

A ONE-SIDED PRIME IDEAL PRINCIPLE FOR NONCOMMUTATIVE RINGS

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ABSTRACT. *Completely prime right ideals* are introduced as a one-sided generalization of the concept of a prime ideal in a commutative ring. Some of their basic properties are investigated, pointing out both similarities and differences between these right ideals and their commutative counterparts. We prove the *Completely Prime Ideal Principle*, a theorem stating that right ideals that are maximal in a specific sense must be completely prime. We offer a number of applications of the Completely Prime Ideal Principle arising from many diverse concepts in rings and modules. These applications show how completely prime right ideals control the one-sided structure of a ring, and they recover earlier theorems stating that certain noncommutative rings are domains (namely, proper right PCI rings and rings with the right restricted minimum condition that are not right artinian). In order to provide a deeper understanding of the set of completely prime right ideals in a general ring, we study the special subset of *comonoform right ideals*.

1. INTRODUCTION

Prime ideals form an important part of the study of commutative algebra. While there are many reasons why this is so, in this paper we will focus on the fact that *prime ideals control the structure of commutative rings*. The following two theorems due to I. S. Cohen ([4, Thm. 2] and [4, Thm. 7], respectively) illustrate the kinds of structure theorems that we shall consider.

Theorem 1.1 (Cohen's Theorem). *A commutative ring R is noetherian iff every prime ideal of R is finitely generated.*

Theorem 1.2 (Cohen). *A commutative ring $R \neq 0$ is a Dedekind domain iff every nonzero prime ideal of R is invertible.*

(Notice that Cohen originally stated Theorem 1.2 for commutative *domains* $R \neq 0$, but it is not hard to extend his result to arbitrary commutative rings.) While prime two-sided ideals are studied in noncommutative rings, it is safe to say that they do not control the structure of noncommutative rings in the sense of the two theorems above. Part of the trouble is that many complicated rings have few two-sided ideals. Some of the most dramatic examples are the simple rings, which have only one prime ideal but often have interesting one-sided structure.

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In this paper we propose a remedy to this situation by studying *completely prime right ideals* (introduced in §2), which are a certain type of “prime one-sided ideal” illuminating the structure of a noncommutative ring. The idea of prime one-sided ideals is not new, as evidenced by numerous attempts to define such objects in the literature (for instance, see [1], [16], and [24]). However, a common theme among earlier versions of one-sided prime ideals is that they were produced by simply deforming the defining condition of a prime ideal in a commutative ring ($ab \in \mathfrak{p}$ implies a or b lies in \mathfrak{p}). Our approach is slightly less arbitrary, as it is inspired by the systematic analysis in [21] of results from commutative algebra in the vein of Theorems 1.1 and 1.2 above. As a result, these one-sided primes are accompanied by a ready-made theory producing a number of results that relate them to the one-sided structure of a ring.

A key part of the classical proofs of Cohen’s theorems above is to show that an ideal of a commutative ring that is maximal with respect to *not* being finitely generated (respectively, invertible) is prime. Many other famous theorems from commutative algebra state that an ideal maximal with respect to a certain property must be prime. The *Prime Ideal Principle* of [21] unified a large number of these results, making use of the fundamental notion of *Oka families of ideals* in commutative rings. In §3, after introducing Oka families of right ideals, we present the *Completely Prime Ideal Principle* 3.4 (CPIP). This result generalizes the Prime Ideal Principle of [21] to one-sided ideals of a noncommutative ring. It formalizes a one-sided “maximal implies prime” philosophy: *right ideals that are maximal in certain senses tend to be completely prime*. The CPIP is our main tool connecting completely prime right ideals to the (one-sided) structure of a ring. For instance, it allows us to provide a noncommutative generalization of Cohen’s Theorem 1.1 in Theorem 3.8.

In order to effectively apply the CPIP, we investigate how to construct examples of right Oka families (from classes of cyclic modules that are *closed under extensions*) in §4. Most of the applications of the Completely Prime Ideal Principle are given in §5. Some highlights include a study of point annihilators of modules over noncommutative rings, conditions for a ring to be a domain, a simple proof that a right PCI ring is a domain, and a one-sided analogue of Theorem 1.2.

Finally in §6 we turn our attention to a special subset of the completely prime right ideals of a ring, the set of *comoniform right ideals*. These right ideals are more well-behaved than completely prime right ideals generally are. They enjoy special versions of the “Prime Ideal Principle” and its “Supplement,” which again allow us to produce results that relate these right ideals to the one-sided structure of a noncommutative ring. Their existence is also closely tied to the well-studied right Gabriel filters from the theory of noncommutative localization.

Conventions. Throughout this paper, the symbol “:=” is used to mean that the left-hand side of the equation is defined to be equal to the right-hand side. All rings are assumed to be associative with unit element, and all subrings, modules and ring homomorphisms are assumed to be unital. Fix a ring R . We say that R is a *semisimple ring* if R_R is a semisimple module. We say R is *Dedekind-finite* if every right invertible element is invertible; this is equivalent to the condition that R_R is not isomorphic to a proper direct summand of itself (see [20, Ex. 1.8]). We write $I_R \subseteq R$ (resp. $I \triangleleft R$) to mean that I is a right (resp. two-sided) ideal in R . The term *ideal* always refers to a two-sided ideal of R , with the sole exception of the phrase “Completely Prime Ideal Principle” (Theorem 3.4). We let $\text{Spec}(R)$ denote

the set of prime (two-sided) ideals of R . An element of R is *regular* if it is not a left or right zero-divisor. Given a family \mathcal{F} of right ideals in R , we let \mathcal{F}' denote the complement of \mathcal{F} within the set of all right ideals of R , and we let $\text{Max}(\mathcal{F}')$ denote the set of maximal elements of \mathcal{F}' . Given an R -module M_R , we let $\text{soc}(M)$ denote the socle of M (the sum of all simple submodules of M). Finally, we use “f.g.” as shorthand for “finitely generated.”

2. COMPLETELY PRIME RIGHT IDEALS

In this section we define the completely prime right ideals that we shall study, and we investigate some of their basic properties. Throughout this paper, given an element m and a submodule N of a right R -module M_R , we write

$$m^{-1}N := \{r \in R : mr \in N\},$$

which is a right ideal of R . Except in §6, we only deal with this construction in the form $a^{-1}I$ for an element a and a right ideal I of R . In commutative algebra, this ideal is usually denoted by $(I : a)$.

Definition 2.1. A right ideal $P_R \subsetneq R$ is *completely prime* if for any $a, b \in R$ such that $aP \subseteq P$, $ab \in P$ implies that either $a \in P$ or $b \in P$ (equivalently, for any $a \in R$, $aP \subseteq P$ and $a \notin P$ imply $a^{-1}P = P$).

