.

Conversely, to show a solution is unstable it suffices to produce a positive subsolution to the problem vanishing on the boundary:

$$L_u(\bar{w}) > 0, \quad \bar{w}|_{\partial\Omega} = 0.$$

Indeed, using $\bar{w}$ as a perturbation we have second-order change in energy

$$\int_{\Omega} -L_u(\bar{w})\bar{w}\, dx < 0.$$

This technique is useful, for example, in showing that stable solutions homogeneous of degree 1 are planes in low dimension. The idea is that if the cone is singular, one can construct some positive subsolution to the linearized equation using derivatives of the solution itself (which have favorable homogeneity) and cutting them off radially.

3 Elliptic Systems

We now turn our attention to vector-valued mappings. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u = (u^1, ..., u^m) : \Omega \to \mathbb{R}^m$. In the following Du denotes the matrix whose α_{th} row is ∇u^α.

The linearization Du is in $\mathbb{R}^{n\times m}$. Let $F : \mathbb{R}^{n\times m} \to \mathbb{R}$ with $F \geq 0$. The general problem we wish to study is

$$\min_{u\in\mathcal{A}} \int_\Omega F(Du)\, dx$$

over some class of admissible functions $\mathcal{A}$.

3.1 Basics

3.1.1 Existence

The main tools for direct methods in calculus of variations are lower semicontinuity and compactness.

Proposition 7. *Assume that F is convex and that Du_k converges weakly to Du in L^1. Then*

$$\liminf_{k\to\infty} \int_\Omega F(Du_k)\, dx \geq \int_\Omega F(Du)\, dx.$$

Proof. The key point is the convexity inequality

$$F(Du_k) \geq F(Du) + DF(Du) \cdot (Du_k - Du).$$

Integrating and taking limits, we are done if the last term goes to 0. For weak L^1 convergence we test against bounded continuous functions, so a technical step is to restrict to sets where $|Du|$ is bounded. Let A_M be the set $\{|Du| > M\}$ and integrate over A_M, then take limits:

$$\liminf \int_\Omega F(Du_k)\, dx \geq \int_{A_M} F(Du)\, dx.$$

Taking $M \to \infty$ and using monotone convergence we are done. □

If Du_k are uniformly bounded in L^q for $q > 1$, there is a weakly convergent subsequence in L^q (hence L^1) since for $q > 1$ we know L^q is reflexive and separable. A more general condition is that Du_k are uniformly integrable (integrals are uniformly small over sets of small measure, heuristically prevents weak convergence to a Dirac point mass). One can easily find a subsequence inducing a bounded linear map on $C_b(\Omega)$ (using the countable dense subset and diagonal trick), but the dual of C_b consists of bounded measures. Uniform integrability implies that this measure is absolutely continuous with respect to Lebesgue, so Du_k converge weakly to some function in L^1.

One useful criterion implying uniform integrability of Du_k is the following:

$$\lim_{|p|\to\infty} \frac{F(p)}{|p|} = \infty \text{ (coercivity condition)}$$

and

$$\int_\Omega F(Du_k)\,dx < C$$

uniformly in k. Indeed, if $\int_{A_k} |Du_k|\,dx > \epsilon$ for a sequence of sets A_k with $|A_k| \to 0$, then $|Du_k|$ must get very large on these sets, giving $\int_{A_k} F(Du_k) > M_k\epsilon$ which is a contradiction for k large (the argument modulo details).

The previous arguments give the following existence and uniqueness theorem:

Theorem 8. *Let Ω be a bounded Lipschitz domain, $\mathcal{A} = \{u \in W^{1,q}(\Omega) : u|_{\partial\Omega} = g\}$, and let F be strictly convex satisfying the coercivity condition and the growth condition*

$$0 \leq F(p) \leq C(1 + |p|^q).$$

Then there is a unique minimizer to the variational problem in $\mathcal{A}$.

The regularity on Ω is necessary to say what the boundary values are. The growth condition on F ensures that the energy is finite, and coercivity gives weak convergence of a minimizing sequence, as mentioned above. Finally, convexity of F gives semicontinuity so that the limit is a minimizer, and the strict convexity of F ensures that the minimizer is unique. (If there were two minimizers, then any convex combination is also a minimizer.)

3.1.2 Quasiconvexity

An important observation is that for any vector mapping $u : \mathbb{R}^n \to \mathbb{R}^n$ and any compact perturbation ϕ of u in Ω, we have

$$\int_\Omega \det(Du)\,dx = \int_\Omega \det D(u + \phi)\,dx$$

but the determinant is not a convex function on $\mathbb{R}^{n\times n}$. (Indeed, $\det(tM) = t^n \det(M)$ which we can arrange not to be convex). To see this in $\mathbb{R}^2$, integrate by parts:

$$\begin{aligned}\int_\Omega (u_1^1u_2^2 - u_1^2u_2^1)\,dx &= \int_\Omega u^2(u_{12}^1 - u_{21}^1)\,dx + \int_{\partial\Omega} u^2(u_1^1\nu_2 - u_2^1\nu_1)\,ds \\ &= \int_{\partial\Omega} g^2 g_T^1\,ds\end{aligned}$$

where T is tangential to $\partial\Omega$. This quantity depends only on the boundary data. To see this in higher dimensions it is more convenient to differentiate and then integrate by parts. To first order,

$$\det(Du + \epsilon D\phi) = \det Du + \epsilon \det Du(Du)^{ij}\phi_i^j.$$

The following interesting computation shows that $\det Du(Du)^{ij}$ is divergence-free:

$$\partial_i(\det Du(Du)^{ij}) = \det Du\left((Du)^{lk}(Du)^{ij}u_{il}^k - (Du)^{ik}(Du)^{lj}u_{li}^k\right) = 0.$$

Thus, via integration by parts we see that $\int_\Omega \det Du\,dx$ is the the same for all maps u with the same boundary data.

Thus, we can add a det to F and the problem will not change. Other functions that will not change F are for example linear functions, since

$$\int_\Omega p^i \cdot \nabla u^i \, dx = \int_{\partial\Omega} g^i p^i \cdot \nu \, ds$$

which depends only on the boundary data.

In the previous section we established that convexity of F gives lower semicontinuity with respect to weak convergence. There is another condition on F, weaker than convexity, that is in fact equivalent to lower semicontinuity. This is known as quasiconvexity:

$$\int_\Omega F(p + D\varphi) \, dx \geq \int_\Omega F(p) \, dx$$

for all $\varphi \in C_0^1(\Omega)$. Quasiconvexity is exactly the statement that linear maps are minimizers (an obvious consequence of convexity). This is an interesting global condition that somehow exactly captures the idea that one can only lose wiggles in the limit. We have seen that some examples of quasi-convex functions include det and linear functions, as well as convex functions.

