Partial Regularity for Singular Solutions to the Monge-Ampère Equation

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Abstract

We prove that solutions to the Monge-Ampère inequality

$$\det D^2 u \geq 1$$

in $\mathbb{R}^n$ are strictly convex away from a singular set of Hausdorff $(n-1)$-dimensional measure zero. Furthermore, we show this is optimal by constructing solutions to $\det D^2 u = 1$ with singular set of Hausdorff dimension as close as we like to $n-1$. As a consequence we obtain $W^{2,1}$ regularity for the Monge-Ampère equation with bounded right-hand side and unique continuation for the Monge-Ampère equation with sufficiently regular right-hand side. © 2014 Wiley Periodicals, Inc.

1 Introduction

In this paper we investigate the Hausdorff dimension of the set where Alexandrov solutions (see Section 2 for the precise definition) to

$$\det D^2 u \geq 1$$

are not strictly convex. Recall that we say that a convex function u is strictly convex at x_0 if there exists L_{x_0}, a supporting tangent plane at x_0, such that

$$\{u = L_{x_0}\} = x_0.$$

Our main theorem is:

THEOREM 1.1. *Assume u is an Alexandrov solution to*

$$\det D^2 u \geq 1$$

in $B_1 \subset \mathbb{R}^n$. Then u is strictly convex away from a singular set Σ with

$$\mathcal{H}^{n-1}(\Sigma) = 0.$$

We show this is optimal by constructing solutions to $\det D^2 u = 1$ with singular set of Hausdorff dimension as close as we like to $n-1$. This result is interesting especially for $n \geq 3$ since it is well-known that in two dimensions solutions to $\det D^2 u \geq 1$ are strictly convex.

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Previous results on the singularities of convex functions include those of Alberti, Ambrosio, and Cannarsa (see [1, 2]), who show that the nondifferentiability set of a semiconvex function is $n-1$ rectifiable. Theorem 1.1 may be viewed as a strengthening of this result when we have positive lower *and* upper bounds on $\det D^2u$, in which case Caffarelli's regularity theory (see below) gives differentiability at points of strict convexity. (In fact, if $\det D^2u = 1$ in the Alexandrov sense, then Σ is precisely the set where u fails to be a classical solution.) However, it is important to note that points in Σ may still be points of differentiability for u (see, for example, the Pogorelov solution to $\det D^2u = 1$ below), and without an upper bound on $\det D^2u$ the points of nondifferentiability for u may not be in Σ (take, for example, $u = |x|^2 + |x_n|$, which solves $\det D^2u \geq 1$ and is strictly convex everywhere).

Theorem 1.1 has several applications to the regularity theory for singular solutions to the Monge-Ampère equation with bounded right-hand side, which we now describe.

Caffarelli developed a regularity theory of solutions to

$$\det D^2u = f \quad \text{in } \Omega,\ \lambda \leq f \leq \Lambda,$$

at points where u is strictly convex. We briefly summarize the main results. We define a section of u at x with height h and slope p by

$$S^u_{h,p}(x) = \{y \in \Omega : u(y) < u(x) + p \cdot (y-x) + h\}$$

for some subgradient p at x. If u is strictly convex at x, then we can find a subgradient p such that the supporting plane of this slope touches only at x, and then take h small enough that $S^u_{h,p}(x) \Subset \Omega$. In this setting, Caffarelli [4, 5] showed that

(1) u is strictly convex in $S^u_{h,p}(x)$ and $u \in C^{1,\alpha}_{\text{loc}}(S^u_{h,p}(x))$;
(2) if $f \in C^\alpha(\Omega)$ then $u \in C^{2,\alpha}_{\text{loc}}(S^u_{h,p}(x))$; and
(3) for every $q > 1$ there is some $\epsilon(q) > 0$ such that if $|f-1| < \epsilon$ then $u \in W^{2,q}_{\text{loc}}(S^u_{h,p}(x))$.

However, these regularity theorems fail at points where u is not strictly convex. Consider the well-known Pogorelov examples on $B_1 \subset \mathbb{R}^n$, $n \geq 3$, which degenerate along $x' = (x_1, \ldots, x_{n-1}) = 0$. One constructs these examples by seeking solutions of the form $|x'| + |x'|^\beta g(x_n)$ and $|x'|^\alpha f(x_n)$. The first is

$$|x'| + |x'|^{n/2}(1 + x_n^2),$$

which solves $\lambda \leq \det D^2u \leq \Lambda$ but is merely Lipschitz. The second is

$$|x'|^{2-2/n}(1 + x_n^2),$$

which solves $\det D^2u = f$ with f strictly positive and smooth, but is only $C^{1,\alpha}$ for $\alpha = 1 - \frac{2}{n}$ and $W^{2,p}$ for $p < \frac{n(n-1)}{2}$.

In [6], Caffarelli generalizes these examples to solutions that degenerate along subspaces of any dimension less than $\frac{n}{2}$ and shows that it is not possible to find solutions degenerating on subspaces of dimension $\frac{n}{2}$ or higher. We provide a short proof in the next section (see Lemma 2.3). If u agrees with a linear function L on a k-dimensional set, we say that $\{u = L\}$ is a k-dimensional singularity. Our proof of Theorem 1.1 in fact shows that the collection of k-dimensional singularities has Hausdorff $(n-k)$-dimensional measure zero (see Remark 3.4).

Since we cannot hope for C^1-regularity or $W^{2,p}$-regularity of singular solutions to $\lambda \leq \det D^2u \leq \Lambda$ for large-p, it is natural to ask what we can show about the integrability of the second derivatives. De Philippis, Figalli, and Savin [7, 8] recently showed $W^{2,1+\epsilon}$-regularity of strictly convex solutions to $\lambda \leq \det D^2u \leq \Lambda$, where ϵ depends only on λ, Λ, and n. Our main theorem rules out the possibility that the second derivatives concentrate on Σ:

THEOREM 1.2. *Let u be an Alexandrov solution to*

$$\lambda \leq \det D^2u \leq \Lambda$$

in $B_1 \subset \mathbb{R}^n$. Then $u \in W^{2,1}_{\mathrm{loc}}(B_1)$.