Our use of the term “completely prime” is justified by the next result, which characterizes the two-sided ideals that are completely prime as right ideals. Recall that an ideal $P \triangleleft R$ is said to be *completely prime* if the factor ring R/P is a domain (equivalently, $P \neq R$ and for all $a, b \in R$, $ab \in P \implies a \in P$ or $b \in P$); for instance, see [18, p. 194].

Proposition 2.2. *For any ring R , an ideal $P \triangleleft R$ is completely prime as a right ideal iff it is a completely prime ideal. In particular, an ideal P is completely prime as a right ideal iff it is completely prime as a left ideal.*

Proof. For an ideal $P \triangleleft R$, we tautologically have $aP \subseteq P$ for all $a \in R$. So such $P \neq R$ is completely prime as a right ideal iff for every $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$, which happens precisely when P is a completely prime ideal. \square

Corollary 2.3. *If R is a commutative ring, then an ideal $P \triangleleft R$ is completely prime as a (right) ideal iff it is a prime ideal.*

Thus completely prime right ideals extend the notion of completely prime ideals in noncommutative rings, and these right ideals also directly generalize the the concept of a prime ideal of a commutative ring. Some readers may wonder whether it would be better to reserve the term “completely prime” for a right ideal $P \subsetneq R$ satisfying the following property: for all $a, b \in R$, if $ab \in P$ then either $a \in P$ or $b \in P$. (Such right ideals have been studied, for instance, in [1].) Let us informally refer to such right ideals as “extremely prime.” We argue that one merit of completely prime right ideals is that they occur in situations where extremely prime right ideals are absent. We shall show in Corollary 2.10 that every maximal right ideal of a ring is completely prime, thus proving that every nonzero ring has a completely prime right ideal. On the other hand, there are many examples of nonzero rings that do not have any extremely prime right ideals. We thank T. Y. Lam for helping to formulate the following result and G. Bergman for simplifying its proof.

Proposition 2.4. *Let R be a simple ring that has a nontrivial idempotent. Then R has no extremely prime right ideals.*

Proof. Assume for contradiction that $I \subsetneq R$ is an extremely prime right ideal. For any idempotent $e \neq 0, 1$ of R we have $RfR = R$, where $f = 1 - e \neq 0$. Since I is extremely prime and $(eRf)^2 = 0 \subseteq I$, we have $eRf \subseteq I$. Hence $eR = eRfR \subseteq I$, so that $e \in I$. Similarly, $f \in I$. Hence $1 = e + f \in I$, which is a contradiction. \square

Explicit examples of such rings are readily available. Let k be a division ring. Then we may take R to be the matrix ring $\mathbb{M}_n(k)$ for $n > 1$. Alternatively, if V_k is such that $\dim_k(V) = \alpha$ is any infinite cardinal, then we may take R to be the factor of $E := \text{End}_k(V)$ by its unique maximal ideal $M = \{g \in E : \dim_k(g(V)) < \alpha\}$ (see [19, Ex. 3.16]). The latter example is certainly neither left nor right noetherian. More generally, large classes of rings satisfying the hypothesis of Proposition 2.4 include simple von Neumann regular rings that are not division rings, as well as purely infinite simple rings (a ring R that is not a division ring is *purely infinite simple* if, for every $r \in R$, there exist $x, y \in R$ such that $xry = 1$; see [2, §1]).

It follows from Proposition 2.2 that we can omit the modifiers “left” and “right” when referring to two-sided ideals that are completely prime. The same result shows that the completely prime right ideals can often be “sparse” among two-sided ideals in noncommutative rings. For example, there exist many rings that have no completely prime ideals, such as simple rings that are not domains. Also, it is noteworthy that a prime ideal $P \triangleleft R$ of a noncommutative ring is not necessarily a completely prime right ideal. Thus completely prime right ideals generalize the notion of prime ideals in commutative rings in a markedly different way than the more familiar two-sided prime ideals of noncommutative ring theory. (For further evidence of this idea, see Proposition 2.11.) The point is that these two types of “primes” give insight into different facets of a ring’s structure, with completely prime right ideals giving a better picture of the right-sided structure of a ring as argued throughout this paper.

Below are some alternative characterizations of completely prime right ideals that help elucidate their nature. The *idealizer* of a right ideal $J_R \subseteq R$ is the subring of R given by

$$\mathbb{I}_R(J) := \{x \in R : xJ \subseteq J\}.$$

This is the largest subring of R in which J is a (two-sided) ideal. It is a standard fact that $\text{End}_R(R/J) \cong \mathbb{I}_R(J)/J$.

Proposition 2.5. *For a right ideal $P_R \subsetneq R$, the following are equivalent:*

- (1) P is completely prime;
- (2) For $a, b \in R$, $ab \in P$ and $a \in \mathbb{I}_R(P)$ imply either $a \in P$ or $b \in P$;
- (3) Any nonzero $f \in \text{End}_R(R/P)$ is injective;
- (4) $E := \text{End}_R(R/P)$ is a domain and ${}_E(R/P)$ is torsionfree.

Proof. Characterization (2) is merely a restatement of the definition given above for completely prime right ideals, so we have (1) \iff (2).

(1) \implies (3): Let P be a completely prime right ideal, and let $0 \neq f \in \text{End}_R(R/P)$. Choose $x \in R$ such that $f(1 + P) = x + P$. Because $f \neq 0$, $x \notin P$. Also, because f is an R -module homomorphism, $(1 + P)P = 0$ implies that $(x + P)P = 0$, or $xP \subseteq P$. Then because P

is completely prime, $x^{-1}P = P$. But this gives $\ker f = (x^{-1}P)/P = 0$, so f is injective as desired.

(3) \implies (1): Assume that any nonzero endomorphism of R/P is injective. Suppose $x, y \in R$ are such that $xP \subseteq P$ and $xy \in P$. Then there is an endomorphism f of R/P given by $f(r+P) = xr+P$. If $x \notin P$ then $f \neq 0$, making f injective. Then $f(y+P) = xy+P = 0+P$ implies that $y+P = 0+P$, so that $y \in P$. Hence P is a completely prime right ideal.

(3) \iff (4): This equivalence is still true if we replace R/P with any nonzero module M_R . If $E = \text{End}_R(M)$ is a domain and ${}_E M$ is torsionfree, then it is clear that every nonzero endomorphism of M is injective. Assume conversely that all nonzero endomorphisms of M are injective. Given $f, g \in E \setminus \{0\}$ and $m \in M \setminus \{0\}$, injectivity of g gives $g(m) \neq 0$ and injectivity of f gives $f(g(m)) \neq 0$. In particular $fg \neq 0$, proving that E is a domain. Because g and m above were arbitrary, we conclude that M is a torsionfree left E -module. \square

(As a side note, we mention that modules for which every nonzero endomorphism is injective have been studied by A. K. Tiwary and B. M. Pandeya in [33]. In [34], W. Xue investigated the dual notion of a module for which every nonzero endomorphism is surjective. These were respectively referred to as modules with the properties (*) and (**). In our proof of (3) \iff (4) above, we showed that a module M_R satisfies (*) iff $E := \text{End}_R(M)$ is a domain and ${}_E M$ is torsionfree. One can also prove the dual statement that $M \neq 0$ satisfies (**) iff E is a domain and ${}_E M$ is divisible.)