3.1.3 Lower Semicontinuity

We begin with a simple lemma saying that quasi-convexity is equivalent to testing against periodic functions.

Lemma 2. *F is quasi-convex if and only if for all φ periodic with any period λ we have*

$$\int_{Q_\lambda} F(p + D\varphi) \, dx \geq \int_{Q_\lambda} F(p) \, dx.$$

Proof. That this condition implies quasi-convexity is obvious; take a large box containing Ω and extend φ periodically outside of this box.

For the other direction, the key thing is that this condition holds at all scales, allowing us to apply Lipschitz rescaling. Let φ be periodic of period 1 and let

$$\varphi_\delta(x) = \delta\varphi(x/\delta)$$

for small δ. Let η be a cutoff function in Q_1 that is 1 away from a δ neighborhood of the boundary, which we denote $Q_{1-\delta}$. We compute

$$\int_{Q_1} F(p + D(\eta\varphi_\delta)) = \delta^{-n} \int_{Q_\delta} F(p + D\varphi(x/\delta)) \, dx + \int_{Q_1 - Q_{1-\delta}} F(p + \delta D\eta\varphi(x/\delta)) \, dx$$

up to a small counting error depending on δ. The first term is $\int_{Q_1} F(p + D\varphi) \, dx$ and the second goes to 0 since F is bounded in this region ($|D\eta| < C/\delta$) and the volume goes to 0. Applying quasi-convexity to this and letting $\delta \to 0$ we obtain the desired result. □

An easy consequence is that if the functional of interest is lower semicontinuous with respect to weak convergence, then it must be quasi-convex. Indeed, suppose $\int_\Omega F(Du)\,dx$ is not quasi-convex. Then there is some linear map p and a compact perturbation φ such that

$$\int_\Omega (F(p+D\varphi)-F(p))\,dx \le -\delta < 0.$$

Extend φ periodically outside of a large cube containing Ω and consider the Lipschitz rescalings

$$\varphi_\lambda(x) = \frac{1}{\lambda}\varphi(\lambda x).$$

Then $\varphi_\lambda \to 0$ uniformly and φ_λ are uniformly Lipschitz with $D(\varphi_\lambda(x))$ converging weakly to 0. However,

$$\int_\Omega F(P+D\varphi_\lambda)\,dx \to \int_\Omega F(p+D\varphi)\,dx < \int_\Omega F(p)\,dx - \delta$$

so lower semicontinuity is not satisfied.

We proceed to the other direction.

Theorem 9. *Assume that F is quasiconvex, that $0 \le F(p) \le C(1+|p|^q)$ with $q>1$, and Du_k converge weakly to Du in L^q then*

$$\liminf_{k\to\infty} \int_\Omega F(Du_k)\,dx \ge \int_\Omega F(Du)\,dx.$$

Proof. We begin by assuming that $u = p\cdot x$ is a plane and that

$$\int_\Omega |v-p\cdot x|^q\,dx < \epsilon, \quad |p|, \int_\Omega |Dv|^q\,dx < M.$$

We will show that

$$\int_\Omega F(Dv)\,dx \ge \int_\Omega F(p)\,dx - \mu(\epsilon, M)$$

where μ tends to 0 with ϵ. To see this, let $w = p\cdot x + \psi(v - p\cdot x)$ be a compact perturbation of $p\cdot x$ with $\psi = 1$ in $B_{1-\delta}$. Moreover, divide remaining annulus into k equal slices with ψ going from 1 to 0 in one of them. We compute $Dw = Dv$ in $B_{1-\delta}$ and outside we have

$$|Dw|^q \le M^q + |Dv|^q + \frac{k}{\delta}|v-p\cdot x|^q.$$

Assume we have chosen an annulus where $|Dv|^q$ integrates to no larger than M/k. Using the growth condition on F we conclude that

$$\int_{B_1} F(Dw) \ge \int_{B_{1-\delta}} F(Dv) - C(\delta M + M/k + \frac{k}{\delta}\epsilon).$$

Taking $\delta = \epsilon^{1/4q}$ and $k = \epsilon^{-1/4q}$ the last term is $\mu(\epsilon, M)$. The trick here is that for Dv (think it is Du_k) we have that $\int |Dv|^q$ is bounded but not necessarily small, so we have to use a slicing method to take care of it.

If u is not a linear map $p \cdot x$ but instead u is close to a linear map in the sense that

$$\int |Du - p|^q < \epsilon$$

(which by Poincaré implies that $\int |u - p \cdot x|^q < \epsilon$ up to a constant change in u, closeness of derivative to a constant in average implies closeness of u to a linear function in average), the same proof goes through.

To finish the proof notice that almost every point in B_1 is a Lebesgue point for Du so by rescaling we may assume that in these balls, Du is close to a constant matrix in average. We also need $|Du| < M$ in these balls so we work on this set. Take a Vitali covering. Finally, we need $\int |Du_k|^q$ to be bounded in the covering balls. This uses the uniform boundedness principle (weak convergence implies $\|u_k\|_{L^q} < L$) so it is bounded in all but a tiny measure of these balls. Then take limits in ϵ and M to complete the proof. □

3.1.4 Rank One Convexity

Quasiconvexity implies that F is convex along rank 1 matrices:

$$F\left(\frac{A+B}{2}\right) \leq \frac{1}{2}\left(F(A) + F(B)\right)$$

for all matrices A, B with $B - A$ being rank 1. Here is the heuristic proof. Let $\phi^1 = \rho(x_1)$ with ρ a periodic zigzag function which has derivative 1 half of the time and -1 the other half. Then

$$\frac{1}{2}(F(e_1 \otimes e_1) + F(-e_1 \otimes e_1)) = \int_{Q_1} F(D\phi)\, dx \geq F(0)$$

by quasiconvexity.

The key geometric observation is that if we regard a perturbation $p \cdot x + \phi$ as a perturbation of the plane, then we can regard ϕ as piecewise linear transformations which agree along the hyperplanes of discontinuity in $D\phi$. This means that the difference of these linear maps must be rank 1, and by quasiconvexity the "average" of F at these maps is larger than F at the average, giving rank-1 convexity.

The rank 1 matrices in $\mathbb{R}^{n \times m}$ do not form a subspace, but a cone with some singularity at 0. They take the form $\lambda p \otimes q$ with $p \in \partial B_1^n$ and $q \in \partial B_1^m$, and $\lambda \in \mathbb{R}$. This gives

$$m + n - 1$$

degrees of freedom, so this cone has dimension $m + n - 1$. Note that in the case $m = 1$ or $n = 1$ we conclude that the rank 1 cone has the same dimension as $\mathbb{R}^{mn}$. (This is obvious; all nonzero vectors have rank 1.) In particular, for scalar equations or maps which are curves in $\mathbb{R}^n$ convexity is the same as rank 1 convexity and quasiconvexity.

