REGULARITY RESULTS FOR THE EQUATION $u_{11}u_{22} = 1$

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Abstract. We study the equation $u_{11}u_{22} = 1$ in $\mathbb{R}^2$. Our results include an interior C^2 estimate, classical solvability of the Dirichlet problem, and the existence of non-quadratic entire solutions. We also construct global singular solutions to the analogous equation in higher dimensions. At the end we state some open questions.

1. Introduction

In this paper we study the equation

$$(u_{11}u_{22})^{1/2} = 1 \tag{1}$$

together with its higher-dimensional versions. We assume $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous and convex when restricted to lines in the coordinate directions. On this class of functions the equation (1) is elliptic and concave, and the ellipticity constants may degenerate when $D^2u \to \infty$. Our interest in (1) comes from the study of interior C^2 estimates for concave equations.

Equation (1) shares several interesting features with the complex Monge-Ampère equation. One is that solutions are not convex. Another is that solutions have two different types of invariances:

$$u(x_1, x_2) + a\,x_1x_2 \quad \text{for any constant } a,$$

and

$$u(\lambda x_1, \lambda^{-1}x_2) \quad \text{for any constant } \lambda \neq 0.$$

Notice that any Hessian $D^2u(x_0)$ can be mapped into I after using these invariances. In some sense equation (1) can be viewed as an interpolation between the Laplace equation and the real Monge-Ampère equation.

There are however some key differences between the equation we consider and the real Monge-Ampère equation. Calabi's theorem states that solutions to the real Monge-Ampère equation are very rigid: the only global solutions are quadratic polynomials. In contrast, there are nontrivial global solutions to (1) which at infinity have subquadratic growth along the axes and superquadratic growth along the diagonals (see Theorem 1.7). Another important difference concerns continuity estimates near $\partial\Omega$. The Dirichlet problem for (1) is well posed if the intersection of Ω with any horizontal or vertical line is a single segment. If we assume that $u = 0$ on $\partial\Omega$ then solutions of (1) do not have any uniform modulus of continuity near the boundary. On the other hand, for the real Monge-Ampère equation, uniform Hölder estimates are a consequence of Alexandrov's estimate.

There are many important nonlinear concave equations for which it is not known whether a Pogorelov-type interior estimate holds (that is, if $u = 0$ on $\partial\Omega$, whether $D^2u(x)$ is bounded in terms of the distance from x to $\partial\Omega$ and the diameter of Ω). Equation (1) can be viewed as a simplified model for such equations. In this work we establish a pure interior C^2 estimate in 2D (see Theorem 1.1). We plan to investigate the higher dimensional case in future work, and this could provide some insight into other similar equations.

We now state our results. The first is an interior a priori C^2 estimate in 2D.

Theorem 1.1. *Assume that $u \in C^4(B_1)$ solves (1) in $B_1 \subset \mathbb{R}^2$. Then*

$$\|u\|_{C^2(B_{1/2})} < C(\|u\|_{L^\infty(B_1)}).$$

The proof of Theorem 1.1 is a maximum principle argument for a second-order quantity. We introduce a cutoff function motivated by the partial Legendre transform, which takes the Monge-Ampère equation to the Laplace equation in $2D$. As a corollary we obtain a Liouville theorem for solutions with quadratic growth.

Corollary 1.2. *Assume that u is a smooth solution to (1) on $\mathbb{R}^2$. If in addition $|u| < C(1 + |x|^2)$ for some constant $C > 0$, then u is a quadratic polynomial.*

Our second result is the classical solvability of the Dirichlet problem. We say that a continuous function w is coordinate-convex on $\mathbb{R}^n$ if w is convex when restricted to lines in the coordinate directions. We say that a smooth function w is uniformly coordinate-convex if $w_{ii} \geq c > 0$ on $\mathbb{R}^n$ for each $i = 1, ..., n$. Finally, we say that a domain $\Omega \subset \mathbb{R}^n$ is uniformly coordinate-convex if Ω is a connected component of $\{w < 0\}$ for some smooth, uniformly coordinate-convex function w. We show:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, uniformly coordinate-convex domain. Let $\varphi \in C^\infty(\mathbb{R}^2)$. Then there exists a unique coordinate-convex solution in $C^\infty(\Omega) \cap C(\overline{\Omega})$ to*

$$(u_{11}u_{22})^{1/2} = 1 \text{ in } \Omega, \quad u|_{\partial\Omega} = \varphi.$$

Remark 1.4. The domain Ω need not be smooth. Consider for example a connected component of $\{|x|^4 + |x|^2 - 4x_1x_2 < 0\} \subset \mathbb{R}^2$.

As a consequence of Theorem 1.1 and Theorem 1.3 we obtain local C^∞ regularity and derivative estimates of all orders for viscosity solutions of (1).