One consequence of characterization (3) above is that the property of being completely prime depends only on the quotient module R/P . (Using language to be introduced in Definition 4.3, the set of all completely prime right ideals in R is closed under similarity.) It is a straightforward consequence of (2) above that *a completely prime right ideal P is a completely prime ideal in the subring $\mathbb{I}_R(P)$* . This is further evidenced from (4) because $\mathbb{I}_R(P)/P \cong \text{End}_R(R/P)$ is a domain.

One might wonder whether the torsionfree requirement in condition (4) above is necessary. That is, can a cyclic module C_R have endomorphism ring E which is a domain but with ${}_E C$ not torsionfree? The next example shows that this is indeed the case.

Example 2.6. For an integer $n > 1$, consider the following ring and right ideal:

$$R := \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/(n) \\ 0 & \mathbb{Z} \end{pmatrix} \supseteq J_R := \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix}.$$

It is easy to show that $\mathbb{I}_R(J) = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$, so that $E := \text{End}_R(R/J) \cong \mathbb{I}_R(J)/J \cong \mathbb{Z}$ is a domain acting on $R/J \cong (\mathbb{Z}, \mathbb{Z}/(n))_R$ by (left) multiplication. But ${}_E(R/J)$ has nonzero torsion submodule isomorphic to $(0, \mathbb{Z}/(n))_R$.

For a prime ideal P in a commutative ring R , the factor module R/P is indecomposable (i.e., has no nontrivial direct summand). This property persists for completely prime right ideals.

Corollary 2.7. *If P is a completely prime right ideal of R , then the right R -module R/P is indecomposable.*

Proof. By Proposition 2.5 the ring $E := \text{End}_R(R/P)$ is a domain. Thus E has no nontrivial idempotents, proving that R/P is indecomposable. \square

For a commutative ring R and $P \in \text{Spec}(R)$, the module R/P is not only indecomposable, it is uniform. (A module $U_R \neq 0$ is *uniform* if every pair of nonzero submodules of U has nonzero intersection.) However, the following example shows that for a completely prime right ideal P in a general ring R , R/P need not be uniform as a right R -module.

Example 2.8. Let k be a division ring and let R be the following subring of $\mathbb{M}_3(k)$:

$$R = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

Notice that R has precisely three simple right modules (since the same is true modulo its Jacobson radical), namely $S_i = k$ ($i = 1, 2, 3$) with right R -action given by multiplication by the (i, i) -entry of any matrix in R .

Let $P \subseteq R$ be the right ideal consisting of matrices in R whose first row is zero. Notice that R/P is isomorphic to the module $V := (k \ k \ k)$ of row vectors over k with the usual right R -action. Then V has unique maximal submodule $M = (0 \ k \ k)$, and M in turn has precisely two nonzero submodules, $U = (0 \ k \ 0)$ and $W = (0 \ 0 \ k)$. Having cataloged all submodules of V , let us list the composition factors:

$$V/M \cong S_1, \quad M/W \cong U \cong S_2, \quad M/U \cong W \cong S_3.$$

Notice that S_1 occurs as a composition factor of every nonzero factor module of V , and it does not occur as a composition factor of any proper submodule of V . Thus every nonzero endomorphism of $V \cong R/P$ is injective, proving that P is a completely prime right ideal. However, V contains the direct sum $U \oplus W$, so $R/P \cong V$ is not uniform.

Proposition 2.2 shows how frequently completely prime right ideals occur among two-sided ideals. The next few results give us further insight into how many completely prime right ideals exist in a general ring. The first result gives a sufficient condition for a right ideal to be completely prime. Recall that a module is said to be *cohopfian* if all of its injective endomorphisms are automorphisms. For example, it is straightforward to show that any artinian module is cohopfian (see [19, Ex. 4.16]). The following is easily proved using Proposition 2.5(3).

Proposition 2.9. *If a right ideal $P \subseteq R$ is such that $E := \text{End}_R(R/P)$ is a division ring, then P is completely prime. The converse holds if R/P is cohopfian.*

Corollary 2.10. (A) *A maximal right ideal $\mathfrak{m}_R \subseteq R$ is a completely prime right ideal.*

(B) *For a right ideal P in a right artinian ring R , the following are equivalent:*

- (1) *P is a completely prime right ideal;*
- (2) *$\text{End}_R(R/P)$ is a division ring;*
- (3) *P is a maximal right (equivalently, maximal left) ideal in its idealizer $\mathbb{I}_R(P)$.*

Proof. Part (A) follows from Schur's Lemma and Proposition 2.9. For part (B), (1) \iff (2) follows from Proposition 2.9 (every cyclic right R -module is artinian, hence cohopfian), and (2) \iff (3) follows easily from the canonical isomorphism $\mathbb{I}_R(P)/P \cong \text{End}_R(R/P)$. \square

Because every nonzero ring has a maximal right ideal (by a familiar Zorn's lemma argument), part (A) above applies to show that a nonzero ring always has a completely prime

right ideal. (This fact was already mentioned at the beginning of this section.) The same cannot be said for completely prime two-sided ideals or the aforementioned “extremely prime” right ideals! On the other hand, Example 2.8 shows that a completely prime right ideal in an artinian ring need not be maximal, so we cannot hope to strengthen part (B) very drastically.

To get another indication of the role of completely prime right ideals, we may ask the following natural question: when is every proper right ideal of a ring completely prime? It is straightforward to verify that a commutative ring in which every proper ideal is prime must be a field. On the other hand, there exist nonsimple noncommutative rings in which every proper ideal is prime. (For example, take $R = \text{End}(V_k)$ where V is a right vector space of dimension at least \aleph_0 over a division ring k ; see [19, Ex. 10.6]). In contrast, the behavior of completely prime right ideals is much closer to that of prime ideals in the commutative case.

Proposition 2.11. *For a nonzero ring R , every proper right ideal in R is completely prime iff R is a division ring.*

Proof. (“Only if”) For an arbitrary $0 \neq a \in R$, it suffices to show that a is right invertible. Assume for contradiction that $aR \neq R$. Then the right ideal $J = a^2R \subseteq aR \subsetneq R$ is proper and hence is completely prime. Certainly $a \in \mathbb{I}_R(J)$. Because $a^2 \in J$, we must have $a \in J$ (recall Proposition 2.5(2)). But also the ideal $0 \neq R$ is completely prime, so R is a domain by Proposition 2.2. Then $a \in J = a^2R$ implies $1 \in aR$, contradicting that $aR \neq R$. \square

An inspection of the proof above actually shows that a nonzero ring R is a division ring iff the endomorphism ring of every nonzero cyclic right R -module is a domain, iff R is a domain and the endomorphism ring of every cyclic right R -module is reduced. We mention here that certain other “prime right ideals” studied previously do not enjoy the property proved above. For instance, K. Koh showed [16, Thm. 4.2] that all proper right ideals I of a ring R satisfy

$$(aRb \in I \implies a \in I \text{ or } b \in I) \quad \text{for all } a, b \in R$$

precisely when R is simple. In this sense, a ring may have “too many” of these prime-like right ideals.