The case $n = 1$ is particularly simple. We have that Ω is an interval so the boundary data are just a pair of points in $\mathbb{R}^m$. This has the same boundary data as the line connecting these points, so if F is strictly convex the connecting line is the unique minimizer.

3.2 Polyconvexity, Quasiconvexity, and Rank One Convexity

The quasiconvexity condition

$$\int_{Q_1} F(p + D\varphi)\, dx \geq F(p)$$

for all $\varphi \in C^\infty(\mathbb{T}^n)$ is a difficult condition to understand and even to verify because of its nonlocal nature. As a first step towards understanding it, consider the simplest possible perturbation we can make: If in the domain we move in the direction of a vector $\xi \in \mathbb{R}^n$, perturb $p \cdot x$ in the direction $\eta \in \mathbb{R}^m$ back and forth so that half of the time we perturb in direction η and the other half in direction $-\eta$. The quasiconvexity inequality gives

$$\frac{F(p + \eta \otimes \xi) + F(p - \eta \otimes \xi)}{2} \geq F(p),$$

exactly that F is convex when we move the the direction of rank 1 matrices $\eta \otimes \xi$ (which acts on $x \in \mathbb{R}^n$ by $\eta \otimes \xi(x) = (\xi \cdot x)\eta$.

Rank-one matrices are 1-homogeneous (a cone) in $M^{m \times n}$ of dimension $m + n - 1$. To see this, we can assume that $\eta \otimes \xi$ is such that $|\eta|, |\xi| = 1$ (removing one dimension) and this leaves $(m-1) + (n-1)$ dimensions left. An easy but important observation is that we can move between any two matrices by adding only rank-one matrices each time.

As a toy problem illustrating that this is very far from convexity, consider functions $f(x, y)$ which are convex in x and y separately, but not jointly. One example is xy (a saddle). If a convex function is 0 on a collection of points, it is ≤ 0 inside the convex hull of these points. The situation is far different; for our example, we take f to be 0 except in the $x > 0, y > 0$ quadrant where we take xy. Alternatively, if we take 4 points where such a function is 0 we can only look at horizontal and vertical lines to use convexity, getting that $f < 0$ in some smaller sub-box, a "reduced convex hull."

The differential version of rank-one convexity is

$$F_{\alpha i, \beta j} \eta^\alpha \eta^\beta \xi_i \xi_j \geq 0 \quad (\lambda |\eta|^2 |\xi|^2).$$

This is the (strict) Legendre-Hadamard condition.

3.2.1 Quadratic Functions

Quadratic functions which are rank-1 convex are quasiconvex, a really remarkable thing since for any perturbation Du it is not in general rank 1. The key idea is that u can be written as a superposition of rank-1 perturbations and for quadratic functionals the cross terms will cancel.

Proof. We write $F(p) = A^{ij}_{\alpha\beta} P^\alpha_i P^\beta_j$. Using the Fourier transform is the key trick (and the only proof I know of this result):

$$\int_{B_1} A^{ij}_{\alpha\beta} u^\alpha_i u^\alpha_j \, dx = \int_{\mathbb{R}^n} A^{ij}_{\alpha\beta} \xi_i \xi_j (\hat{u^\alpha})(\overline{\hat{u^\beta}}) d\xi$$
$$= \int_{\mathbb{R}^n} A^{ij}_{\alpha\beta} \xi_i \xi_j (w^\alpha w^\beta + v^\alpha v^\beta) \, d\xi$$
$$\geq \lambda \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}|^2 \, d\xi$$
$$= \lambda \int_{B_1} |Du|^2.$$

Thus, this estimate tells us that rank 1 quadratic functions are in fact “as good as” convex ones! □

Alternatively, we can write any periodic perturbation (say in $n = m = 2$) as a Fourier series,

$$w^\alpha = \sum_{i,j} C^\alpha_{ij} \sin(ix) \sin(jy)$$

and easily check that Dw is a sum of orthogonal rank 1 perturbations, where the contributions from cross terms to the energy are 0.

A natural conclusion is that for small perturbations (where F is roughly quadratic), we have rank-1 convexity giving quasiconvex inequalities. The way one might hope to give an alternative proof is to add on null Lagrangians involving determinants to make this thing convex (which turns out to be possible for $2 \times n$ or $m \times 2$ matrices, but not for 3×3 matrices). This condition is actually another local condition known as polyconvexity:

$$F(p) = G(t_1, ..., t_s)$$

where t_i are subdeterminants of size i and G is convex in its arguments. (For example, in 3×3 matrices, $G(P, cof(P), \det P)$ would be it). It is easy to verify that such F are quasiconvex using that determinants are null Lagrangians:

$$F(p + D\phi) \geq F(p) + G_{t_i}(p)(t_i(D\phi)),$$

and the last terms integrate to 0 since they are null Lagrangians. (For example, if m or n is 1 we have convexity, F lies above linear functions).

Thus,

Polyconvexity → Quasiconvexity → Rank One Convexity.

The goal would be to show when these are the same for existence/ lower semicontinuity theory. Sverak has a counterexample to rank one convexity implying quasiconvexity (and for 3×3 we know it doesn't give polyconvexity even for quadratic functions...) if the target has dimension 3 or higher. It remains open whether these are equivalent in $M^{2\times 2}$ (or more generally if the target has dimension 2).

3.3 Sverak's Example

In this section we give an example of a function $F : \mathbb{R}^{2\times 3} \to \mathbb{R}$ which is rank-1 convex but not quasiconvex. We will build F by looking at simple rank-1 perturbations which are superposed. The idea is that any periodic perturbation φ is a superposition of rank-1 perturbations by Fourier analysis (constant in $n-1$ directions).

First build φ^1 such that

$$D\varphi^1 = \pm f_1 \otimes e_1 = \pm E_1.$$

Then $D\varphi^1$ splits 0 into two pieces, $\pm E_1$, with equal probability $\frac{1}{2}$ (split between the left/right sides of Q_1. Now build φ^2 such that

$$D\varphi^2 = \pm f_2 \otimes e_2 = \pm E_2.$$

Then the sum of the two splits 0 into 4 parts of equal probability, $\pm E_1 \pm E_2$ in the 4 quadrants of the box. Furthermore, $E_1 \perp E_2$. An important observation (EXERCISE) is that if we superpose ANY two rank-1 perturbations like above we will get a similar splitting. Note that rank-one convexity guarantees that we cannot contradict the quasiconvexity inequality at this point, since we have a convex combination with equal weights. We require an asymmetric perturbation.