We also show that Theorem 1.2 is optimal by proving that the examples giving optimality of Theorem 1.1 are not in $W^{2,1+\epsilon}$ for ϵ as small as we like.

A second consequence of Theorem 1.1 is that the points of strict convexity for u form a connected set when f is bounded away from 0 (see Lemma 6.3). If f is sufficiently regular, we obtain unique continuation for the Monge-Ampère equation:

THEOREM 1.3. *Assume that u and v are Alexandrov solutions to*

$$\det D^2u = \det D^2v = f$$

in an open connected set $\Omega \subset \mathbb{R}^n$, with $f \in C^{1,\alpha}(\Omega)$ strictly positive. If $u = v$ on an open subset of Ω, then $u \equiv v$ in Ω.

To our knowledge, these are the first Sobolev regularity and unique continuation results for singular solutions to the Monge-Ampère equation.

The paper is organized as follows. In Section 2 we present basic geometric properties of the sections of solutions to $\det D^2u \geq 1$. In particular, we present an important estimate on the volume growth of sections that are not compactly contained and relate the volume of compactly contained sections to the Monge-Ampère mass of these sections. In Section 3 we use these results at singular points together with the useful technique of replacing u by $u + \frac{1}{2}|x|^2$ to prove Theorem 1.1. In Section 4 we construct, for any δ, a solution to $\det D^2u = 1$ with a singular set of Hausdorff dimension $n - 1 - \delta$, which shows that our main theorem is optimal. In Section 5 we use Theorem 1.1 to prove Theorem 1.2, and we show that the examples constructed in Section 4 are not in $W^{2,1+\epsilon}$ for ϵ as small as we like, which shows that $W^{2,1}$ regularity is optimal. Finally, in Section 6 we prove Theorem 1.3 by applying a classical unique continuation theorem in the set of strict convexity.

In future work we intend to present a more precise, quantitative version of our main theorem to obtain $L \log L$ estimates for the second derivatives of singular solutions to $\lambda \leq \det D^2 u \leq \Lambda$.

2 Preliminaries

We first recall the precise definition of Alexandrov solutions. Any convex function $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has an associated Borel measure Mv, called the Monge-Ampère measure, defined by

$$Mv(A) = |\partial v(A)|$$

where $|\partial v(A)|$ represents the Lebesgue measure of the image of the subgradients of v in A (see [10]). (We say $p \in \mathbb{R}^n$ is a subgradient of v at x if it is the slope of some supporting hyperplane to the graph of v at x). If $v \in C^2$, then

$$|\partial v(A)| = \int_A \det D^2 v \, dx.$$

Given a Borel measure μ, we say that v is an Alexandrov solution to

$$\det D^2 v = \mu$$

if $Mv = \mu$.

For a convex function v defined on $\Omega \subset \mathbb{R}^n$, we define a section $S^v_{h,p}(x)$ by

$$S^v_{h,p}(x) = \{y \in \Omega : v(y) < v(x) + p \cdot (y - x) + h\}$$

for some subgradient p at x. We now present some results on the geometry of the sections.

LEMMA 2.1 (John's Lemma). *If $K \subset \mathbb{R}^n$ is a bounded convex set with nonempty interior, and* 0 *is the center of mass of K, then there exists an ellipsoid E and a dimensional constant $C(n)$ such that*

$$E \subset K \subset C(n)E.$$

We call E the John ellipsoid of K. There is some linear transformation A such that $A(B_1) = E$, and we say that A normalizes K.

The next lemma is an important observation about the volume growth of sections that may not be compactly contained in Ω:

LEMMA 2.2. *Assume that* $\det D^2 u \geq 1$ *in a bounded domain $\Omega \subset \mathbb{R}^n$. Then if $S^u_{h,p}(x)$ is any section of u, we have*

$$|S^u_{h,p}(x)| \leq C h^{n/2}$$

for some constant C depending only on n.

PROOF. Assume by translation that 0 is the center of mass of $S^u_{h,p}(x)$. By subtracting a linear function we can assume that

$$p = 0, \quad u|_{\partial S^u_{h,0}(x)} \le 0, \quad \text{and} \quad |\min_{S^u_{h,0}(x)} u| = h.$$

By John's lemma, there is a linear transformation A that normalizes $S^u_{h,0}(x)$. Let

$$\widetilde{u}(x) = |\det A|^{-2/n} u(Ax).$$

It is easy to check that

$$\det D^2\widetilde{u} \ge 1, \quad \widetilde{u}|_{\partial\widetilde{\Omega}} \le 0,$$

where $B_1 \subset \widetilde{\Omega} \subset B_{C(n)}$. Then $\frac{1}{2}(|x|^2 - 1)$ is an upper barrier for $\widetilde{u}$, so

$$|\min_{\widetilde{\Omega}} \widetilde{u}| \ge \frac{1}{2}.$$

Since $|\det A| \ge c(n)|S^u_{h,0}(x)|$, the conclusion follows. □

Caffarelli proved the next proposition in [6]. We provide a short proof using a technique related to our proof of the main theorem.

LEMMA 2.3. *Assume*

$$\det D^2 u \ge 1$$

in $B_1 \subset \mathbb{R}^n$. Then u cannot vanish on a subspace of dimension $\frac{n}{2}$ or higher.