The following is an analogue of the theorem describing $\text{Spec}(R/I)$ for a commutative ring R and an ideal $I \triangleleft R$. We present the result in a more general context than that of completely prime right ideals because it will be applicable to other types of “prime right ideals” that we will consider later.

Remark 2.12. Let \mathcal{P} be a module-theoretic property such that, if V_R is a module and I is an ideal of R contained in $\text{ann}(V)$, then V satisfies \mathcal{P} as an R -module iff it satisfies \mathcal{P} when considered as a module over R/I . For every ring R let $\mathcal{S}(R)$ denote the set of all right ideals $P_R \subseteq R$ such that R/P satisfies \mathcal{P} . Then it follows directly from our assumption on the property \mathcal{P} that there is a one-to-one correspondence

$$\{P_R \in \mathcal{S}(R) : P \supseteq I\} \longleftrightarrow \mathcal{S}(R/I)$$

given by $P \leftrightarrow P/I$.

In particular, we may take \mathcal{P} to be the property “ $V \neq 0$ and every nonzero endomorphism of V is injective.” Then the associated set $\mathcal{S}(R)$ is the collection of all completely prime right

ideals of R , according to characterization (3) of Proposition 2.5. In this case we conclude that *for any ideal $I \triangleleft R$ the completely prime right ideals of R/I correspond bijectively, in the natural way, to the set of completely prime right ideals of R containing I .*

In §§3–5 we will take a much closer look at the existence of completely prime right ideals in rings. To close this section, we explore how completely prime right ideals behave when “pulled back” along ring homomorphisms. One can interpret Remark 2.12 as demonstrating that, under a surjective ring homomorphism $f: R \rightarrow S$, the preimage of any completely prime right ideal of S is a completely prime right ideal of R . The next example demonstrates that this does not hold for arbitrary ring homomorphisms.

Example 2.13. For a division ring k , let $S := \mathbb{M}_3(k)$ and let R be the subring of S defined in Example 2.8. Consider the right ideals

$$Q_S := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ d & e & f \end{pmatrix} \right\} \subseteq S \quad \text{and} \quad P_R := \begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq R.$$

Because $Q_S \cong (k \ k \ k)_S^2$ has composition length 2, it is a maximal right ideal of S and thus is completely prime by Corollary 2.10(A). Let $f: R \rightarrow S$ be the inclusion homomorphism. Then $P = Q \cap R = f^{-1}(Q)$. However $(R/P)_R \cong (0 \ k \ k)_R \cong (0 \ k \ 0)_R \oplus (0 \ 0 \ k)_R$ is decomposable, so Lemma 2.7 shows that P_R is not completely prime.

The above example may seem surprising to the reader who recalls that completely prime (two-sided) ideals pull back along *any* ring homomorphism. The tension between this fact and Example 2.13 is resolved in the following result.

Proposition 2.14. *Let $f: R \rightarrow S$ be a ring homomorphism, let $Q_S \subsetneq S$ be a completely prime right ideal, and set $P_R := f^{-1}(Q)$. If $f(\mathbb{I}_R(P)) \subseteq \mathbb{I}_S(Q)$, then P is a completely prime right ideal of R .*

Proof. Because Q is a proper right ideal of S , P must be a proper right ideal of R . Suppose that $a \in \mathbb{I}_R(P)$ and $b \in R$ are such that $ab \in P$. Then $f(a)f(b) = f(ab) \in f(P) \subseteq Q$ with $f(a) \in f(\mathbb{I}_R(P)) \subseteq \mathbb{I}_S(Q)$. Because Q_S is completely prime, this means that one of $f(a)$ or $f(b)$ lies in Q . Hence one of a or b lies in $f^{-1}(Q) = P$ and P is completely prime. \square

This simultaneously explains our two positive examples of when a completely prime right ideal pulls back along a ring homomorphism. When $Q_S \subseteq S$ above is a two-sided ideal, $\mathbb{I}_S(Q) = S$ and the condition $f(\mathbb{I}_R(P)) \subseteq \mathbb{I}_S(Q)$ is trivially satisfied. On the other hand, if $Q \subseteq \text{im}(f)$ (e.g. if f is surjective), then one can use the fact that $f(P) = Q$ to show that $f(\mathbb{I}_R(P)) \subseteq \mathbb{I}_S(Q)$ again holds.

3. THE COMPLETELY PRIME IDEAL PRINCIPLE

A distinct advantage that completely prime right ideals have over earlier notions of “one-sided primes” is a theorem assuring the existence of completely prime right ideals in a wide array of situations. It states that right ideals that are “maximal” in certain senses must be completely prime. This result is the *Completely Prime Ideal Principle*, or *CPIP*, and it is presented in Theorem 3.4 below.

A number of famous theorems from commutative algebra state that an ideal maximal with respect to a certain property must be prime. A useful perspective from which to study

this phenomenon is to consider a family \mathcal{F} of ideals in a commutative ring R and ask when an ideal maximal in the *complement* \mathcal{F}' of \mathcal{F} is prime. Some well-known examples of such \mathcal{F} include the family of ideals intersecting a fixed multiplicative set $S \subseteq R$, the family of finitely generated ideals, the family of principal ideals, and the family of ideals that do not annihilate any nonzero element of a fixed module M_R . In [21] an *Oka family of ideals* in a commutative ring R was defined to be a set \mathcal{F} of ideals of R with $R \in \mathcal{F}$ such that, for any ideal $I \triangleleft R$ and element $a \in R$, $I + (a) \in \mathcal{F}$ and $(I : a) \in \mathcal{F}$ imply $I \in \mathcal{F}$. The *Prime Ideal Principle* (or PIP) [21, Thm. 2.4] states that for any Oka family \mathcal{F} , an ideal maximal in the complement of \mathcal{F} is prime (in short, $\text{Max}(\mathcal{F}') \subseteq \text{Spec}(R)$). In [21, §3] it was shown that many of the “maximal implies prime” results in commutative algebra (including those mentioned above) follow directly from the Prime Ideal Principle.

The following notion generalizes Oka families to the noncommutative setting.

Definition 3.1. Let R be a ring. An *Oka family of right ideals* (or *right Oka family*) in R is a family \mathcal{F} of right ideals with $R \in \mathcal{F}$ such that, given any $I_R \subseteq R$ and $a \in R$,

$$(3.2) \quad I + aR, a^{-1}I \in \mathcal{F} \implies I \in \mathcal{F}.$$

If R is commutative, notice that this coincides with the definition of an Oka family of ideals in R , given in [21, Def. 2.1]. When verifying that some set \mathcal{F} is a right Oka family, we will often omit the step of showing that $R \in \mathcal{F}$ if this is straightforward.