Finally, we add the perturbation φ^3 such that

$$D\varphi^3 = \pm \frac{1}{\sqrt{2}} f_3 \otimes (e_1 + e_2) = \pm E_3.$$

The key point now is that we have an asymmetric perturbation when we superpose the three. Indeed, we have split the four points $\pm E_1 \pm E_2$ into 8 points where $E_1 + E_2 + E_3$ has $\frac{3}{16}$ probability and the same with $-E_3$ has $\frac{1}{16}$ probability.

We can now build F. It is easy to see that the only rank 1 directions in the subspace spanned by E_1, E_2 and E_3 are E_i themselves. (Any nontrivial combination of them will have full rank). This is the key thing that doesn't hold in higher dimensions (we can make them all perpendicular still if we map into $\mathbb{R}^2$, and issue is that the resulting subspace will contain lots of directions of rank 1 matrices comparitively, making it harder to build F). For matrices with coordinates x, y, z in this subspace if we let

$$F = -xyz$$

we have built a function which is rank-1 linear on this subspace Σ, and is negative exactly at the points to which $D(\varphi^1 + \varphi^2 + \varphi^3)$ gives most weight. Thus, at least on this subspace we have our example.

The final step is to extend this function to all of $\mathbb{R}^{2\times 3}$ without breaking rank 1 convexity. First, we may convexify F a little on Σ without changing violation of the quasiconvexity inequality. Second, outside of a large ball we may make F convex easily (take the maximum of what we have and a very convex function lying beneath it in B_{10}). Third, extend F as follows:

$$F(p) = f(p_\Sigma) + K|p_{\Sigma^\perp}|^2$$

for $K >> 1$. Then for any $p_0 \in \Sigma$ and q a rank 1 direction, if $p_0 + q$ lies close to Σ we can use convexity there, and if not we have convexity in the perpendicular directions to cancel any negative effects from the tangential direction, so we're good. (Translation invariance of F in the perpendicular direction).

3.4 Linear Theory

We start with the constant coefficient case. Let $u : B_1 \to \mathbb{R}^m$ be the unique minimizer of

$$\int_{B_1} A^{ij}_{\alpha\beta} u^\alpha_i u^\beta_j$$

with $u = \phi$ on ∂B_1 and $u, \phi \in H^1(B_1)$, with

$$A^{ij}_{\alpha\beta}\eta^\alpha\eta^\beta\xi_i\xi_j \le \lambda|\eta|^2|\xi|^2.$$

(The Legendre-Hadamard condition). Since this is a quadratic functional, previous sections imply the existence and uniqueness of such a minimizer which satisfies the Euler-Lagrange equation

$$A^{ij}_{\alpha\beta}u^\alpha_{ij} = 0$$

in B_1. The Caccioppoli inequality is easy to verify, (in all of this the key is the Fourier transform inequality) giving

$$\|u\|_{H^1(B_{1/2})} \le C\|u\|_{L^2(B_1)} \le C(\|\phi\|_{H^1}).$$

By linearity the derivatives satisfy the same equation so we may iterate to obtain

$$\|u\|_{H^k(B_{1/2})} \le C(k)\|u\|_{L^2(B_{1/2})}$$

giving the familiar interior estimate we know for harmonic functions. (Rescaling should give Liouville theorems too).

Observe that we can only get the Caccioppoli inequality with the Legendre-Hadamard condition provided $a^{ij}_{\alpha\beta}$ are constant (or, by a quantitative analysis of errors if $a^{ij}_{\alpha\beta}$ are close to constant).

We can now do a perturbation Schauder theory which is easier than the nondivergence case (which requires the Harnack inequality to compare solutions to the given equation and to the approximate constant-coefficient equation.)

Theorem 10. *Assume that*

$$\partial_j(a^{ij}_{\alpha\beta}u^\alpha_{ij}) = \partial_j(f^j_\beta)$$

with the data in C^μ and the L-H condition. Then u is $C^{1,\mu}$ with a quantitative estimate.

Proof. The key point is the following iterative inequality: If $\int_{B_r}|u-l|^2 \le r^{n+2(1+\alpha)}$, then there exists ρ universal and a linear $\tilde{l}$ such that

$$\int_{B_{\rho r}}|u-\tilde{l}|^2 \le (\rho r)^{n+2(1+\alpha)}$$

provided the oscillations of f, a^{ij} are less than δ universal and $|l| < 1$. The oscillation decay of f and a^{ij} at all scales takes care of the iteration.

Proceed by rescaling and comparison with constant-coeffient solution. By subtracting a linear function and tuning our scaling,

$$r^{1+\mu}\tilde{u}(x/r) = u - l,$$

we may assume that we are in B_1 with the same hypotheses and $l = 0$. Let w be the solution to the constant-coefficient equation

$$a^{ij}_{\alpha\beta}(0) w^\alpha_{ij} = 0$$

with $w = u$ on $\partial B_{1/2}$. The equation for $u - w$ schematically looks like

$$D(AD(u-w)) = Df + (A - A(0))Du$$

and in L^2 the rhs is bounded by δ by hypothesis and the Cacciopoli inequality (small perturbations version). We conclude via IBP and LH condition that

$$\int_{B_{1/2}} |D(u-w)|^2 \, dx \leq C\delta.$$

Since w is smooth there is some L so that $|w - L|_{B_r} < Cr^2 < r^{1+\mu}/3$ for r small, giving (with previous and Poincare inequality) that

$$\int_{B_r} |u - L|^2 \, dx \leq C\delta + r^{2(1+\mu)}/3.$$

Taking $r = \rho$ small and δ small the right side is $\rho^{2(1+\mu)}$ and we are done. □

Now a remark on the original nonlinear minimization problem:

$$\min_{u \in H^1, u|_{\partial B_1} = \phi} \int_{B_1} F(Du) \, dx.$$

For existence assume that F is quasiconvex and satisfies the appropriate coercivity and growth $C(1 + |p|^2)$ conditions. The Euler-Lagrange equation is

$$\partial_j (F_{\beta j}(Du)) = 0.$$

Differentiating we obtain

$$\partial_j (F_{\alpha i, \beta j}(Du)(u^\alpha_i)_j = 0.$$

Assume that F is smooth and uniformly convex (D^2F is comparable to I) even (or maybe that it is locally uniformly convex and u is Lipschitz). Then by the perturbation theory if $u \in C^1$ then Du is C^μ for any $0 < \mu < 1$ (changing by constants doesn't change the equation so we don't need the coefficients in C^μ), and then the coefficients are even better and we bootstrap to C^∞. The problem is that De Giorgi-Nash-Moser doesn't work for systems, so we can't rule out that the derivatives oscillate wildly unlike the scalar equations case. In fact they can; see the counterexamples in a future section.