Corollary 1.5. *Let $u \in C(\overline{B_1})$ be a viscosity solution to (1) in $B_1 \subset \mathbb{R}^2$. Then u is in fact smooth, and we have*

$$\|u\|_{C^k(B_{1/2})} \leq C(k, \|u\|_{L^\infty(B_1)}).$$

Remark 1.6. The Dirichlet problem for (1), and its higher dimensional versions, is uniquely solvable in the class of viscosity solutions e.g. when Ω is uniformly convex and the boundary data φ are smooth. This follows from general theory (see [I]).

To obtain solutions to (1) that are smooth up to $\partial\Omega$, it suffices to obtain boundary C^2 estimates. Caffarelli, Nirenberg and Spruck accomplished this for a large class of Hessian equations in [CNS1], [CNS2]. We can hope that the following Caffarelli-Nirenberg-Spruck type result holds: *Let Ω be a uniformly coordinate-convex, C^3 domain, and $\varphi \in C^3(\partial\Omega)$. Then the Dirichlet problem for (1) is uniquely solvable and the solution u is of class $C^{2,\alpha}(\overline{\Omega})$.* However, the boundary C^2 estimate for (1) seems to be tricky, even in two dimensions (see Remark 3.3). To prove Theorem 1.3 we instead use Theorem 1.1 and an approximation method developed by Lions for the real Monge-Ampère equation [L].

Our next theorem concerns global solutions. Results in the spirit of the Calabi theorem that global solutions to $\det D^2u = 1$ in $\mathbb{R}^n$ are quadratic polynomials are closely connected to regularity questions. Interestingly, (1) admits non-quadratic entire solutions.

Theorem 1.7. *There exist non-quadratic entire solutions to (1).*

By Corollary 1.2, any such solution grows super-quadratically at ∞. The solution we construct grows $\sim |x|^2 \log |x|$ along the diagonal lines $x_1^2 = x_2^2$.

Finally, we show that the local regularity results in two dimensions are false for the analogous problem

$$(\Pi_{i=1}^n u_{ii})^{1/n} = 1, \quad u \text{ coordinate-convex} \tag{2}$$

in higher dimensions $n \geq 3$.

Theorem 1.8. *In dimensions $n \geq 3$ there exist global, non-classical viscosity solutions to (2).*

Our example can be viewed as an analogue of the well-known Pogorelov example for the real Monge-Ampère equation. A new difficulty in our case is that (2) is not rotation-invariant. Another difference is that there are no global singular solutions to the real Monge-Ampère equation $\det D^2 u = 1$. There are global singular solutions to the complex Monge-Ampère equation $\det \partial\bar{\partial} u = 1$ in $\mathbb{C}^n$ for all $n \geq 2$ (see e.g. [B]), in contrast with the real case.

We prove each theorem in a separate section below. We delay a standard argument in the proof of Theorem 1.3 to the Appendix. In the last section we state some interesting open problems motivated by this work.

2. Interior C^2 Estimate

In this section we prove Theorem 1.1. We begin by showing that solutions to (1) are strictly convex on horizontal and vertical lines.

Lemma 2.1. *Assume that u is a viscosity solution of (1) in $B_1 \subset \mathbb{R}^2$. Then*

$$u(e_2/2) + u(-e_2/2) - 2u(0) \geq \delta > 0, \tag{3}$$

where δ depends only on $\|u\|_{L^\infty(B_1)}$.

Remark 2.2. This result is special to $2D$; see the example in Section 5.

Proof. Assume by way of contradiction that the lemma is false. Then there exists a uniformly bounded sequence of solutions to (1) that violate (3) for arbitrarily small δ. By coordinate-convexity, this sequence is locally uniformly Lipschitz and thus converges locally uniformly to a bounded viscosity solution v of (1) in B_1. After subtracting a linear function, we may assume that $v(0, x_2) = 0$ for $|x_2| \leq 1/2$.

Now take

$$g_\lambda(x_1, x_2) = \lambda \left(x_1(\log(x_1^{-1}))^{1/2}\right)(4x_2^2 - 1) + \lambda^{-1}x_1,$$

with $\lambda > 0$ small. A short computation gives

$$0 \leq (g_\lambda)_{11}(g_\lambda)_{22} \leq 6\lambda$$

in $R := (0, 1/4) \times (-1/2, 1/2)$. In addition, for λ small we have $g_\lambda \geq v$ on ∂R by the local Lipschitz regularity of v (which follows from coordinate-convexity). By the maximum principle, $v \leq g_\lambda$ in R for λ small. However, $\partial_1 g_\lambda \to -\infty$ near $(0, 0)$, contradicting the coordinate-convexity of v. $\square$

We now prove Theorem 1.1. Lemma 2.1 is used to justify our choice of cutoff function.

Proof of Theorem 1.1: It suffices to show

$$u_{11}(0) < C(\|u\|_{C^1(B_1)}). \tag{4}$$

Indeed, the bounds $\|u_{ii}\|_{L^\infty(B_{3/4})} < C(\|u\|_{L^\infty(B_1)})$ for $i = 1, 2$ follow by the invariance of (1) under $(x_1, x_2) \to (x_2, x_1)$ and a standard covering argument. The equation (1) then becomes uniformly elliptic in $B_{3/4}$, and the full C^2 estimate follows by classical theory of uniformly elliptic PDE in $2D$, or by the concavity of the equation (see Remark 2.4 below).