Remark 3.3. The fact that this definition is given in terms of the “closure property” (3.2) makes it clear that the collection of Oka families of right ideals in a ring R is closed under arbitrary intersections. Thus the set of right Oka families of R forms a complete lattice under the containment relation.

Without delay, let us prove the noncommutative analogue of the Prime Ideal Principle [21, Thm. 2.4], the Completely Prime Ideal Principle (CPIP).

Theorem 3.4 (Completely Prime Ideal Principle). *Let \mathcal{F} be an Oka family of right ideals in a ring R . Then every $I \in \text{Max}(\mathcal{F}')$ is a completely prime right ideal.*

Proof. Let $I \in \text{Max}(\mathcal{F}')$. Notice that $I \neq R$ since $R \in \mathcal{F}$. Assume for contradiction that there exist $a, b \in R \setminus I$ such that $aI \subseteq I$ and $ab \in I$. Because $a \notin I$ we have $I \subsetneq I + aR$. Additionally $I \subseteq a^{-1}I$, and $b \in a^{-1}I$ implies $I \subsetneq a^{-1}I$. Since $I \in \text{Max}(\mathcal{F}')$ we find that $I + aR, a^{-1}I \in \mathcal{F}$. Because \mathcal{F} is a right Oka family we must have $I \in \mathcal{F}$, a contradiction. \square

In the original setting of Oka families in commutative rings, a result called the “Prime Ideal Principle Supplement” [21, Thm. 2.6] was used to recover results such as Cohen’s Theorem [4, Thm. 2] that a commutative ring R is noetherian iff its prime ideals are all finitely generated. As with the Completely Prime Ideal Principle above, there is a direct generalization of this fact for noncommutative rings. The idea of this result is that for *certain* right Oka families \mathcal{F} , in order to test whether \mathcal{F} contains all right ideals of R , it is sufficient to test only the completely prime right ideals. We first define the one-sided version of a concept introduced in [21].

Definition 3.5. A *semifilter of right ideals* in a ring R is a family \mathcal{F} of right ideals such, for all right ideals I and J of R , if $I \in \mathcal{F}$ and $J \supseteq I$ then $J \in \mathcal{F}$.

Theorem 3.6 (Completely Prime Ideal Principle Supplement). *Let \mathcal{F} be a right Oka family in a ring R such that every nonempty chain of right ideals in \mathcal{F}' (with respect to inclusion) has an upper bound in \mathcal{F}' (for example, if every right ideal in \mathcal{F} is f.g.). Let \mathcal{S} denote the set of completely prime right ideals of R .*

- (1) *Let \mathcal{F}_0 be a semifilter of right ideals in R . If $\mathcal{S} \cap \mathcal{F}_0 \subseteq \mathcal{F}$, then $\mathcal{F}_0 \subseteq \mathcal{F}$.*
- (2) *For $J_R \subseteq R$, if all right ideals in \mathcal{S} containing J (resp. properly containing J) belong to \mathcal{F} , then all right ideals containing J (resp. properly containing J) belong to \mathcal{F} .*
- (3) *If $\mathcal{S} \subseteq \mathcal{F}$, then all right ideals of R belong to \mathcal{F} .*

Proof. For (1), let \mathcal{F}_0 be a semifilter of right ideals and suppose that $\mathcal{S} \cap \mathcal{F}_0 \subseteq \mathcal{F}$. Assume for contradiction that there exists a right ideal $I \in \mathcal{F} \setminus \mathcal{F}_0 = \mathcal{F}' \cap \mathcal{F}_0$. The hypothesis on \mathcal{F}' allows us to apply Zorn's lemma to find a right ideal $P \supseteq I$ with $P \in \text{Max}(\mathcal{F}')$, so $P \in \mathcal{S}$ by the Completely Prime Ideal Principle 3.4. Because \mathcal{F}_0 is a semifilter containing I , we also have $P \in \mathcal{F}_0$. It follows that $P \in \mathcal{S} \cap \mathcal{F}_0 \setminus \mathcal{F}$, contradicting our hypothesis.

Parts (2) and (3) follow from (1) by taking the semifilter \mathcal{F}_0 to be, respectively, the set of all right ideals containing J , the set of all right ideals properly containing J , or the set of all right ideals of R . \square

We will largely refrain from applying the Completely Prime Ideal Principle and its Supplement until §5, when we will have enough tools to efficiently construct right Oka families. However, it seems appropriate to at least give one classical application to showcase these ideas at work. In [25, Appendix], M. Nagata gave a simple proof of Cohen's Theorem 1.1 using the following lemma: if an ideal I and an element a of a commutative ring R are such that $I + (a)$ and $(I : a)$ are finitely generated, then I itself is finitely generated. This statement amounts to saying that the family of finitely generated ideals in a commutative ring is an Oka family. Nagata cited a paper [26] of K. Oka as the inspiration for this result. (In [21, p. 3007] it was pointed out that Oka's Corollaire 2 is the relevant statement.) This was the reason for the use of the term *Oka family* in [21]. More generally, in [21, Prop. 3.16] it was shown that, for any infinite cardinal α , the family of all ideals generated by a set of cardinality $< \alpha$ is Oka. The following generalizes this collection of results to the noncommutative setting. We let $\mu(M)$ denote the smallest cardinal μ such that the module M_R can be generated by a set of cardinality μ .

Proposition 3.7. *Let α be an infinite cardinal, and let $\mathcal{F}_{<\alpha}$ be the set of all right ideals $I_R \subseteq R$ with $\mu(I) < \alpha$. Then $\mathcal{F}_{<\alpha}$ is a right Oka family, and any right ideal maximal with respect to $\mu(I) \geq \alpha$ is completely prime. In particular, the set of finitely generated right ideals is a right Oka family; hence a right ideal maximal with respect to not being finitely generated is prime.*

Proof. We first show that $\mathcal{F}_{<\alpha}$ is a right Oka family. Let $I_R \subseteq R$, $a \in R$ be such that $I + aR$, $a^{-1}I \in \mathcal{F}_{<\alpha}$. From $\mu(I + aR) < \alpha$ it is straightforward to verify that there is a right ideal $I_0 \subseteq I$ with $\mu(I_0) < \alpha$ such that $I + aR = I_0 + aR$. It follows that $I = I_0 + a(a^{-1}I)$. Because $\mu(I_0) < \alpha$ and $\mu(a(a^{-1}I)) \leq \mu(a^{-1}I) < \alpha$, we see that $\mu(I) < \alpha + \alpha = \alpha$. Thus $I \in \mathcal{F}_{<\alpha}$, proving that $\mathcal{F}_{<\alpha}$ is right Oka.