3.5 Partial Regularity

We turn to partial regularity. Assume first for simplicity that F is smooth uniformly convex with uniformly bounded second derivatives:

$$\frac{1}{C}I \leq D^2F \leq CI.$$

Assume that u is a minimizer for $\int F(Du)$. If we can guarantee that u is very close to a plane in measure, i.e.

$$\frac{1}{r^{n+2}} \int_{B_r} |u - L|^2 \, dx < \epsilon^2$$

then u is $C^{1,\alpha}$ in $B_{r/2}$ (think $(u-L)/r$ is small in average). We just need to show that there is some small universal ρ and linear L' such that

$$\frac{1}{(\rho r)^{n+2}} \int_{B_{\rho r}} |u - L'|^2 \, dx < \frac{\epsilon^2}{2}.$$

One proves this via compactness.

Take the rescaling

$$u(x) = L(x) + \epsilon r u_\epsilon(x/r)$$

so that $Du(x) = p + \epsilon Du_\epsilon(x/r)$. Then u_ϵ minimizes the functional $\int F_\epsilon(Du_\epsilon)$ where

$$F_\epsilon(a) = \frac{1}{\epsilon^2}(F(p + \epsilon a) - F(p) - DF(p) \cdot \epsilon a)$$

which converges locally in C^2 to the uniformly convex quadratic functional $D^2F(p)$. We are thus in the situation

$$\int_{B_1} |u_\epsilon|^2 \, dx < 1$$

with u_ϵ solving the Euler-Lagrange equation

$$\int DF_\epsilon(Du_\epsilon) \cdot D\phi = 0.$$

We want to show that there is some ρ universal such that for ϵ small enough,

$$\frac{1}{\rho^n} \int_{B_\rho} |u_\epsilon - L| < \frac{1}{2}\rho^2.$$

We proceed compactness. Assume that the conclusion is false for a sequence of ϵ's going to 0. By the Caccioppoli inequality (this is important),

$$\int_{B_{1/2}} |Du_\epsilon|^2 \, dx \leq C.$$

Hence, u_ϵ converge in L^2 and Du_ϵ weakly in L^2 to u, Du. We test what happens in the limit.

$$0 = \int DF_\epsilon(Du_\epsilon) \cdot D\phi = \int D^2F(p)(Du_\epsilon, D\phi) + (DF_\epsilon - D^2F(p))(Du_\epsilon) \cdot D\phi.$$

The first term tends to $\int D^2F(p)(Du, D\phi)$ by weak convergence (uses bilinearity). The second term will go to zero where $|Du_\epsilon|$ is bounded, and where $|Du_\epsilon|$ is large is a small set and doesn't bother (linear growth of DF_ϵ).

For solutions to the constant coefficient equation we have

$$\int_{B_\rho} |u - L_\rho|^2 \, dx < C\rho^{n+4} < \frac{1}{4}\rho^{n+2}$$

for ρ small, so by L^2 convergence we get a contradiction.

Remark 12. *Actually, minimizers of $\int_{B_1} F(Du)\, dx$ solve the Euler-Lagrange system*

$$div(\nabla F(Du)) = 0,$$

which upon passing a derivative gives that

$$\partial_i \left(A^{ij}_{\alpha\beta}(v) v^\beta_j \right) = 0$$

where v are derivatives of u. For equations of this form, all we need for the coefficients to be close to constant is that v is close to constant (we need $v \in C^\alpha$, not its derivatives), i.e.

$$\int_{B_r} |v - v_{B_r}|^2 \leq \epsilon r^n.$$

This follows from (in fact, by Caccioppoli is equivalent to)

$$r^2 \int_{B_r} |Dv|^2 \leq \epsilon r^n,$$

which in turn gives $\mathcal{H}^{n-2}$ as the dimension of the singular set. Contrast this to the above argument where we needed Du close to constant.

Actually, in a future section we show that for systems of the form

$$\partial_i \left(A^{ij}_{\alpha\beta}(v) v^\beta_j \right) = 0$$

where $A(M, M) \geq \lambda |M|^2$, the Caccioppoli inequality gives a reverse Hölder inequality saying that Dv is $L^{2+\delta}$. The smallness condition is then guaranteed if

$$\int_{B_r} |Dv|^{2+\delta} \, dx < \epsilon r^{n-2-\delta},$$

giving that the singular set has Hausdorff dimension strictly less than $n - 2$.

Remark 13. *There is also a version of the Caccioppoli inequality for minimizers when F is uniformly quasiconvex with uniformly bounded second derivatives:*

$$\int F(p + D\phi) \geq \int F(p) + \lambda \int |D\phi|^2,$$

with quadratic growth $D^2F < CI$ (see the next section). One obtains by the above arguments that minimizers are smooth away from a closed singular set of Lebesgue measure zero.

However, there is not a Caccioppoli inequality for the derivatives of minimizers as in the uniformly convex case, so we cannot conclude that the derivatives are weakly compact in H^1. In particular, we cannot improve the Hausdorff dimension of the singular set as above.

Remark 14. *For certain functions (for example involving determinant) F will have faster growth at infinity:*

$$|D^2F(p)| \leq C(1+|p|^{q-2})$$

for some $q > 2$. Then the correct structure condition for partial regularity is

$$\int F(p+D\psi) \geq \int F(p) + \gamma\left(\int |D\psi|^2 + \int |D\psi|^q\right).$$

A prototype equation would be

$$F(p) = |p|^2 + |p|^n + |\det p|$$

for $p \in \mathbb{R}^{n\times n}$. The first two terms are what gives the correct notion of strict quasiconvexity, and the last term $|\det p|$ is just quasiconvex.

3.6 Caccioppoli inequality for quasiconvex functionals

The following key observation of Evans allows one to show almost everywhere smoothness of minimizers of quasiconvex functionals.

We say F is uniformly quasiconvex if

$$\int_\Omega F(p+D\varphi)\,dx \geq \int_\Omega (F(p) + \gamma|D\varphi|^2)\,dx$$

for all compactly supported smooth deformations φ. This says that we pay proportional to the Dirichlet energy to perturb away from a plane. One might expect, then, that if u is a minimizer to $\int_{B_1} F(Du)\,dx$ then one can control the Dirichlet energy on the interior in terms of how far u deviates from a plane on the boundary.