We may assume after subtracting a linear function that

$$u(0) = u_2(0) = 0. \tag{5}$$

Let

$$Lv = \sum_{i=1}^{2} \frac{v_{ii}}{u_{ii}}$$

denote the linearized equation. By differentiating (1) once we obtain

$$L(u_k) = 0, \quad k = 1, 2. \tag{6}$$

Differentiating (1) twice in the e_1 direction we get

$$L(u_{11}) = \frac{1}{u_{11}^2} u_{111}^2 + u_{11}^2 u_{122}^2. \tag{7}$$

Let η be a C^2 function such that $\eta(0) > 0$ and the connected component $\mathcal{U}$ of $\{\eta > 0\}$ containing the origin is compactly contained in $B_{3/4}$. (We will choose an appropriate η later). Let

$$M := \log u_{11} + \frac{\sigma}{2} u_1^2 + \log \eta,$$

with $\sigma > 0$. Then M attains its maximum in $\mathcal{U}$ at some point x_0. At x_0 we have

$$0 = M_i = \frac{u_{11i}}{u_{11}} + \sigma u_1 u_{1i} + \frac{\eta_i}{\eta}, \quad i = 1, 2 \tag{8}$$

and

$$0 \geq L(M) = \frac{1}{u_{11}} L(u_{11}) - \frac{u_{11i}^2}{u_{11}^2 u_{ii}} + \sigma u_{11}(1 + u_{12}^2) + \frac{L(\eta)}{\eta} - \frac{\eta_i^2}{\eta^2 u_{ii}}.$$

Here we used the equation and its derivative (6). Using the twice-differentiated equation (7) to simplify the first two terms we obtain

$$\sigma u_{11}(1 + u_{12}^2) - \frac{u_{112}^2}{u_{11}^2} u_{11} \leq \frac{\eta_i^2}{\eta^2 u_{ii}} - \frac{L(\eta)}{\eta}.$$

Using condition (8) we can estimate the second term from below by $-2\sigma^2 u_1^2 u_{12}^2 u_{11} - 2\frac{\eta_2^2}{\eta^2} u_{11}$. Multiplying by $\eta^2 u_{11}$ we arrive at

$$\sigma \eta^2 u_{11}^2 (1 + (1 - 2\sigma u_1^2) u_{12}^2) \leq \eta_1^2 + 3\eta_2^2 u_{11}^2 - \eta L(\eta) u_{11}.$$

We now specify η. By (5) and Lemma 2.1 (appropriately rescaled), we have $u(\pm e_2/2) \geq \delta > 0$. We claim that

$$\eta := 1 - A(x_1^2 + u_2^2)/2$$

satisfies the required conditions for some $A(\|u\|_{C^1(B_1)})$. Indeed, $u(x_1, 0) < C|x_1|$ and $u(x_1, \pm 1/2) > \delta - C|x_1|$, where $C = \|u\|_{C^1(B_1)}$. Coordinate-convexity implies that $|u_2|(x_1, \pm 1/2) > \delta$ for $|x_1| < \delta/4C$. Thus, $A(x_1^2 + u_2^2)/2 > 1$ on the boundary of $(-\delta/4C, \delta/4C) \times (-1/2, 1/2)$ for A large depending on $\|u\|_{C^1(B_1)}$.

Our choice of η gives

$$\eta_1^2 = A^2(x_1 + u_2 u_{12})^2, \quad \eta_2^2 u_{11}^2 = A^2 u_2^2.$$

Using the linearized equation (6) we have

$$-\eta L(\eta) u_{11} = A\,\eta(2 + u_{12}^2).$$

Putting these together we obtain

$$\sigma u_{11}^2 \eta^2 (1 + (1 - 2\sigma u_1^2) u_{12}^2) \leq C(\|u\|_{C^1(B_1)})(1 + u_{12}^2).$$

By choosing σ small depending on $\sup_{B_1} |u_1|$ we obtain

$$\eta^2 u_{11}^2(x_0) < C(\|u\|_{C^1(B_1)}). \tag{9}$$

We conclude that

$$u_{11}(0) \leq \eta u_{11} e^{\sigma u_1^2/2}(0) \leq \eta u_{11} e^{\sigma u_1^2/2}(x_0) \leq C(\|u\|_{C^1(B_1)}).$$

□

Remark 2.3. Our choice of η is motivated by the partial Legendre transform, which takes the Monge-Ampère equation to the Laplace equation in $2D$. Roughly, to estimate u_{11} from above we'd like to estimate u_{22} from below. This is the same as obtaining C^2 estimates for the partial Legendre transform "taken in the e_2 direction." Since the transformed coordinates are $(p_1, p_2) = (x_1, u_2)$, it is natural to seek cutoff functions depending on x_1 and u_2.