PROOF. Suppose u vanishes on

$$\{x_{k+1} = \cdots = x_n = 0\} \cap B_1.$$

By subtracting a linear function of the form $a_{k+1}x_{k+1} + \cdots + a_n x_n$, we may assume that $u(te_n) = o(t)$. Then $S^u_{h,0}(0)$ has length $R(h)h$ in the e_n direction, where $R(h) \to \infty$ as $h \to 0$. Furthermore, $S^u_{h,0}(0)$ has length exceeding $\frac{1}{C}h$ in the $e_{n-k}, \ldots, e_{n-1}$ directions, where C is the Lipschitz constant of u in $B_{1/2}$. Finally, $S^u_{h,0}(0)$ contains the unit ball in the subspace spanned by $\{e_1, \ldots, e_k\}$. We conclude that

$$|S^u_{h,0}(0)| \ge C^{-k} R(h) h^{n-k},$$

which contradicts Lemma 2.2 as $h \to 0$ for $k \ge \frac{n}{2}$. □

Remark 2.4. Lemma 2.3 implies in particular that every solution to $\det D^2 u \ge 1$ in two dimensions is strictly convex. Furthermore, it follows easily that any solution to $\det D^2 u \ge 1$ on some domain in $\mathbb{R}^n$ cannot agree with a linear function l on any set of affine dimension $k \ge \frac{n}{2}$. Indeed, if not we could subtract l, find some point in the (k-dimensional) interior of $\{u = 0\}$, translate to 0, and rescale to get into the setting of Lemma 2.3.

We conclude the section with the following variant of Alexandrov's maximum principle. In the following $c(n), C(n)$ denote small and large constants depending only on n, and their values may change from line to line.

LEMMA 2.5. *Let v be any convex function on a bounded domain $\Omega \subset \mathbb{R}^n$ with $v|_{\partial\Omega} = 0$. Then*

$$M v(\Omega)\, |\Omega| \geq c(n) |\min_{\Omega} v|^n.$$

PROOF. By translation assume that the center of mass of Ω is 0. Let A normalize Ω and let

$$\widetilde{v}(x) = (\det A)^{-2/n} v(Ax).$$

Then

$$M\widetilde{v}(\widetilde{\Omega}) = (\det A)^{-1} M v(\Omega) \quad \text{with } B_1 \subset \widetilde{\Omega} \subset B_{C(n)}.$$

The maximum of $|\widetilde{v}|$ is achieved at some point $\widetilde{x} \in \widetilde{\Omega}$. Let K be the function whose graph is the cone generated by $(\widetilde{x}, \widetilde{v}(x))$ and $\partial B_{C(n)}$. By convexity,

$$M\widetilde{v}(\widetilde{\Omega}) \geq |\partial K(\widetilde{x})|.$$

Since $\partial K(\widetilde{x})$ contains a ball of radius at least $c(n)|\min_{\widetilde{\Omega}} \widetilde{v}|$, we have

$$|\partial K(\widetilde{x})| \geq c(n) |\min_{\widetilde{\Omega}} \widetilde{v}|^n \geq c(n) |\det A|^{-2} |\min_{\Omega} v|^n.$$

Finally, $|\det A| \leq C(n)|\Omega|$ so the conclusion follows. □

3 Proof of Theorem 1.1

In this section assume that

$$\det D^2 u \geq 1$$

in $B_1 \subset \mathbb{R}^n$. Fix $x \in \Sigma$ and a subgradient p at x. By translation and subtracting a linear function assume that $x = p = 0$. Then $\{u = 0\}$ contains a line segment of some length l. By Lemma 2.2,

$$|S^u_{h,0}(0)| \leq C(n) h^{n/2}$$

for all $h > 0$.

Letting $v = u + \frac{1}{2}|x|^2$, it follows that

$$|S^v_{h,0}(0)| \leq \frac{C(n)}{l} h^{\frac{n+1}{2}}$$

for all h small. In fact, for any $x_0 \in \Sigma$ and subgradient p_0 to v at x_0, we have

$$|S^v_{h,p_0}(x_0)| < C h^{\frac{n+1}{2}}$$

for some C that may depend on x_0 and p_0. Indeed, p_0 can be written as $p + x_0$ for some subgradient p of u at x_0, and one easily checks that

$$S^v_{h,p_0}(x_0) = S^{u+\frac{1}{2}|x-x_0|^2}_{h,p}(x_0),$$

so by subtracting a linear function with slope p and translating we are in the situation described above.

Theorem 1.1 thus follows from the following more general result:

THEOREM 3.1. *Let v be any convex function on $B_1 \subset \mathbb{R}^n$ with sections $S^v_{h,p}$, and let Σ_v denote the set of points x such that for all subgradients p at x, there is some $C_{x,p}$ such that*

$$|S^v_{h,p}(x)| < C_{x,p} h^{\frac{n+1}{2}}$$

for all h small. Then

$$\mathcal{H}^{n-1}(\Sigma_v) = 0.$$

PROOF OF THEOREM 1.1. Let $v = u + \frac{1}{2}|x|^2$. By the discussion preceding the statement of Theorem 3.1, $\Sigma \subset \Sigma_v$. The conclusion follows from Theorem 3.1. □

We briefly discuss the main ideas of the proof. Fix $x \in \Sigma_v$ and a subgradient p at x. In the following analysis c, C will denote small and large constants depending on n and $C_{x,p}$. If $S^v_{h,p}(x) \Subset B_1$ then the definition of Σ_v and Lemma 2.5 give

$$Mv(S^v_h(x)) \geq ch^{\frac{n-1}{2}} = c(h^{1/2})^{n-1} \tag{3.1}$$

for all h small.

An important technique of the proof is to replace v by $v + \frac{1}{2}|x|^2$. Since adding a quadratic can only decrease section volume, we have

$$\Sigma_v \subset \Sigma_{v+\frac{1}{2}|x|^2},$$

and it suffices to prove Theorem 3.1 for this case. Then all of the sections are compactly contained in B_1 for h small, and the diameter of sections is at most $h^{1/2}$. By replacing the sections $S^v_{h,p}(x)$ by $B_{\sqrt{h}}(x)$ and using a covering argument, we easily obtain that Σ_v has Hausdorff dimension at most $n-1$.

Lemmas 3.2 and 3.3 improve this result as follows. We aim to rule out behavior like

$$|x|^2 + |x_n|,$$

which has a singular hyperplane. For this example, the sections at $\{x_n = 0\}$ have the correct growth when we take supporting slopes with no x_n-component, but the sections are too large when we take supporting slopes with x_n-component 1.