This is in fact true. Assume that F is uniformly quasiconvex with bounded second derivatives. By subtracting a plane (which doesn't change minimizers) we may assume that $F(0) = 0$ and $\nabla F(0) = 0$. Let η be a cutoff function between B_r and B_ρ for some $0 < r < \rho < 1$. By uniform quasiconvexity we pay the Dirichlet energy to deform 0 to ηu:

$$\int_{B_r} |Du|^2\,dx \leq C\int_{B_r} F(Du)\,dx + C\int_{B_\rho - B_r} F(D(\eta u))\,dx.$$

On the other hand, since u is a minimizer, the deformation $(1-\eta)u$ has larger energy than u:

$$\int_{B_r} F(Du)\,dx \leq \int_{B_\rho - B_r} [F(D((1-\eta)u)) - F(Du)]\,dx.$$

Combining these inequalities and using that F has bounded second derivatives we obtain

$$\int_{B_r} |Du|^2\,dx \leq C\int_{B_\rho - B_r} |Du|^2\,dx + \frac{C}{(\rho-r)^2}\int_{B_1} |u|^2\,dx.$$

By iterating this inequality we get the usual Caccioppoli inequality. Let

$$a(r) = \int_{B_r} |Du|^2\,dx,$$

and assume by multiplying by a small constant that $\int_{B_1} |u|^2 < \epsilon$ small. For an increasing sequence r_k starting at $r_0 = \frac{1}{2}$ the above inequality can be written

$$a(r_{k+1}) \geq (1+\delta)a(r_k) - \frac{\epsilon}{(r_{k+1}-r_k)^2}.$$

We claim that if $a(r_0) = 1$ then for ϵ small we can choose $r_k \leq 1$ so that $a(r_k)$ increases geometrically, contradicting that u is H^1 and giving the usual Caccioppoli inequality. Indeed, we have

$$a(r_{k+1}) \geq (1+\delta/2)a(r_k) + \left(\frac{\delta}{2}a(r_k) - \frac{\epsilon}{(r_{k+1}-r_k)^2}\right).$$

Define the radii by $(r_{k+1}-r_k)^2 = \frac{2\epsilon}{\delta a(r_k)}$. Then one proves inductively that $a(r_k) \geq (1+\delta/2)^k$, so the radii decrease geometrically and are all less than 1 if ϵ is small.

3.7 Counterexamples

The general question is whether minimizers of

$$\int F(Du)$$

for F uniformly convex, with quadratic growth at infinity (guaranteeing Cacciopoli inequality) are regular. The Euler Lagrange equation is

$$\partial_j(F_{\beta j}(Du)) = 0,$$

which upon passing a derivative gives

$$F_{\beta j,\alpha i}(Du)(u_k)_i^\alpha = 0,$$

an equation of the form

$$\partial_j(a_{\alpha\beta}^{ij}(x)v_i^\alpha) = 0$$

for some $a_{\alpha\beta}^{ij}$ making a strictly quadratic functional without any regularity apriori. Once $v = Du$ is continuous, we are in business by the linear theory. However, this is in general not true.

Example 1 (De Giorgi): We search for a simple homogeneous map fixing ∂B_1 (so it's radial) which minimizes a functional of the desired type. Consider

$$u = \frac{x}{|x|^{1+\gamma}}$$

for γ small. First, note that $|Du|^2|_{\partial B_\epsilon} = C\epsilon^{-2\gamma-2}$, grows slower than $|B_\epsilon|$ decays only if $n \geq 3$, so we need $n \geq 3$ for this map to be H^1. Second, since u is radial with homogeneity $-\gamma$, in polar coordinates it is easy to compute

$$Du = r^{-\gamma-1}\text{diag}(1, 1, ..., 1, -\gamma).$$

It is then obvious that in these coordinates, taking $A = \mathrm{diag}(\gamma/(n-1), ..., \gamma/(n-1), 1)$ the combination of derivatives

$$tr(ADu) = 0.$$

It is thus reasonable to guess that u is a minimizer of the functional

$$\int_{B_1} (tr(ADu))^2 \, dx$$

over all maps which are the identity on ∂B_1. (To do this, check that the Euler-Lagrange equation is satisfied away from 0 and use that $u \in H^1$ to take care of behavior near 0). Since this functional is not uniformly convex, just make a small modification: minimize

$$\int_{B_1} (tr(ADu))^2 + \delta |Du|^2.$$

One checks that $u = \frac{x}{|x|^{1+\sigma}}$ is a minimizer for appropriate $\sigma(\gamma)$. (A remains the same).

Philosophically, the reason we don't have regularity is that the maximum principle doens't hold for systems. If u maps onto a line, the key point is that if we cut u off at a point (which is itself a minimizer) then the energy will decrease; u cannot map outside of what the boundary maps to. This suggests that for regularity of systems, if we could say energy decreased when we "project maps" to lower dimensional subspaces we might be able to say something.

Example 2 (Giusti, Giaquinta): A natural question is whether the dependence of the $a^{ij}_{\alpha\beta}$ on certain quantities involving u is special and might give regularity. This is not the case. Take

$$u = \frac{x}{|x|}$$

the radially constant function and do a small modification of what we did before to get coefficients depending on u.

Example 3 (Necas): The real question is whether we can find counterexamples which are actually derivatives of minimizers of $\int F(Du)$ for uniformly convex, quadratically growing F. (The previous ones constructed $a^{ij}_{\alpha\beta}$ but these don't necessarily come from the initial problem.) For this, we look for a homogeneous degree 1 function (so the derivative has some discontinuity at 0). Take

$$u(x) = \frac{x \otimes x}{|x|}$$

from $\mathbb{R}^n$ to $\mathbb{R}^{n^2}$. For n large, Necas constructed explicitly a convex smooth F so that u solves the Euler-Lagrange system. By modifying these techniques Hao, Leonardi and Necas showed that

$$u(x) = \frac{x \otimes x}{|x|} - \frac{|x|}{n} I$$

is a counterexample for $n = 5$.

Example 4 (Sverak, Yan): The above example takes values that are symmetric and traceless, so the target in fact has dimension $n(n+1)/2 - 1$. The best example to date is due to Sverak and Yan, who show that u is a minimizer for $n = 3$, $m = 5$. They do this by constructing a quadratic null Lagrangian L such that $\nabla L(Du(\partial B_1)) = \nabla F(Du(\partial B_1))$ for some smooth uniformly convex F on $M^{5\times 3}$. The Euler-Lagrange system is then automatically satisfied.

It is open (stated as a problem in the book of Giaquinta) whether there are examples in lower dimensional targets ($\mathbb{R}^3$ to $\mathbb{R}^3$ or $\mathbb{R}^3$ to $\mathbb{R}^2$).