Remark 2.4. Estimates for all the higher derivatives in terms of $\|u\|_{L^\infty(B_1)}$ follow from uniform ellipticity and either classical $2D$ theory (see e.g. [GT], Chapter 17) or the theory of concave equations (see e.g. [CC], Chapter 6).

More precisely, a derivative u_e of u solves the uniformly elliptic equation $L(u_e) = 0$. Such equations enjoy interior $C^{1,\alpha}$ estimates in $2D$, giving $C^{2,\alpha}$ estimates for u. Schauder theory can be used to estimate all higher derivatives. Alternatively, the full C^2 estimates can be obtained using the concavity of the equation and the weak Harnack inequality of Krylov-Safonov for the second derivatives u_{ee} of u. Then $C^{2,\alpha}$ estimates follow by the Evans-Krylov theorem, and higher regularity as before.

Corollary 1.2 follows from Theorem 1.1 by considering the rescalings $R^{-2}u(Rx)$ and applying Remark 2.4. We will remove the assumption that u is C^4 in the next section by solving the Dirichlet problem.

3. The Dirichlet Problem

In this section we prove Theorem 1.3. We use an idea of Lions based on solving global approximating problems [L].

Let $C_b^\infty(\mathbb{R}^2)$ be the space of smooth functions ψ on $\mathbb{R}^2$ satisfying $\|\psi\|_{C^k(\mathbb{R}^2)} < \infty$ for all k. Assume without loss of generality that $\varphi \in C_0^\infty(\mathbb{R}^2)$. Finally, let ρ be a smooth nonnegative function that vanishes on $\overline{\Omega}$, is positive on $\mathbb{R}^2 \backslash \overline{\Omega}$, and is 1 outside a neighborhood of $\overline{\Omega}$. (We note that such a function exists for any bounded domain Ω.) The key proposition is:

Proposition 3.1. *For all $\epsilon > 0$, there exists a solution in $C_b^\infty(\mathbb{R}^2)$ of*

$$(u_{11}^\epsilon - \epsilon^{-1}(u^\epsilon - \varphi)\rho)(u_{22}^\epsilon - \epsilon^{-1}(u^\epsilon - \varphi)\rho) = 1 \tag{10}$$

with $u_{ii} - \epsilon^{-1}(u^\epsilon - \varphi)\rho > 0$ for $i = 1, 2$.

Heuristically, the additional terms in (10) "penalize" the solution for deviating from φ outside Ω.

Proof of Proposition 3.1*:* It suffices to obtain a global C^2 estimate for solutions in $C_b^\infty(\mathbb{R}^2)$ of

$$\sum_{i=1}^{2} \log(u_{ii} - \epsilon^{-1}(u - \varphi)\rho) = \log(f), \tag{11}$$

where $f \in C_b^\infty(\mathbb{R}^2) \cap \{f \geq 1\}$, and the estimate depends only on ϵ and $\|f\|_{C^2(\mathbb{R}^n)}$. Here we assume $u_{ii} > \epsilon^{-1}(u - \varphi)\rho$ for $i = 1, 2$. Global estimates for the higher derivatives of u then follow by classical uniformly elliptic theory (see Remark 2.4). The existence in $C_b^\infty(\mathbb{R}^2)$ of solutions to (10) follows easily by the method of continuity. For the details of this argument, see the Appendix (Section 7).

Let w be a defining function of Ω (that is, Ω is a connected component of $\{w < 0\}$ and $w_{ii} \geq c > 0$ on $\mathbb{R}^2$) and we let $\tilde{w} \in C_0^\infty(\mathbb{R}^2)$ agree with w in a neighborhood of $\overline{\Omega}$. Below, C and K will denote large constants depending of $\|f\|_{C^2(\mathbb{R}^2)}$ and ϵ.

C^0 **Estimate:** Let $A^{ii} = 1/(u_{ii} - \epsilon^{-1}(u - \varphi)\rho)$. Note that

$$A^{ii}u_{ii} - \epsilon^{-1}\left(\sum_{i=1}^{2} A^{ii}\right)\rho\, u = 2 - \epsilon^{-1}\left(\sum_{i=1}^{2} A^{ii}\right)\rho\, \varphi.$$

It is easy to check that $C\tilde{w} - h$ is a subsolution to this equation when $h \geq K$, for appropriate large constants C, K. (Here h is constant). For some h, we have that $\inf_{\mathbb{R}^2}(u - (C\tilde{w} - h)) = 0$. Assume by way of contradiction that $h \geq K$. Then by the maximum principle, $u > -h$ outside a large ball and approaches $-h$ at a sequence of points going to ∞. By sliding a paraboloid with Hessian $-I$ centered at a point far from the origin where u is close to $-h$ from below until it touches u, we can find a point where $D^2u > -I$ and $\rho = 1$, but $u < 1 - K$. This contradicts the equation for K large. We conclude that $u \geq C\tilde{w} - K$.

For the estimate from above, use $K - C\tilde{w}$ as a barrier and argue in the same way.