In the first lemma we use that the sections are small for *all* supporting planes at $x \in \Sigma_v$ to show that v must grow much faster than quadratically in at least two directions, unlike the example above:

LEMMA 3.2. *Assume that $v = v_0 + \frac{1}{2}|x|^2$ for some convex function v_0. Fix $x \in \Sigma_v$. For a supporting slope p of v at x, let*

$$d_1(h) \geq d_2(h) \geq \cdots \geq d_n(h)$$

denote the axis lengths of the John ellipsoid of the section $S^v_{h,p}(x)$. Then

$$\frac{d_{n-1}(h)}{h^{1/2}} \to 0 \quad \text{as } h \to 0.$$

In the second lemma we use the above observation about the Monge-Ampère mass of v (inequality (3.1)) in the directions where v grows much faster than quadratically from x. Since we replaced v by $v + \frac{1}{2}|x|^2$, we also know that v grows at least quadratically in the remaining directions. This allows us to cover Σ_v with balls in which the Monge-Ampère mass of v is much larger than the radius to the $n-1$, giving the desired improvement.

LEMMA 3.3. *Assume that $v = v_0 + \frac{1}{2}|x|^2$ for some convex function v_0. Fix $x \in \Sigma_v$. For any $\epsilon > 0$, there is a sequence $r_k \to 0$ such that*

$$Mv(B_{r_k}(x)) > \frac{1}{\epsilon} r_k^{n-1}.$$

The proof of Theorem 3.1 follows easily from Lemmas 3.2 and 3.3.

PROOF OF THEOREM 3.1. Since $\Sigma_v \subset \Sigma_{v+(1/2)|x|^2}$, we may assume without loss of generality that v has the form $v_0 + \frac{1}{2}|x|^2$ with v_0 convex.

Fix ϵ small. By Lemma 3.3, for each $x \in \Sigma_v$ we can choose an arbitrarily small r such that

$$Mv(B_r(x)) > \frac{1}{\epsilon} r^{n-1}.$$

Cover $\Sigma_v \cap B_{1/2}$ with such balls and choose a Vitali subcover $\{B_{r_i}(x_i)\}_{i=1}^N$, i.e., a disjoint subcollection such that $B_{3r_i}(x_i)$ cover $\Sigma_v \cap B_{1/2}$. Then

$$\begin{aligned}\sum_{i=1}^N (3r_i)^{n-1} &\leq C\epsilon \sum_{i=1}^N Mv(B_{r_i}(x_i)) \\ &\leq C\epsilon,\end{aligned}$$

since v is locally Lipschitz and the B_{r_i} are disjoint. This means exactly that

$$\mathcal{H}^{n-1}(\Sigma_v \cap B_{1/2}) = 0.$$

The above reasoning also gives $\mathcal{H}^{n-1}(\Sigma_v \cap B_{1-\beta}) = 0$ for any β small, but not necessarily for $\beta = 0$, since we only know v is locally Lipschitz. To get

$$\mathcal{H}^{n-1}(\Sigma_v \cap B_1) = 0,$$

use that $\Sigma_v \cap B_1 = \bigcup_{k=1}^\infty \{\Sigma_v \cap B_{1-1/k}\}$ and apply countable subadditivity. □

We now prove Lemmas 3.2 and 3.3.

PROOF OF LEMMA 3.2. By translating and subtracting a linear function, assume that $x = p = 0$. Assume by way of contradiction that we can find $h_k \to 0$ and some $\delta > 0$ such that

$$d_{n-1}(h_k) > \delta h_k^{1/2} \tag{3.2}$$

for all k. We first show that v is trapped by two tangent planes at 0.

Let $x_{1,k}$ and $x_{2,k}$ be the points on $\partial S^v_{h_k,0}(0)$ where the hyperplanes perpendicular to the shortest axis of the John ellipsoid become tangent to $\partial S^v_{h_k,0}(0)$, and let $p_{1,k}$ and $p_{2,k}$ denote subgradients at these points. Since

$$d_1(h_k)d_2(h_k)\cdots d_n(h_k) < Ch_k^{\frac{n+1}{2}},$$

we have by the inequality (3.2) that $d_n(h_k) < \frac{C}{\delta^{n-1}}h_k$ for all k. By this observation and convexity we can rotate and pass to a subsequence such that

$$p_{1,k} \to c_1(\delta)e_n, \quad p_{2,k} \to -c_2(\delta)e_n.$$

Then v is trapped by the planes $\pm c(\delta)x_n$. We conclude that

$$S^v_{h_k,0}(0) \subset \{|x_n| < C(\delta)h_k\}.$$

To complete the proof, we show that the volumes of sections obtained with tilted supporting planes are too large. Take the largest a such that $v \geq ax_n$, and consider the sections

$$S_k = S^v_{(1+aC(\delta))h_k,ae_n}(0).$$

Then S_k engulf $S^v_{h_k,0}(0)$. Furthermore,

$$\sup\{|x_n| : x \in S_k\} = R_k h_k,$$

where $R_k \to \infty$ as $k \to \infty$. Indeed, if not, then for some small ϵ and a sequence $b_i \to 0$, we would have $v(x', b_i) > (a+\epsilon)b_i$ for all x'. Convexity and $v(0) = 0$ imply that $v > (a+\epsilon)x_n$ for all $x_n > b_i$, which in turn implies that

$$v > (a+\epsilon)x_n,$$

contradicting the definition of a.

Finally, let $x_k = (x'_k, R_k h_k) \in S_k$ be the point in S_k furthest in the e_n direction. Since v grows at least quadratically away from every tangent plane, we have

$$|x'_k| < C(\delta, a)h_k^{1/2}. \tag{3.3}$$

Explicitly, since ae_n is a subgradient at 0 and v is of the form $v_0 + \frac{1}{2}|x|^2$ with v_0 convex, we have that ae_n is a subgradient of v_0 at 0, giving

$$ax_n + \frac{1}{2}|x|^2 \leq v \leq C(\delta, a)h_k + ax_n \quad \text{in } S_k,$$

giving the desired bound on $|x'_k|$.