If I is a right ideal maximal with respect to $\mu(I) \geq \alpha$ then $I \in \text{Max}(\mathcal{F}')$ and the CPIP 3.4 implies that I is completely prime. The last sentence follows when we take $\alpha = \aleph_0$. \square

This leads to a noncommutative generalization of Cohen's Theorem 1.1 for completely prime right ideals.

Theorem 3.8 (A noncommutative Cohen's Theorem). *A ring R is right noetherian iff all of its completely prime right ideals are finitely generated.*

Proof. This follows from Proposition 3.7 and the CPIP Supplement 3.6(3) applied to $\mathcal{F} = \mathcal{F}_{<\aleph_0}$, noticing that \mathcal{F} tautologically consists of f.g. right ideals. \square

Notice how quickly the last two results were proved using the CPIP and its Supplement! This highlights the utility of right Oka families as a framework from which to study such problems. Of course, other generalizations of Cohen's Theorem have been proven in the past. In a forthcoming paper [30], we will apply the methods of right Oka families developed here to improve upon our generalization of Cohen's Theorem. We will also develop noncommutative generalizations of the theorems of Kaplansky which say that a commutative ring is a principal ideal ring iff its prime ideals are principal, iff it is noetherian and its maximal ideals are principal. To avoid repetition, we shall wait until [30] to compare our version of Cohen's Theorem to earlier generalizations in the literature.

The generalization of Cohen's Theorem 1.1 in Theorem 3.8 above does not hold if we replace the phrase "completely prime" with "extremely prime" (as defined in §2). Indeed, using Proposition 2.4 we showed that there exist rings R that are not right noetherian with no extremely prime right ideals. But for such R , it is vacuously true that every extremely prime right ideal of R is finitely generated! This strikingly illustrates the idea that completely prime right ideals control the right-sided structure of a general ring better than extremely prime right ideals.

For any cardinal β , we can also define a family $\mathcal{F}_{\leq\beta}$ of all right ideals I such that $\mu(I) \leq \beta$. Letting β^+ denote the successor cardinal of β , we see that $\mathcal{F}_{\leq\beta} = \mathcal{F}_{<\beta^+}$, so we have not sacrificed any generality in the statement of Proposition 3.7. In particular, taking $\beta = \aleph_0$ we see that the family of all countably generated right ideals is a right Oka family. The "maximal implies prime" result in the case where R is commutative and $\beta = \aleph_0$ was noted in Exercise 11 of [15, p. 8]. The case of larger infinite cardinals α for commutative rings was proved by Gilmer and Heinzer in [9, Prop. 3]. (In [21, p. 3017], we mistakenly suggested that this had not been previously observed in the literature.)

One might wonder whether the obvious analogue of Cohen's Theorem for right ideals with generating sets of higher cardinalities is also true, in light of Proposition 3.7. However, in the commutative case Gilmer and Heinzer [9] have already settled this in the negative. The rings which serve as their counterexamples are (commutative) valuation domains.

It is well-known that Cohen's Theorem 1.1 can be used to prove that if R is a commutative noetherian ring, then the power series ring $R[[x]]$ is also noetherian. In [24], G. Michler proved a version of Cohen's Theorem and gave an analogous application of this result to power series over noncommutative rings. Here we show that our version of Cohen's Theorem can be applied in the same way.

Corollary 3.9. *If a ring R is right noetherian, then the power series ring $R[[x]]$ is also right noetherian.*

Proof. Let P be a completely prime right ideal of $S := R[[x]]$; by Theorem 3.8 it suffices to show that P is finitely generated. Let $C_R \subseteq R$ be the right ideal of R consisting of all

constant terms of all power series in P . Then P is finitely generated. Choose power series $f_1, \dots, f_n \in P$ whose constant terms generate C , and set $I := \sum f_j R \subseteq P$. If $x \in P$ then it is easy to see that $P = I + xS$ is finitely generated. So assume that $x \notin P$. In this case, we claim that $P = I$. Again we will have P finitely generated and the proof will be complete. Given $h \in P$, the constant term of $h_0 := h$ is equal to the constant term of some $g_0 = \sum a_{0j} f_j \in I$, where $a_{0j} \in R$. Then $h_0 - g_0 = xh_1$ for some $h_1 \in S$. Notice that $xh_1 = h_0 - g_0 \in P$. Because P is completely prime with $xP = Px \subseteq P$ and $x \notin P$, it follows that $h_1 \in P$. One can proceed inductively to find $g_i = \sum_{j=1}^n a_{ij} f_j$ ($a_{ij} \in R$) such that $h_i = g_i + xh_{i+1}$. Hence $h = \sum_{j=1}^n (\sum_{i=0}^{\infty} a_{ij} x^i) f_j \in I$. \square

Before moving on, we mention a sort of “converse” to the CPIP 3.4 characterizing exactly which families \mathcal{F} of right ideals are such that $\text{Max}(\mathcal{F}')$ consists of completely prime right ideals. It turns out that a weak form of the Oka property (3.2) characterizes these families.

Proposition 3.10. *Let \mathcal{F} be a family of right ideals in a ring R . All right ideals in $\text{Max}(\mathcal{F}')$ are completely prime iff, for all $I_R \subseteq R$ where every right ideal $J \supseteq I$ lies in \mathcal{F} and for all elements $a \in \mathbb{I}_R(I)$, the Oka property (3.2) is satisfied.*

Proof. First suppose that \mathcal{F} satisfies property (3.2) for all I and a described above. Then the proof of the CPIP 3.4 applies to show that any right ideal in $\text{Max}(\mathcal{F}')$ is completely prime. Conversely, suppose that $\text{Max}(\mathcal{F}')$ consists of completely prime right ideals, and let $I_R \subseteq R$ and $a \in \mathbb{I}_R(I)$ be as described above. Assume for contradiction that $I \notin \mathcal{F}$. It follows that $I \in \text{Max}(\mathcal{F}')$, so I is a completely prime right ideal. Because $I \notin \mathcal{F}$ and $I + aR \in \mathcal{F}$, we see that $a \notin I$. But then the remark at the end of Definition 2.1 shows that $I = a^{-1}I \in \mathcal{F}$, a contradiction. We conclude that in fact $I \in \mathcal{F}$, completing the proof. \square

So for a commutative ring R , this result classifies precisely which families \mathcal{F} of ideals satisfy $\text{Max}(\mathcal{F}') \subseteq \text{Spec}(R)$. (Note: here we can replace $a \in \mathbb{I}_R(I)$ by $a \in R$.) In fact, the above result was first discovered in the commutative setting by T. Y. Lam and the present author during the development of [21], though the result did not appear there.