C^1 **Estimate:** This follows easily from the C^0 estimate and the semiconcavity of u in coordinate directions. (Recall that $u_{ii} > \epsilon^{-1}(u - \varphi)\rho$ for $i = 1, 2$.)

C^2 **Estimate:** We have $u_{kk} > -C$ for $k = 1, 2$ by the C^0 estimate. Differentiating (11) twice and using the C^1 estimate gives

$$A^{ii}(u_{kk})_{ii} - \epsilon^{-1}\left(\sum_{i=1}^{2} A^{ii}\right)\rho u_{kk} \geq -C\sum_{i=1}^{2} A^{ii}.$$

(Here we dropped a positive expression that is quadratic in third derivatives on the right side. The positivity is a consequence of the concavity of the equation). Using a barrier of the form $-C\tilde{w} + K$ and arguing as in the C^0 estimate gives an upper bound for u_{kk}. The equation (11) then becomes uniformly elliptic, so the full C^2 bound follows from classical theory (see Remark 2.4). $\square$

We now prove Theorem 1.3 by taking a limit of the solutions u^ϵ from Proposition 3.1.

Proof of Theorem 1.3. The proof is a refinement of the C^0 estimate from Proposition 3.1. We have

$$A^{ii}u^\epsilon_{ii} - \epsilon^{-1}\left(\sum_{i=1}^{2} A^{ii}\right)\rho u^\epsilon = 2 - \epsilon^{-1}\left(\sum_{i=1}^{2} A^{ii}\right)\rho\varphi,$$

where $A^{ii} = 1/(u^\epsilon_{ii} - \epsilon^{-1}(u^\epsilon - \varphi)\rho)$. Let $\tilde{w}_\delta \in C_0^\infty(\mathbb{R}^2)$ agree with w in a small neighborhood of $\overline{\Omega}$ and satisfy $|\tilde{w}_\delta| < \delta$ on $\mathbb{R}^2\backslash\Omega$. Then $C\tilde{w}_\delta + \varphi - 2C\delta$ is a subsolution of the above equation for ϵ small depending on δ. By the maximum principle, $(\varphi - u^\epsilon)^+$ converges uniformly to zero on $\mathbb{R}^2\backslash\Omega$ as $\epsilon \to 0$. (The unboundedness of the domain is not an issue, since we are working in $C_b^\infty(\mathbb{R}^2)$; argue as in the C^0 estimate from the proof of Proposition 3.1).

Similarly, $-C\tilde{w}_\delta + \varphi + 2C\delta$ is a super-solution of this equation for ϵ small depending on δ. We conclude by the maximum principle that u^ϵ converge uniformly to φ on $\mathbb{R}^2\backslash\Omega$.

Since u^ϵ solve (1) in Ω and converge uniformly on $\partial\Omega$, we have by the maximum principle that $\{u^\epsilon\}$ is Cauchy in $C^0(\mathbb{R}^2)$. The u^ϵ thus converge as $\epsilon \to 0$ to a continuous function on $\mathbb{R}^2$ that agrees with φ on $\mathbb{R}^2\backslash\Omega$.

Finally, by Theorem 1.1, in any $\Omega' \subset\subset \Omega$, we have derivative estimates of all orders for u^ϵ that are independent of ϵ. We conclude that the limit is smooth in Ω and solves (1) classically. The uniqueness follows from the maximum principle. □

Corollary 1.5 follows from Theorem 1.1 and Theorem 1.3 by approximating the boundary data with smooth functions, solving the Dirichlet problem, and taking a limit.

Remark 3.2. In higher dimensions, the above techniques show that we can approximate any viscosity solution to the equation

$$(\Pi_{i=1}^n u_{ii})^{\frac{1}{n}} = 1 \text{ in } \Omega \subset \mathbb{R}^n \text{ bounded, uniformly coordinate-convex,} \quad u|_{\partial\Omega} = \varphi$$

by smooth solutions (with different boundary data). In the case $n \geq 3$ the classical solvability remains open due to the lack of an interior C^2 estimate (which is false without e.g. hypotheses on boundary data; see the example in Section 5).

Remark 3.3. An interesting question is whether C^2 estimates hold on $\partial\Omega$, even in the simple case that $\Omega = B_1 \subset \mathbb{R}^2$ and φ is smooth. It seems that the main difficulty is to estimate the mixed second derivatives at points with a horizontal or vertical tangent line.

Boundary gradient and tangential second derivative estimates are standard. At points with a horizontal or vertical tangent line, we can also estimate the normal second derivative. To see this, assume for simplicity that $\partial\Omega = \{(x_1, x_1^2/2)\}$ near 0, and that $u(0) = 0$, $\nabla u(0) = 0$. By the equation it suffices to bound $u_{11}(0)$ from below. By subtracting a multiple of x_1x_2 we may assume that the cubic part in the expansion of the boundary data vanishes. If $u_{11}(0) = 0$, then $u \sim x_1^4$ along $\partial\Omega$. It follows that $\{u < h\}$ contains a box Q centered on the x_2 axis with area $\sim h^{3/4} >> h$ for h small. It is easy to construct a convex quadratic polynomial P such that $P > h$ on ∂Q, $P = 0$ in the center of Q, and $P_{11}P_{22} << 1$. By coordinate convexity, $u \geq 0$ in the center of Q, so this contradicts the maximum principle.