Recall that

$$S^v_{h_k,0}(0) \subset \{|x_n| < C(\delta)h_k\} \cap B_{h_k^{1/2}}(0).$$

Take any two points y, z in $\{|x_n| < C(\delta)h_k\} \cap B_{h_k^{1/2}}(0)$ a distance $\delta h_k^{1/2}$ apart, take the lines from these points to $(x'_k, R_k h_k)$, and denote the intersections of these lines with $\{x_n = C(\delta)h_k\}$ by $\tilde{y}$ and $\tilde{z}$. Since $|y_n - z_n| < Ch_k$ and $|y - z| > \delta h_k^{1/2}$,

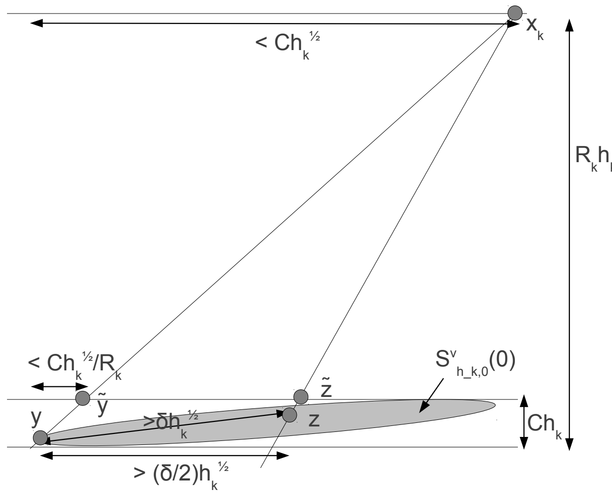

FIGURE 3.1. The cone above $\{x_n = Ch_k\}$ generated by x_k and the John ellipsoid of $S^v_{h_k,0}(0)$ has a base containing a ball of radius at least $(\delta/2 - C(a,\delta)/R_k)h_k^{1/2}$.

it is obvious that $|y' - z'| > \frac{\delta}{2}h_k^{1/2}$ for k large. By similar triangles and inequality (3.3), we also have

$$|y' - \tilde{y}'| = \frac{C}{R_k}|y' - x_k'| \leq \frac{C}{R_k}h_k^{1/2},$$

and we have the same bound on $|z' - \tilde{z}'|$ (see Figure 3.1). We conclude that

$$|\tilde{y} - \tilde{z}| \geq |y' - z'| - |y' - \tilde{y}'| - |z' - \tilde{z}'| \geq (\delta/2 - C/R_k)h_k^{1/2}. \tag{3.4}$$

Since $d_i(h_k) > \delta h_k^{1/2}$ for all $i \leq n-1$, inequality (3.4) (applied to the center of the John ellipsoid for $S^v_{h_k,0}(0)$ and the $(n-1)$-dimensional ball of radius δh_k it contains) implies that S_k contains the cone with vertex $(x_k', R_k h_k)$ and base containing a ball of radius $(\delta/2 - C(a,\delta)/R_k)h_k^{1/2}$ on the hyperplane $\{x_n = C(\delta)h_k\}$.

We conclude that

$$|S_k| \geq c(\delta, a)R_k h_k^{\frac{n+1}{2}},$$

contradicting our definition of Σ_v for k large. □

PROOF OF LEMMA 3.3. Fix a subgradient p at x and let $d_1(h), d_2(h), \ldots, d_n(h)$ be defined as in the statement of Lemma 3.2. Let

$$I = \min\left\{i : \frac{d_i(h)}{h^{1/2}} \to 0 \text{ as } h \to 0\right\}.$$

Fix δ small. Then we can find a sequence $h_k \to 0$ and η depending only on p such that

$$d_I(h_k) < \delta h_k^{1/2} \tag{3.5}$$

and

$$d_i(h_k) > \eta h_k^{1/2} \tag{3.6}$$

for all $i < I$. Rotate the axes so that the e_i are the axes for the John ellipsoid of $S^v_{h_k,p}(x)$ and assume by translation that $x = 0$.

Take the restriction of v to the subspace spanned by $e_I, \dots, e_n$, and call this restriction w. Let

$$S_k^w = S^v_{h_k,p}(x) \cap \{x_1 = \cdots = x_{I-1} = 0\},$$

the slice of the section $S^v_{h_k,p}(x)$ in this subspace. Then since

$$d_1(h_k)d_2(h_k)\cdots d_n(h_k) \leq C h_k^{\frac{n+1}{2}}$$

and v grows at most quadratically in the first $I-1$ directions (inequality (3.6)), we have

$$|S_k^w|_{\mathcal{H}^{n-I+1}} \leq \frac{C}{\eta^{I-1}} h_k^{\frac{n+2-I}{2}}.$$

Using this and Lemma 2.5,

$$Mw(S_k^w) \geq c\eta^{I-1} h_k^{\frac{n-I}{2}}. \tag{3.7}$$

Finally, let $r_k = C(n)d_I(h_k)$, with $C(n)$ taken large enough that

$$S_k^w \subset B_{r_k/2}(x).$$

By strict quadratic growth in all directions, $\partial v(B_{r_k}(x))$ contains a ball of radius $r_k/2$ around every point in $\partial v(S_k^w)$. It follows that

$$\begin{aligned} Mv(B_{r_k}(x)) \geq c(n)Mw(S_k^w)r_k^{I-1} &\geq c h_k^{\frac{n-I}{2}} r_k^{I-1} \text{ (inequality (3.7))} \\ &\geq \frac{c}{\delta^{n-I}} r_k^{n-1} \text{ (inequality (3.5))}. \end{aligned}$$

By Lemma 3.2 we have $I \leq n-1$, so the conclusion follows. □

Remark 3.4. Replacing Σ_v with the set Σ_v^k of points such that

$$|S^v_{h,p}| < C_{x,p} h^{\frac{n+k}{2}}$$

for all h small ($1 \leq k \leq n-1$) and replacing 1 with k in the preceding, one obtains that $\mathcal{H}^{n-k}(\Sigma_v^k) = 0$. If $\det D^2u \geq 1$, such growth happens for $v = u + \frac{1}{2}|x|^2$ at points where u agrees with a linear function on a k-dimensional subspace. This shows that the Hausdorff dimension of the k-dimensional singularities is at most $n-k$. In particular, we recover Lemma 2.3 since for $k \geq \frac{n}{2}$ we would have a k-dimensional singularity with Hausdorff k-dimensional measure 0.