4. RIGHT OKA FAMILIES AND CLASSES OF CYCLIC MODULES

In order to apply the CPIP 3.4, we need an effective tool for constructing right Oka families. The relevant result will be Theorem 4.7 below. This theorem generalizes one of the most important facts about Oka families in commutative rings: there is a correspondence between Oka families in a ring R and certain classes of cyclic R -modules (to be defined below). Throughout this paper we use \mathfrak{M}_R to denote the class of all right R -modules and $\mathfrak{M}_R^c \subseteq \mathfrak{M}_R$ to denote the subclass of cyclic R -modules.

Definition 4.1. Let R be any ring. A subclass $\mathcal{C} \subseteq \mathfrak{M}_R^c$ with $0 \in \mathcal{C}$ is *closed under extensions* if, for every exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of cyclic right R -modules, whenever $L, N \in \mathcal{C}$ it follows that $M \in \mathcal{C}$.

Specifically, it was shown in [21, Thm. 4.1] that for any commutative ring R , the Oka families in R are in bijection with the classes of cyclic R -modules that are closed under extensions. This correspondence provided many interesting examples of Oka families in commutative rings. The goal of this section is to show that the Oka families of right ideals

in an arbitrary ring R correspond to the classes of cyclic right R -modules which are closed under extensions.

In a commutative ring R , the correspondence described above was given by associating to any Oka family \mathcal{F} the class $\mathcal{C} := \{M_R : M \cong R/I \text{ for some } I \in \mathcal{F}\}$ of cyclic modules. Then \mathcal{F} is determined by \mathcal{C} because, for an ideal I of R , we may recover I from the isomorphism class of the cyclic module R/I since I is the annihilator of this cyclic module. (In fact, this works for any family \mathcal{F} of ideals in R .) However, in a noncommutative ring there can certainly exist right ideals $I, J \subseteq R$ such that $I \neq J$ but $R/I \cong R/J$ (as right R -modules).

Example 4.2. Any simple artinian ring R has a single isomorphism class of simple right modules. Thus all maximal right ideals $\mathfrak{m}_R \subseteq R$ have isomorphic factor modules R/\mathfrak{m} . But if $R \cong \mathbb{M}_n(k)$ for a division ring k and if $n > 1$ (i.e. R is not a division ring), then there exist multiple maximal right ideals: we may take \mathfrak{m}_i ($i = 1, \dots, n$) to correspond to the right ideal of matrices whose i th row is zero. In fact, over an infinite division ring k even the ring $\mathbb{M}_2(k)$ has infinitely many maximal right ideals! This is true because, for any $\lambda \in k$, the set of all matrices of the form

$$\begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}$$

is a maximal right ideal, and these right ideals are distinct for each value of λ . (Of course, a similar construction also works over the ring $\mathbb{M}_n(k)$ for $n > 2$.)

Therefore we do not expect every family \mathcal{F} of right ideals to naturally correspond to a class of cyclic modules. This prompts the following definition.

Definition 4.3. Two right ideals I and J of a ring R are said to be *similar* if $R/I \cong R/J$ as right R -modules. A family \mathcal{F} of right ideals in a ring R is *closed under similarity* if, for any similar right ideals $I, J \subseteq R$, $I \in \mathcal{F}$ implies $J \in \mathcal{F}$. This is equivalent to $I \in \mathcal{F} \iff J \in \mathcal{F}$ whenever $R/I \cong R/J$.

The notion of similarity dates at least as far back as Jacobson's text [14, pp. 33 & 130] (although he only studied this idea in specific classes of rings). With the appropriate terminology in place, the next fact is easily verified.

Proposition 4.4. *For any ring R , there is a bijective correspondence*

$$\left\{ \begin{array}{l} \text{families } \mathcal{F} \text{ of right} \\ \text{ideals of } R \text{ that are} \\ \text{closed under similarity} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{classes } \mathcal{C} \text{ of cyclic right} \\ \text{R-modules that are closed} \\ \text{under isomorphism} \end{array} \right\}.$$

For a family \mathcal{F} and a class \mathcal{C} as above, the correspondence is given by the maps

$$\begin{aligned} \mathcal{F} &\mapsto \mathcal{C}_{\mathcal{F}} := \{M_R : M \cong R/I \text{ for some } I \in \mathcal{F}\}, \\ \mathcal{C} &\mapsto \mathcal{F}_{\mathcal{C}} := \{I_R \subseteq R : R/I \in \mathcal{C}\}. \end{aligned}$$

We will show that every right Oka family is closed under similarity with the help of the following two-part lemma. The first part describes, up to isomorphism, the cyclic submodules of a cyclic module R/I . The second part is a rather well-known criterion for two cyclic modules to be isomorphic.

Lemma 4.5. *Let R be a ring.*

(A) For any right ideal $I \subseteq R$ and any element $a \in R$, there is an isomorphism

$$R/a^{-1}I \xrightarrow{\sim} (I + aR)/I \subseteq R/I$$

given by $r + a^{-1}I \mapsto ar + I$.

(B) Given two right ideals $I_R, J_R \subseteq R$, $R/I \cong R/J$ iff there exists $a \in R$ such that $I + aR = R$ and $a^{-1}I = J$.

Proof. Part (A) is a straightforward application of the First Isomorphism Theorem. Proofs for part (B) can be found, for instance, in [5, Prop. 1.3.6] or [19, Ex. 1.30]. (In fact, it was already observed in Jacobson's text [14, p. 33], though in the special setting of PIDs.) In any case, the “if” direction follows from part (A) above, and the reader can readily verify the “only if” direction. \square

These elementary observations are very important for us. The reader should be aware that we will freely use the isomorphism $R/a^{-1}I \cong (I + aR)/I$ throughout this paper.

Proposition 4.6. *A family \mathcal{F} of right ideals in a ring R is closed under similarity iff for any $I_R \subseteq R$ and $a \in R$, $I + aR = R$ and $a^{-1}I \in \mathcal{F}$ imply $I \in \mathcal{F}$. In particular, any right Oka family \mathcal{F} is closed under similarity.*

Proof. The first statement follows directly from Lemma 4.5(B), and the second statement follows from Definition 3.1 because every right Oka family contains the unit ideal R . \square

Thus we see that every right Oka family will indeed correspond, as in Proposition 4.4, to *some* class of cyclic right modules; it remains to show that they correspond precisely to the classes that are closed under extensions. We first need to mention one fact regarding module classes closed under extensions. From the condition $0 \in \mathcal{C}$ and the exact sequence $0 \rightarrow L \rightarrow M \rightarrow 0 \rightarrow 0$ for $L_R \cong M_R$, we see that a class \mathcal{C} of cyclic modules closed under extensions is also closed under isomorphisms. We are now ready to prove the main result of this section.