Remark 15. *If the domain is $\mathbb{R}^2$ we actually have regularity by a reverse Hölder inequality (see the next section) which says that minimizers to uniformly convex functionals are $W^{2,2+\delta}$. In particular, in two dimensions minimizers are $C^{1,\alpha}$ and in dimensions $n \leq 4$ minimizers are Hölder continuous. Using similar techniques to those described above, Sverak and Yan constructed unbounded minimizers in dimension $n \geq 5$ of the form $u/|x|^\gamma$ (the smallest dimension where such examples are possible), and a non-Lipschitz (but Hölder) example with $n = 4$, $m = 3$ coming from the Hopf fibration.*

Remark 16. *Without F being uniformly convex etc. we don't have the Caccioppoli inequality for the derivatives, so one cannot do the reverse Holder theory. In addition, solutions to the Euler-Lagrange equations are not necessarily minimizers. Remarkably, Muller and Sverak produced a quasiconvex F and a solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ to the corresponding Euler-Lagrange equation such that u is Lipschitz but not C^1 in any open set. (Compare to the result of Evans which says that minimizers to uniformly quasiconvex functionals are smooth away from a closed set of measure zero).*

3.8 Reverse Hölder Inequality

If $\mathbf{u}$ minimizes $\int_{B_1} F(D\mathbf{u})\,dx$ where F is uniformly convex with bounded second derivatives, then by passing a derivative into the Euler-Lagrange equation we see that $\mathbf{v} = \mathbf{u}_e$ has a Caccioppoli inequality:

$$\int_{B_{1/2}} |D\mathbf{v}|^2\,dx \leq C \int_{B_1} |\mathbf{v} - \mathbf{v}_{B_1}|^2\,dx.$$

Let $f = |D\mathbf{v}|^2$ and $\gamma = 2_*/2 < 1$. The Sobolev inequality gives

$$\frac{1}{|B_r|}\int_{B_r} f\,dx \leq C\left(\frac{1}{|B_r|}\int_{B_{2r}} f^\gamma\,dx\right)^{1/\gamma}.$$

this inequality in fact guarantees that $f \in L^{1+\delta}$, i.e. $\mathbf{u} \in W^{2,2+\delta}$, for some $\delta(n, C, \gamma)$.

As a consequence minimizers are always smooth in the case $n = 2$, a classical result, and minimizers are Hölder continuous in dimensions $n \leq 4$.

We prove this result below.

Theorem 11. *Assume that*

$$\left(\frac{1}{|B_r|}\int_{B_r(x_0)} f^p\right)^{\frac{1}{p}} \leq \frac{C}{|B_{2r}|}\int_{B_{2r(x_0)}} f$$

for some $f \geq 0$, and all $B_{2r(x_0)} \subset B_1$. Then there exists δ small depending on C, n, p such that

$$\left(\int_{B_{1/2}} f^{p+\delta}\right)^{\frac{1}{p+\delta}} \leq \tilde{C}\int_{B_1} f.$$

The rough intuition of this theorem is as follows: if $f \in L^p$ but not in $L^{p+\delta}$ for very small δ, then the distribution function of f decays not much faster than t^{-p}, and we expect that f is very large in L^p.

The key point is that the hypothesis prevents f from having sudden jumps, we must pay in measure to go from 0 to 1. Indeed, if f was 1 on some set and 0 otherwise, take a small ball in which $\{f = 1\}$ has density δ, and the inequality becomes

$$\delta^{\frac{1}{p}} \leq C\delta,$$

a contradiction for δ small. To do this theorem rigorously, we must use this at all scales.

The basic inequality is as follows. Suppose that

$$c\delta \leq \int_{B_1} f \leq \delta$$

for δ previously chosen (can arrange by expanding the "correct balls" around points in $\{f > 1\}$ and assuming $\int f < \epsilon$ small).

The hypothesis gives that

$$\int_{\{f \geq 1\}} f^p \leq \left(\int_{\{f \geq 1\}} f \right)^p \leq C\delta^p << \delta.$$

It follows from this and the previous inequality that

$$\int_{\{\delta/2 < f < 1\}} f^p \geq c(\delta) \int_{B_1} f^p.$$

One rewrites this as

$$\frac{1}{1+c} \int_{B_1} f^p \geq \int_{\{f>1\}} f^p,$$

which upon rescaling and multiplying by constants gives that

$$|\{f > t\}| < ct^{-(p+\tilde{\delta})}.$$

(Of course, we need to do a sort of localization and covering argument to make this rigorous).

3.9 Special Structure: Radial

If we look for minimizers of functionals with convex integrands with certain special structure, we can get everywhere regularity. In particular, if $F(Du) = G(|Du|^2/2)$ then we get a maximum principle for minimizers and their gradients.

Indeed, the Euler-Lagrange equation is

$$\partial_i \left(G' \delta_{ij} u_j^\alpha \right) = 0$$

so $u \cdot e$ has a maximum principle for any $e \in \mathbb{R}^m$. Geometrically, if the graph of u is Σ and the boundary of Σ is one one side of a hyperplane, then it can't cross this hyperplane. If it did, we could project to the hyperplane, decreasing $|Du|$ and thus the energy.

Passing a derivative we get

$$\partial_i \left(G' \delta_{ij} u^\alpha_{kj} + G'' u^\alpha_i u^\beta_j u^\beta_{kj} \right) = 0.$$

Let $v = |Du|^2/2$. Multiplying the equation by u^α_k and summing we obtain

$$\partial_i ((G'\delta_{ij} + G'' u^\alpha_i u^\alpha_j) \, v_j) = (G' \delta^{ij}_{\alpha\beta} + G'' u^\alpha_i u^\beta_j) u^\alpha_{ki} u^\beta_{kj}.$$

Since $F(M) = G(|M|^2/2)$ is uniformly convex, this equation is of the form

$$\partial_i (a^{ij} v_j) \geq c |D^2 u|^2.$$

Multiplying by the fundamental solution cut-off we get

$$r^{2-n} \int_{B_r} |D^2 u|^2 \, dx \to 0$$

as $r \to 0$, so the gradients are close to constant in average, and a blow-up argument gives everywhere regularity.

The point is that the gradients satisfy a very good differential inequality which gives oscillation decay in average. It would be interesting to find a geometric way to understand this. In the case $m = 1$, there is a comparison principle and translation invariance (for any convex integrand), and comparing a minimizer with translations of itself shows that the gradients have a maximum principle. In a similar spirit, sup-convolution produces a subsolution, and if the gradient has a maximum we get a competitor that shouldn't exist. So a natural question is, if F is radial, can we produce comparison hypersurfaces by a higher-codimension version of sup-convolution?