4 Examples

In this section we construct examples of solutions to $\det D^2u = 1$ in $\mathbb{R}^3$ such that Σ has Hausdorff dimension as close to 2 as we like. A small modification produces the analagous examples in $\mathbb{R}^n$.

For this section, fix $\delta > 0$ small. We construct our examples in several steps, which we briefly describe:

(1) First, we construct functions w with

$$\det D^2 w \geq 1$$

in $\mathbb{R}^3$ that degenerate along $\{x_1 = x_2 = 0\}$ and behave like $x_1^{2-\delta}$ along the x_1-axis.

(2) Next, we construct a standard $S \subset [-1,1]$ with Hausdorff dimension close to 1 and a convex function v on $[-1,1]$ such that for any $x \in S$, there is a tangent line such that v separates from this line faster than $r^{2-\delta}$.

(3) Finally, we get our example by solving the Dirichlet problem

$$\det D^2u = 1 \quad \text{in } \Omega = \{|x'| < 1\} \times [-1,1], \;\; u|_{\partial\Omega} = C(\delta)(v(x_1) + |x_2|),$$

and comparing with w at points in $S \times \{0\} \times \{\pm 1\}$.

In the following analysis c and C will denote small and large constants depending on δ.

CONSTRUCTION OF w. We look for a convex function $w(x_1, x_2, x_3)$ with the homogeneity

$$w(x_1, x_2, x_3) = \frac{1}{\lambda} h(\lambda^{1/\alpha} x_1, \lambda^{1/\beta} x_2)(1 + x_3^2),$$

where α and β satisfy $1 < \alpha, \beta < 2$ and

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{2}.$$

(It is easy to check that $\geq \frac{3}{2}$ is necessary for such a function to have $\det D^2 w$ bounded below). Note that this rescaling preserves the curves $x_2 = m x_1^{\alpha/\beta}$.

Let $f(x)$ denote $1 + x^2$. An obvious candidate for w is

$$w(x_1, x_2, x_3) = (x_1^\alpha + x_2^\beta) f(x_3).$$

One checks that

$$\begin{aligned} \det D^2 w = {} & |x_1|^{2\alpha-2}|x_2|^{\beta-2}\big(2\alpha\beta(\alpha-1)(\beta-1)f^2 - 4\alpha^2\beta(\beta-1)f x_3^2\big) \\ & + |x_1|^{\alpha-2}|x_2|^{2\beta-2}\big(2\alpha\beta(\alpha-1)(\beta-1)f^2 - 4\alpha\beta^2(\alpha-1)f x_3^2\big). \end{aligned}$$

Take $\alpha = 2 - \delta$. Then for $|x_3|$ small depending on δ we have

$$\det D^2 w \geq c(\delta)(|x_1|^{2\alpha-2}|x_2|^{\beta-2} + |x_1|^{\alpha-2}|x_2|^{2\beta-2}).$$

Along the curves $x_2 = m x_1^{\alpha/\beta}$, we compute

$$\det D^2 w \geq c(\delta)(|m|^{\beta-2} + |m|^{2\beta-2}) \geq c(\delta),$$

since $1 < \beta < 2$.

Thus, up to rescaling the x_3-axis and multiplying by a constant, we have

$$\det D^2 w \geq 1 \quad \text{in } \Omega = \{|x'| < 1\} \times [-1, 1].$$

CONSTRUCTION OF S. Let $\epsilon > 0$ be a small constant we will choose shortly depending on δ. Construct a self-similar set in $[-\frac{1}{2}, \frac{1}{2}]$ as follows: First, remove an open interval of length $\gamma = 1 - 2^{-3\epsilon}$ from the center. Proceed inductively by removing intervals a fraction γ of each of those that remains. Denote the centers of the intervals removed at stage k by $\{x_{i,k}\}_{i=1}^{2^{k-1}}$ and the intervals by $I_{i,k}$. Finally, let

$$S = [-\tfrac{1}{2}, \tfrac{1}{2}] - \textstyle\bigcup_{i,k} I_{i,k}.$$

It is easy to check that $|I_{i,k+1}| = \gamma 2^{-(1+3\epsilon)k}$ and that S has Hausdorff dimension $\frac{1}{1+3\epsilon}$.

CONSTRUCTION OF v. Let

$$v_0(x) = \begin{cases} |x|, & |x| \leq 1, \\ 2|x| - 1, & |x| > 1. \end{cases}$$

We add rescalings of v_0 together to produce the desired function:

$$v(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} 2^{-2(1+2\epsilon)k} v_0(2\gamma^{-1} 2^{(1+3\epsilon)k}(x - x_{i,k})).$$

We now check that v satisfies the desired properties:

(1) v is convex, as the sum of convex functions. Furthermore,

$$|v(x)| \leq C \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} 2^{-(1+\epsilon)k} \leq C \sum_{k=1}^{\infty} 2^{-\epsilon k},$$

so v is bounded.

(2) Let $x \in S$. We aim to show that v separates from a tangent line more than $r^{2-\delta}$ a distance r from x. By subtracting a line assume that $v(x) = 0$ and that 0 is a subgradient at x. Assume further that $x + r < \frac{1}{2}$ and that $2^{-(1+3\epsilon)k} < r \leq 2^{-(1+3\epsilon)(k-1)}$. There are two cases to examine:

Case 1. There is some $y \in (x + r/2, x + r) \cap S$. Then by the construction of S it is easy to see that there is some interval $I_{i,k+2}$ such that $I_{i,k+2} \subset (x, x + r)$. On this interval, v grows by

$$2^{-2(1+2\epsilon)(k+2)} \geq c r^{2\frac{1+2\epsilon}{1+3\epsilon}} = c r^{2-\delta},$$

where we choose ϵ so that

$$\delta = \frac{2\epsilon}{1 + 3\epsilon}.$$

Case 2. Otherwise, there is an interval $I_{i,j}$ of length exceeding $\frac{r}{2}$ such that $(x+\frac{r}{2}, x+r) \subset I_{i,j}$. In particular, $j \leq k+2$. Then at the left point of $I_{i,j}$, the slope of v jumps by at least $2^{-(1+\epsilon)(k+2)}$. It follows that at $x+r$, v is at least

$$\frac{r}{2}2^{-(1+\epsilon)(k+2)} \geq c r^{2-\delta}.$$

Thus, v has the desired properties.