To bound $u_{12}(0)$ it is natural to consider a tangential derivative $u_\tau = u_1 + x_1 u_2$. However, the right hand side of the linearized equation for u_τ is $2u_{12}/u_{11}$, which is not controlled by the right hand sides for the usual quantities u_1^2 and $|x|^2$. One can instead get estimates that degenerate near 0 by observing that the tangential derivative $x_1u_1+2x_2u_2$ solves the linearized equation with constant right hand side. This leads to the bounds

$$C^{-1}x_1^2 \leq u_{11} \leq C, \quad C^{-1} \leq u_{22} \leq Cx_1^{-2}, \quad |u_{12}| \leq Cx_1^{-1}$$

on $\partial\Omega$ near 0. It is unclear how to get bounds that extend all the way to the origin.

4. Entire Solutions

In this section construct non-quadratic entire solutions to (1) in $\mathbb{R}^2$. We search for solutions of the form

$$u(x_1, x_2) = f(x_1)f(x_2).$$

It suffices to find a global solution to the ODE

$$ff'' = 1, \quad f(0) = 1, \quad f'(0) = 0. \tag{12}$$

It is easy to check that the solution to (12) is given by

$$f(s) = H^{-1}(\sqrt{2}|s|),$$

where

$$H(t) = \int_1^t \frac{1}{\sqrt{\log(x)}}\, dx.$$

The function f is positive, convex, even and analytic, and

$$f \sim \sqrt{2}|s|\sqrt{\log|s|}$$

for s large. In particular, $u \sim r^2 \log r$ on the diagonal lines $x_1^2 = x_2^2$, and $u \sim r\sqrt{\log r}$ on the coordinate axes, for r large.

Remark 4.1. There are also explicit solutions to (1) in the box $\Omega := [-1, 1]^2$ with $u|_{\partial\Omega} = 0$, of the form $u = g(x_1)g(x_2)$. Here $g < 0$ on $(-1, 1)$ and solves $g''g = -1$, $g(\pm 1) = 0$. A direct computation gives $g = \lambda_0^{-1}G^{-1}(\lambda_0|x|)$ for some $\lambda_0 > 0$, where $G(t) = \int_{-1}^t [\log(x^{-2})]^{-1/2}\, dx$, $-1 \leq t \leq 0$. Notice that $|\nabla u| \to \infty$ on $\partial\Omega$. We used a variant of this solution to establish strict coordinate convexity for solutions to (1) in 2D (see Lemma 2.1).

5. Singular Solutions in Higher Dimensions

In this section we construct a non-classical global (viscosity) solution in $\mathbb{R}^3$ to the equation

$$u_{11}u_{22}u_{33} = 1, \quad u \text{ coordinate-convex.}$$

This shows that the interior regularity results for $2D$ are false in higher dimensions.

Our example is inspired by the Pogorelov example for the real Monge-Ampère equation [P]. We search for solutions of the form

$$u(x_1, x_2, x_3) = w(x_1, x_2)h(x_3).$$

The problem reduces to constructing solutions to

$$ww_{11}w_{22} = 1 \text{ on } \mathbb{R}^2, \quad h^2h'' = 1 \text{ on } \mathbb{R},$$

with $h > 0$ convex and $w \geq 0$ coordinate-convex. The main difficulty is that the equation for w is not rotation-invariant.

We first solve for h. The solution with initial conditions $h(0) = 1$, $h'(0) = 0$ is

$$h(s) = G^{-1}(\sqrt{2}|s|),$$

where

$$G(t) = \int_1^t \left(\frac{x}{x-1}\right)^{1/2} dx.$$

In particular, h is positive, convex, globally defined and analytic, and $h \sim \sqrt{2}|s| - \frac{1}{2}\log|s|$ for $|s|$ large.

We now construct a positive coordinate-convex solution to $ww_{11}w_{22} = 1$ in $\{x_2 > 0\}$, where w is homogeneous of degree $4/3$, even over the x_2 axis, and $w_1(1,\,1) = w_2(1,\,1)$. We can extend to a global solution on $\mathbb{R}^2$ by taking reflections over the lines $x_2 = \pm x_1$. (The solution we construct is in fact analytic outside the origin; see Remark (5.1)).