Remark 17. *In the scalar case the following computation is useful. Let G be a function on $\mathbb{R}^n$. Start by differentiating the Euler-Lagrange equation:*

$$\partial_i (F_{ij}(\nabla u) u_{kj}) = 0.$$

Multiply by $G_k(\nabla u)$ and sum over k to obtain

$$\partial_i (F_{ij}(\nabla u) \partial_j (G(\nabla u))) = tr(D^2 F \cdot D^2 u \cdot D^2 G \cdot D^2 u) \geq c |D^2 u|^2$$

for any uniformly convex function of the gradient (e.g. the Legendre transform F^*).

3.10 Quasiconformal Mappings

If a mapping $u : \mathbb{R}^2 \to \mathbb{R}^2$ has components which are the components of a holomorphic function, then

$$|Du|^2 = 2 \det Du.$$

In particular, to first order, these mappings take circles to circles (the eigenvalues of Du are the same). Here we describe a generalization, known as quasiconformal mappings, which locally take cirlces to ellipses. We say u is K-quasiconformal if

$$|Du|^2 \leq 2K \det Du.$$

It is clear that $K \geq 1$ (or else $Du = 0$) and if $K = 1$ then u is analytic. The geometric interpretation is that $Du(x)$ takes circles to ellipses of eccentricity bounded below by

$$\alpha := K - \sqrt{K^2 - 1} = K\left(\frac{1}{2K^2} + \frac{1}{8K^4} + ...\right) \geq \frac{1}{2K}.$$

Indeed, if $Du(x) = \text{diag}(1, \alpha)$ is K-quasiconformal for $\alpha < 1$ we compute

$$1 + \alpha^2 = 2K\alpha$$

and solve for α.

Quasiconformal maps arise as the gradient maps for solutions to nondivergence uniformly elliptic PDE in 2 dimensions. The quasiconformal inequality is invariant under rotations, so we may assume that at a point our equation satisfies

$$au_{xx} + u_{yy} = 0,$$

with $1 \leq a \leq \Lambda$. Using this equation we have

$$u_{xx}^2 + u_{xy}^2 \leq au_{xx}^2 + u_{xy}^2 = -\det D^2u,$$
$$u_{xy}^2 + u_{yy}^2 \leq au_{xy}^2 + u_{yy}^2 = -a\det D^2u.$$

Adding the two inequalities we obtain

$$|D^2u|^2 \leq -2\left(\frac{1+a}{2}\right)\det D^2u,$$

giving that ∇u is a $\frac{1+\Lambda}{2}$-quasiconformal mapping.

The main theorem about quasiconformal mappings is a monotonicity formula, from which α-Hölder regularity follows. Let

$$E_u(r) = \frac{\int_{B_r} |Du|^2\, dx}{r^{2\alpha}}$$

and note that under the rescaling

$$u \to \lambda^{-\alpha}u(\lambda x) := u_\lambda$$

we have

$$E_{u_\lambda}(1) = E_u(\lambda),$$

and in particular E_u is constant for α-homogeneous functions u.

By this scaling invariance, if we show that $E_u'(1) \geq 0$ then we have that E_u is increasing. We compute

$$E_u'(1) = \int_{\partial B_1} |Du|^2\, d\theta - 2\alpha \int_{B_1} |Du|^2\, dx.$$

Using K-quasiconformality and integrating by parts we obtain

$$E_u'(1) \geq \int_{\partial B_1} \left(|Du|^2 - 2(1+\alpha^2)(u^1 - c)(D_T u^2)\right)\, d\theta$$

where T is the unit tangent to the circle and c is any constant. Take c to be the average of u^1 on the circle. Then the Wirtinger inequality says

$$\int_{\partial B_1} (u^1 - c)^2 \, d\theta \leq \int_{\partial B_1} |D_T u^1|^2 \, d\theta$$

(expand in a Fourier series and use Parseval), heuristically the best-case scenario is that the function is sinusoidal with period 2π, otherwise it has more oscillations that make the right side larger. By the basic inequality $a^2 + b^2 \geq 2ab$ and the previous inequality we thus have

$$\begin{aligned} E_u'(1) &\geq \int_{\partial B_1} \left(|Du|^2 - (1+\alpha^2)|D_T u|^2\right) d\theta \\ &= \int_{\partial B_1} \left(|Du \cdot N|^2 - \alpha^2 |Du \cdot T|^2\right) d\theta \end{aligned}$$

where N is the unit normal. Finally, by K-quasiconformality the integrand is positive (the ratio of eigenvalues of Du is at least α), proving the monotonicity.

One immediately concludes that K-quasiconformal mappings are Hölder continuous with $\alpha = K - \sqrt{K^2 - 1}$. This α is in fact optimal, as can be seen by examining the mappings

$$u(x) = |x|^\alpha \frac{x}{|x|}.$$

Indeed, in polar coordinates $Du = \text{diag}(\alpha, 1)$ on the unit circle, and since u is homogeneous of degree α we have

$$Du = r^{\alpha - 1} \text{diag}(\alpha, 1).$$

As a consequence of the above discussion we obtain $C^{1,\beta}$ estimates for solutions to uniformly elliptic PDE in two dimensions for $\beta\left(\frac{\Lambda}{\lambda}\right)$ explicit.

Remark 18. *If we have*

$$a^{11} u_{11} + 2a^{12} u_{12} + u_{22} = 0$$

then by formally differentiating we get

$$\partial_1(a^{11}(u_1)_1) + \partial_1(2a^{12}(u_1)_2) + \partial_2(u_1)_2 = 0$$

showing that in two dimensions the derivatives satisfy a divergence-form elliptic equation. Thus, we may apply the Harnack inequality to get the same regularity result.

Philosophically, the theory of quasiconformal mappings is a quantitative version of the idea that gradient maps of solutions to elliptic PDE in two dimensions are analytic in an ellipsoid geometry. This is (superficially speaking) the perspective taken to show Harnack inequalities for linearized Monge-Ampere.

As a simple toy example to illustrate, the constant in the Harnack inequality grows like $C^{\sqrt{\Lambda/\lambda}}$, which is sharp in view of the example

$$u(x, y) = e^{\sqrt{\Lambda} x} \cos(y).$$

This can be predicted by scaling. If

$$tr(A \cdot D^2 u) = 0$$

for some $I \leq A \leq \Lambda I$ constant then u is harmonic in an ellipsoidal geometry where the eccentricity is bounded below by $\frac{1}{\sqrt{\Lambda}}$. Indeed,

$$\tilde{u}(x) = u(\sqrt{A}x)$$

is harmonic. Thus, the "balls" are

$$|\sqrt{A}x| < 1,$$

an ellipsoid with smallest axis $\frac{1}{\sqrt{\Lambda}}$. By applying the harmonic Harnack inequality $\sqrt{\Lambda}$ times we get the expected dependence of the Harnack constant on Λ.