CONSTRUCTION OF u. We recall the following lemma on the solvability of the Monge-Ampère equation (see [10, 11]).

LEMMA 4.1. *If Ω is open, bounded, and convex, μ is a finite Borel measure on Ω, and g is continuous and convex in $\overline{\Omega}$, then there exists a unique convex solution $u \in C(\overline{\Omega})$ to the Dirichlet problem*

$$\det D^2 u = \mu, \quad u|_{\partial\Omega} = g.$$

Let $g(x_1, x_2, x_3) = C(v(x_1) + |x_2|)$ for a constant C that we will choose shortly, and let u solve the Dirichlet problem

$$\det D^2 u = 1 \quad \text{in } \Omega = \{|x'| < 1\} \times [-1, 1], \ u|_{\partial\Omega} = g.$$

Take $z = (z_1, 0, 0)$ for $z_1 \in S$, and let a_z be a subgradient of v at z_1. Let

$$w_z(x) = g(z) + a_z(x_1 - z_1) + w(x - z).$$

Since

$$w(x-z) \leq C_0(|x_1 - z_1|^{2-\delta} + |x_2|^{\beta})$$

for some C_0, we can take C large so that

$$\begin{aligned} g(x_1, x_2, \pm 1) &\geq g(z) + a_z(x_1 - z_1) + C(|x_1 - z_1|^{2-\delta} + |x_2|) \\ &\geq w_z(x_1, x_2, \pm 1) \end{aligned}$$

on the top and bottom of Ω. Furthermore, since g is independent of x_3 and for any fixed x' we know w_z takes its maxima at $(x', \pm 1)$, we have $g \geq w_z$ on all of $\partial\Omega$. Thus, $u \geq w_z$ in all of Ω. Since u takes the value $g(z)$ at $(z_1, 0, \pm 1)$ and $w_z(z_1, 0, x_3) = g(z)$ for all $|x_3| < 1$, we have by convexity that $u = g(z)$ along $(z_1, 0, x_3)$.

We conclude that Σ contains $S \times \{0\} \times (-1, 1)$, which has Hausdorff dimension $1 + \frac{1}{1+3\epsilon} = 2 - \frac{3}{2}\delta$.

Remark 4.2. To get the analagous example in $\mathbb{R}^n$, take

$$u(x_1, x_2, x_3) + x_4^2 + \cdots + x_n^2.$$

Observe that this solution has exactly the behavior described by Lemma 3.2, which says that u must grow faster than quadratically in two directions. In the next section we show that for any ϵ, these examples are not in $W^{2,1+\epsilon}$ for δ small enough.

5 $W^{2,1}$ Regularity

In this section we obtain $W^{2,1}$ regularity for singular solutions to the Monge-Ampère equation. Furthermore, by examining the examples in the previous section we show that we cannot improve this result to $W^{2,1+\epsilon}$ regularity for an ϵ depending on λ, Λ, and n.

The following result of De Philippis, Figalli, and Savin (see [8]) gives $W^{2,1+\epsilon}$ regularity of solutions to $\lambda \leq \det D^2 u \leq \Lambda$ in compactly contained sections:

THEOREM 5.1. *Assume that*

$$\lambda \leq \det D^2 u \leq \Lambda \text{ in } \Omega \quad \text{and} \quad S_h(x) \Subset \Omega.$$

Then $u \in W^{2,1+\epsilon}(S_{h/2}(x))$ for some ϵ depending only on λ, Λ, and n.

$W^{2,1}$ regularity then follows from our main theorem.

PROOF OF THEOREM 1.2. We will show that $u \in W^{2,1}(B_{1/2})$. Local $W^{2,1}$ regularity follows from a standard covering argument.

Theorem 5.1 gives local $W^{2,1}$ regularity on $B_1 - \Sigma$. By Theorem 1.1, for any $\eta > 0$ we can cover $\Sigma \cap B_{1/2}$ by balls $\{B_{r_i}(x_i)\}$ with $r_i < \frac{1}{4}$ such that

$$\sum_{i=1}^{\infty} r_i^{n-1} < \eta.$$

Let $A = \bigcup_{i=1}^{\infty} B_{r_i}(x_i)$. Since u is a convex function, the second derivatives are controlled by Δu. It follows that

$$\int_A \|D^2 u\| dx \leq \int_A \Delta u \, dx \leq \sum_{i=1}^{\infty} \int_{\partial B_{r_i}} u_\nu \, ds$$

$$\leq C \sum_{i=1}^{\infty} r_i^{n-1} \leq C\eta,$$

where C is the Lipschitz constant of u in $B_{3/4}$. This shows that the second derivatives cannot concentrate on Σ. □

We now examine the integrability of Δu for the examples constructed in the previous section. Fix a small δ. We will show that for some ϵ small depending on δ, we have $\Delta u \notin L^{1+\epsilon}$. (Note that this ϵ is not related to the one from the previous section.)