Let

$$w(x_1,\, x_2) = x_2^{4/3} g(x_2^{-1}x_1)$$

in $\{x_2 > 0\}$, so that w is $4/3$-homogeneous and $w(t,\,1) = g(t)$. The equation for w reduces to the ODE

$$g\,g''\left(t^2g'' - \frac{2}{3}tg' + \frac{4}{9}g\right) = 1. \tag{13}$$

We first claim that there exists a global even, convex solution g_1 to (13) with the initial conditions $g_1(0) = 1$ and $g_1'(0) = 0$. The existence and uniqueness in a neighborhood of 0 (say $|t| < \epsilon$) follows from the fact that

$$xz(t^2z - 2/3ty + 4/9x) - 1 = 0 \tag{14}$$

defines z as a smooth function of $(x,\,y,\,t)$ in a neighborhood of $(x,\,y,\,z,\,t) = (1,\,0,\,9/4,\,0)$ by the implicit function theorem. The solution is even by the invariance of (13) under reflection and the initial conditions, and convex since $g_1''(0) > 0$. To complete the argument, note that for $x,\,t > 0$ the positive solution to (14) is given by

$$z(x,\,y,\,t) = \frac{1}{3t^2}\left(\left(ty - \frac{2}{3}x\right) + \left(\frac{9t^2}{x} + \left(ty - \frac{2}{3}x\right)^2\right)^{1/2}\right). \tag{15}$$

This function is smooth and uniformly Lipschitz in $x,\,y$ in the region $\{t \geq \epsilon\} \cap \{x \geq 1\}$. We have $g_1 \geq 1$ on any interval of existence around 0 by the initial conditions and convexity, which combined with the previous observation gives long-time existence.

We next observe that for $\lambda > 0$ the rescalings

$$g_\lambda(t) := \lambda^{-2/3} g_1(\lambda t)$$

solve (13). This invariance comes from the invariance of $ww_{11}w_{22} = 1$ under $(x_1,\,x_2) \to (\lambda^{1/2}x_1,\,\lambda^{-1/2}x_2)$. We will choose $\lambda_0 > 0$ such that $g := g_{\lambda_0}$ satisfies

$$g'(1) - 2/3g(1) = 0,$$

which implies that $w_1(1,\,1) = w_2(1,\,1)$.

To that end we let

$$h_\lambda(t) := tg_\lambda'(t) - 2/3 g_\lambda = \lambda^{-2/3} h_1(\lambda t).$$

It suffices to show that $h_1(\lambda_0) = 0$ for some $\lambda_0 > 0$. (Then we have $h_{\lambda_0}(1) = 0$, so letting $g = g_{\lambda_0}$ would complete the construction).

Since

$$h_1(t) = \frac{d}{d\lambda} g_\lambda|_{\lambda=1}(t)$$

and g_λ solve (13) for all $\lambda > 0$, the function h_1 solves the linearized equation

$$\left(g_1 + t^2(g_1\, g_1'')^2\right) h_1'' - \frac{2}{3}(g_1 g_1'')^2 t h_1' + \left(g_1'' + \frac{4}{9}(g_1 g_1'')^2\right) h_1 = 0. \tag{16}$$

Since $h_1(0) = -2/3$ we have that h_1 is convex in a neighborhood of 0. Note that h_1 is even. The equation (16) prevents h_1' from becoming zero before h_1 reaches 0. We conclude from (16) that h_1 is convex in the interval around 0 where it is negative, and thus crosses zero at some time $\lambda_0 > 0$. This completes the construction.

Remark 5.1. We have in addition that

$$g(t) = t^{4/3} g(1/t).$$

Indeed, one checks that $\tilde{g} := t^{4/3} g(1/t)$ solves (13) and satisfies $\tilde{g}(1) = g(1)$, $\tilde{g}'(1) = g'(1)$ by the condition $g'(1) - 2/3 g(1) = 0$. We conclude that

$$g \sim at^{4/3} + bt^{-2/3}$$

for t large and for some $a, b > 0$. We also conclude that

$$w = |x_2|^{4/3} g(x_2^{-1} x_1) = |x_1|^{4/3} g(x_1^{-1} x_2)$$

is analytic outside the origin.

6. Open Problems

Here we list some open problems related to our results.

(1) Classify the global solutions to $u_{11}u_{22} = 1$ in $\mathbb{R}^2$.
(2) Find conditions (e.g. on the boundary and boundary data) that guarantee an interior C^2 estimate for $\Pi_{i=1}^n u_{ii} = 1$ in $\mathbb{R}^n$, $n \geq 3$.
(3) Solve the classical Dirichlet problem for $\Pi_{i=1}^n u_{ii} = 1$, $n \geq 3$, on some natural class of domains. (More generally, consider equations for concave symmetric functions of the u_{ii}).
(4) Analyze the structure of the singular set for solutions to $\Pi_{i=1}^n u_{ii} = 1$ in dimensions $n \geq 3$. For example: Are the singularities analytic? Do they propagate to the boundary? What is the Hausdorff dimension of the singular set?

7. Appendix

In the appendix we describe how to obtain existence in $C_b^\infty(\mathbb{R}^2)$ of solutions to (10) using global C^2 estimates for (11).

Let w and $\tilde{w}$ be as in the proof of Proposition 3.1, and let

$$g = \Pi_{i=1}^2((C_0\tilde{w} - K_0)_{ii} - \epsilon^{-1}(C_0\tilde{w} - K_0 - \varphi)\rho),$$

for constants C_0, K_0 chosen large enough that $g > 1$. We would like to solve in $C_b^\infty(\mathbb{R}^2)$ the problems

$$\sum_{i=1}^{2} \log(u_{ii} - \epsilon^{-1}(u-\varphi)\rho) = \log(t + (1-t)g), \tag{17}$$

for all $t \in [0, 1]$.