On any ball B_r, by Hölder's inequality we have

$$\int_{B_r} (\Delta u)^{1+\epsilon} \, dx \geq c(n) r^{-\epsilon n} \left(\int_{B_r} \Delta u \, dx \right)^{1+\epsilon}.$$

Recall from the construction in the previous section that at points in Σ, u grows from its tangent plane faster than $x_2^{\beta} = x_2^{1+\frac{\delta}{4-3\delta}}$ in the x_2 direction (at singular points, a translation and modification of w by a linear function touches u from below). It follows that for $x \in \Sigma$ and l_x a tangent plane to u at x, we have

$$\sup_{\partial B_r(x)} (u - l_x) \geq r^{\beta}.$$

Applying convexity, we get

$$\begin{aligned}\int_{B_r(x)} (\Delta u)^{1+\epsilon}\, dx &\geq c(n) r^{-\epsilon n} \left(\int_{\partial B_r} u_\nu \, ds \right)^{1+\epsilon} \\ &\geq c(n) r^{(n+\beta-2)(1+\epsilon)-\epsilon n} \\ &\geq c(n) r^{n-1-\epsilon+(1+\epsilon)\frac{\delta}{3}}.\end{aligned}$$

Fix η small and cover $S \times \{0\} \times (-1,1)^{n-2}$ with balls of radius $r_i < \eta$. Take a Vitali subcover $\{B_{r_i}\}_{i=1}^{\infty}$. It follows that

$$\int_{B_1} (\Delta u)^{1+\epsilon}\, dx \geq c(n) \sum_{i=1}^{\infty} r_i^{n-1-\epsilon+(1+\epsilon)\frac{\delta}{3}}.$$

Taking $\epsilon = 4\delta$ above, we conclude that

$$\int_{B_1} (\Delta u)^{1+\epsilon}\, dx \geq c(n) \sum_{i=1}^{\infty} r_i^{n-1-3\delta},$$

where the expression on the right goes to ∞ as $\eta \to 0$ because the Hausdorff dimension of $S \times \{0\} \times (-1,1)^{n-2}$ is $n - 1 - \frac{3}{2}\delta$. Thus, Δu is not $L^{1+\epsilon}$ for $\epsilon \geq 4\delta$.

Remark 5.2. In future work we intend to present a more precise version of Theorem 1.1 that gives $L \log L$ regularity of second derivatives of singular solutions to

$$\lambda \leq \det D^2 u \leq \Lambda.$$

6 Unique Continuation

Assume that u, v satisfy the hypotheses of Theorem 1.3. For our proof of unique continuation we rely on the following classical unique continuation theorem for linear equations (see [9]):

THEOREM 6.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a connected open set and $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to the equation*

$$\partial_i (a^{ij}(x) u_j) + b^i(x) u_i + c(x) u = 0,$$

where $a^{ij}(x)$ is Lipschitz and uniformly elliptic and $b^i(x), c(x)$ are bounded measurable. If $u = 0$ on some open subset of Ω, then $u \equiv 0$ in Ω.

In [3], the authors use the same theorem to prove unique continuation for fully nonlinear uniformly elliptic equations. As in [3], we note that Theorem 6.1 also applies to classical solutions of nondivergence equations with Lipschitz coefficients, which may be rewritten in the divergence form above. A more general version of this statement, proved using Carleman estimates, can be found in Hörmander's book [12, theorem 17.2.6].

We will apply this result to the difference of u and v, which solves a linear equation where u and v are sufficiently regular. Indeed, suppose u and v are C^2 in a neighborhood of x, and let w_t be the convex combination $tu+(1-t)v$. Let $(W_t)^{ij}$ be the matrix of cofactors for $D^2 w_t$. Then by expanding $0 = \int_0^1 \frac{d}{dt} \det D^2 w_t \, dt$ we get

$$a^{ij}(x)(u-v)_{ij} = 0 \quad \text{where } a^{ij}(x) = \int_0^1 (W_t)^{ij}(x) dt.$$

The regularity theory of Caffarelli [5] allows us to use this observation at points of strict convexity for solutions to the Monge-Ampère equation:

THEOREM 6.2. *Assume*

$$\det D^2 u = f \quad \textit{in } \Omega, \ u|_{\partial\Omega} = 0,$$

where $f \in C^{1,\alpha}(\Omega)$ *is strictly positive. Then*

$$u \in C^{3,\alpha}(\Omega).$$

Finally, we observe that open sets whose complements have zero Hausdorff $(n-1)$-dimensional measure are connected.

LEMMA 6.3. *Assume* $K \subset \mathbb{R}^n$ *is closed, and assume further that* $\mathcal{H}^{n-1}(K) = 0$. *Then* $\mathbb{R}^n - K$ *is pathwise connected.*

PROOF. Assume by way of contradiction that $D = \mathbb{R}^n - K$ is not pathwise connected. Since D is open, by rotation, translation, and scaling we can assume that the points $\pm Re_n \in D$ cannot be connected by any continous path through D and that

$$\{|x'| < 1\} \times \{\pm R\} \subset D.$$

Let K' be the projection of K onto $\{x_n = 0\}$ and let $B_1' = B_1 \cap \{x_n = 0\}$. If $B_1' - K' \neq \varnothing$, this would violate the contradiction hypothesis because then we could find a point $x' \in B_1'$ such that $(x', t) \in D$ for all $t \in \mathbb{R}$ and take our path to be the straight lines from $-Re_n$ to $(x', -R)$ to (x', R) to Re_n.

We conclude that for any cover of K by balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$, we have

$$\sum_i r_i^{n-1} \geq 1,$$

contradicting that $\mathcal{H}^{n-1}(K) = 0$. □

The proof of unique continuation follows easily from these observations and our main theorem.

PROOF OF THEOREM 1.3. Let Σ_u and Σ_v be the singular sets of u and v respectively, and let $A = \Omega - (\Sigma_u \cup \Sigma_v)$. Since A is dense in Ω, it suffices to show that $u = v$ on A.

By Caffarelli's theory [4], A is an open set. Indeed, for $x \in A$ we can find some p in $\mathbb{R}^n$ and h small such that $S^u_{h,p}(x) \Subset \Omega$, and since f is bounded in this section, Caffarelli gives that u is strictly convex in a neighborhood of x. (The same reasoning gives that v is strictly convex in a neighborhood of x.) By Theorem 1.1, the complement of A has Hausdorff $(n-1)$-dimensional measure zero, so by Lemma 6.3, A is connected. By Theorem 6.2, the difference $u - v$ satisfies the linear equation

$$a^{ij}(x)(u-v)_{ij} = 0$$

on A, where a^{ij} are locally uniformly elliptic and $C^{1,\alpha}$ in A. The conclusion follows from Theorem 6.1. □

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