We first claim the the set of t for which (17) is solvable in $C_b^\infty(\mathbb{R}^2)$ is closed. This follows from the global C^2 estimates for (11). Indeed, by classical uniformly elliptic theory these imply global C^k estimates (for each k) for solutions in $C_b^\infty(\mathbb{R}^2)$ of (17), that are independent of t.

We next claim that the set of t for which (17) is solvable in $C_b^\infty(\mathbb{R}^2)$ is also open. We will use the implicit function theorem. To that end, let X_1 be the open subset of $C^{2,\alpha}(\mathbb{R}^2)$ given by

$$X_1 = \{\psi \in C^{2,\,\alpha}(\mathbb{R}^2) : \inf_{\mathbb{R}^2}(\psi_{ii} - \epsilon^{-1}(\psi - \varphi)\rho) > 0 \text{ for } i = 1,\, 2\},$$

and define $G : X_1 \times [0, 1] \to C^\alpha(\mathbb{R}^2)$ by

$$G(u,\, t) = \sum_{i=1}^{2} \log(u_{ii} - \epsilon^{-1}(u-\varphi)\rho) - \log(t + (1-t)g).$$

Assume that $G(u_0, t_0) = 0$. It is straightforward to check that G is C^1 on $X_1 \times [0, 1]$, and that the linearization G_u at (u_0, t_0) is given by

$$G_u(u_0,\, t_0)(v) = \sum_{i=1}^{2} A^{ii} v_{ii} - \epsilon^{-1}\left(\sum_{i=1}^{2} A^{ii}\right)\rho v,$$

where $A^{ii} = 1/((u_0)_{ii} - \epsilon^{-1}(u_0 - \varphi)\rho)$.

The injectivity of $G_u(u_0, t_0) : C^{2,\,\alpha}(\mathbb{R}^2) \to C^\alpha(\mathbb{R}^2)$ follows from the maximum principle. (We remark again that there is no problem with the domain being unbounded, since we work with globally bounded quantities; one can argue as in the C^0 estimate from the proof of Proposition 3.1). For surjectivity, solve the problems

$$G_u(u_0,\, t_0)(v_R) = f,\; v_R|_{\partial B_R} = 0$$

for each R. The functions $\pm(-C\tilde{w} + K)$ are super- and sub- solutions for C, K large constants depending on $\|f\|_{L^\infty(\mathbb{R}^2)}$, so by the maximum principle, v_R are uniformly bounded. Schauder estimates give uniform $C^{2,\,\alpha}$ bounds for v_R in B_{R-1}. In the limit $R \to \infty$ we obtain a solution in $C^{2,\,\alpha}(\mathbb{R}^2)$ to $G_u(u_0, t_0)v = f$, with $\|v\|_{C^{2,\,\alpha}(\mathbb{R}^2)}$ controlled by $\|f\|_{C^\alpha(\mathbb{R}^2)}$.

By the implicit function theorem (see e.g. Chapter 17 in [GT]), there exist solutions in X_1 to $G(u, t) = 0$ for all t close to t_0. Schauder theory implies that in fact $u \in C_b^\infty(\mathbb{R}^2)$, proving the claim.

Since $G(C_0\tilde{w} - K_0, 0) = 0$, the set of t for which (17) is solvable in $C_b^\infty(\mathbb{R}^2)$ is nonempty. We conclude that the equation (17) is solvable in $C_b^\infty(\mathbb{R}^2)$ for all $t \in [0, 1]$.

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References

[B] Błocki, Z. On the regularity of the complex Monge-Ampère operator. *Complex Geometric Analysis in Pohang* **222** (1997), 181-189. Contemp. Math. Amer. Soc., Providence, RI (1999).

[CC] Caffarelli, L.; Cabré, X. *Fully Nonlinear Elliptic Equations.* Colloquium Publications 43. Providence, RI: American Mathematical Society (1995).

[CNS1] Caffarelli L.; Nirenberg, L.; Spruck, J. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math.* **37** (1984), no. 3, 369-402.

[CNS2] Caffarelli, L.; Nirenberg, L.; Spruck, J. The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. *Acta Math.* **155** (1985), no. 3-4, 261-301.

[GT] Gilbarg, D.; Trudinger, N. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[I] Ishii, H. On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDE's. *Comm. Pure Appl. Math.* **42** (1989), 15-45.

[L] Lions, P. L. Sur les équations de Monge-Ampère. I. *Manuscripta Math.* **41** (1983), no. 1-3, 1-43.

[P] Pogorelov, A. The regularity of the generalized solutions of the equation $\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = \varphi(x_1, x_2, ..., x_n) > 0$. (Russian) *Dokl. Akad. Nauk SSSR* **200** (1971), 534-537.

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