Theorem 4.7. *Given a class \mathcal{C} of cyclic right R -modules that is closed under extensions, the family $\mathcal{F}_{\mathcal{C}}$ is an Oka family of right ideals. Conversely, given a right Oka family \mathcal{F} , the class $\mathcal{C}_{\mathcal{F}}$ of cyclic right R -modules is closed under extensions.*

Proof. First suppose that the given class \mathcal{C} is closed under extensions. Then $R \in \mathcal{F}_{\mathcal{C}}$ because $0 \in \mathcal{C}$. So let $I_R \subseteq R$ and $a \in R$ be such that $I + aR, a^{-1}I \in \mathcal{F}_{\mathcal{C}}$. Then $R/(I + aR)$ and $R/a^{-1}I$ lie in \mathcal{C} . Moreover, we have an exact sequence

$$0 \rightarrow (I + aR)/I \rightarrow R/I \rightarrow R/(I + aR) \rightarrow 0,$$

where $(I + aR)/I \cong R/a^{-1}I$ lies in \mathcal{C} (recall that \mathcal{C} is closed under isomorphisms). Because \mathcal{C} is closed under extensions, $R/I \in \mathcal{C}$. Thus $I \in \mathcal{F}_{\mathcal{C}}$, proving that $\mathcal{F}_{\mathcal{C}}$ is a right Oka family.

Now suppose that \mathcal{F} is a right Oka family. That $0 \in \mathcal{C}_{\mathcal{F}}$ follows from the fact that $R \in \mathcal{F}$. Consider an exact sequence of cyclic right R -modules

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

where $L, N \in \mathcal{C}_{\mathcal{F}}$, so that there exist $A, B \in \mathcal{F}$ such that $L \cong R/A$ and $N \cong R/B$. We may identify M up to isomorphism with R/I for some right ideal $I \subseteq R$. Because L is cyclic and embeds in $M \cong R/I$, we have $L \cong (I + aR)/I$ for some $a \in R$. Hence $R/(I + aR) \cong N \cong R/B$, and Proposition 4.6 implies that $I + aR \in \mathcal{F}$. Note also that

$R/a^{-1}I \cong (I+aR)/I \cong L \cong R/A$, so by Proposition 4.6 we conclude that $a^{-1}I \in \mathcal{F}$. Because \mathcal{F} is a right Oka family we must have $I \in \mathcal{F}$. So $M \cong R/I$ implies that $M \in \mathcal{C}_{\mathcal{F}}$. \square

We examine one consequence of this correspondence. This will require the following lemma, which compares a class $\mathcal{C} \subseteq \mathfrak{M}_R^c$ that is closed under extensions with its closure under extensions in the larger class \mathfrak{M}_R .

Lemma 4.8. *Let \mathcal{C} be a class of cyclic right R -modules that is closed under extensions (as in Definition 4.1), and let $\bar{\mathcal{C}}$ be its closure under extensions in the class \mathfrak{M}_R of all cyclic right R -modules. Then $\mathcal{C} = \bar{\mathcal{C}} \cap \mathfrak{M}_R^c$.*

Proof. Certainly $\mathcal{C} \subseteq \bar{\mathcal{C}} \cap \mathfrak{M}_R^c$. Conversely, suppose that $M \in \bar{\mathcal{C}} \cap \mathfrak{M}_R^c$. Because $M \in \bar{\mathcal{C}}$, there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that each $M_j/M_{j-1} \in \mathcal{C}$. One can then prove by downward induction that the cyclic modules M/M_j lie in \mathcal{C} . So $M \cong M/M_0$ and $M/M_0 \in \mathcal{C}$ imply that $M \in \mathcal{C}$. \square

Corollary 4.9. *Let \mathcal{F} be a right Oka family in a ring R . Suppose that $I_R \subseteq R$ is such that R/I has a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = R/I$$

where each filtration factor is cyclic and of the form $M_j/M_{j-1} \cong R/I_j$ for some $I_j \in \mathcal{F}$. Then $I \in \mathcal{F}$.

Proof. Let $\mathcal{C} := \mathcal{C}_{\mathcal{F}}$, which is closed under extensions by Theorem 4.7. Then the above filtration of the cyclic module $M = R/I$ has filtration factors isomorphic to the $R/I_j \in \mathcal{C}$. From Lemma 4.8 it follows that $R/I \in \mathcal{C}$, and thus $I \in \mathcal{F}_{\mathcal{C}} = \mathcal{F}$. \square

This implies, for instance, that if a right Oka family \mathcal{F} in a ring R contains all maximal right ideals of R , then it contains all right ideals I such that R/I has finite length.

We close this section by applying Theorem 4.7 to produce a second “converse” to the Completely Prime Ideal Principle 3.4, distinct from the one mentioned at the end of §3. This result mildly strengthens the CPIP to an “iff” statement, saying that a right ideal P of a ring R is completely prime iff $P \in \text{Max}(\mathcal{F}')$ for some right Oka family \mathcal{F} . (This was already noted in the commutative case in [22, p. 274].)

Let V_R be an R -module, and define the class

$$(4.10) \quad \mathcal{E}[V] := \{M_R : M = 0 \text{ or } M \not\hookrightarrow V\}.$$

We claim that $\mathcal{E}[V]$ is closed under extensions in \mathfrak{M}_R . Indeed, suppose that $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence in \mathfrak{M}_R with $L, N \in \mathcal{E}[V]$. If $L = 0$ then $M \cong N$, so that $M \in \mathcal{E}[V]$. Otherwise L cannot embed in V . Because $L \hookrightarrow M$, M cannot embed in V , proving $M \in \mathcal{E}[V]$. With the class $\mathcal{E}[V]$ in mind, we prove the second “converse” of the CPIP.

Proposition 4.11. *For any completely prime right ideal $P_R \subseteq R$, there exists an Oka family \mathcal{F} of right ideals in R such that $P \in \text{Max}(\mathcal{F}')$.*

Proof. Let $V_R = R/P$, and let $\mathcal{E}[V]$ be as above. Fixing the class $\mathcal{C} = \mathcal{E}[V] \cap \mathfrak{M}_R^c$, set $\mathcal{F} := \mathcal{F}_{\mathcal{C}}$. By Theorem 4.7, \mathcal{F} is a right Oka family. Certainly $P \notin \mathcal{F}$ since $R/P = V \notin \mathcal{E}[V]$, so it only remains to show the maximality of P . Assume for contradiction that there is a right ideal $I \notin \mathcal{F}$ with $I \supsetneq P$. Then we have a natural surjection $R/P \twoheadrightarrow R/I$, and because $I \notin \mathcal{F}$ we have $0 \neq R/I \hookrightarrow V = R/P$. Composing these maps as

$$R/P \twoheadrightarrow R/I \hookrightarrow R/P$$

gives a nonzero endomorphism $f \in \text{End}(R/P)$ with $\ker f = I/P \neq 0$. This contradicts characterization (3) of Proposition 2.5, so we must have $P \in \text{Max}(\mathcal{F})$ as desired. \square

5. APPLICATIONS OF THE COMPLETELY PRIME IDEAL PRINCIPLE

In this section we will give various applications of the Completely Prime Ideal Principle. Every application should be viewed as a new source of completely prime right ideals in a ring or as an application of the notion of completely prime right ideals (and right Oka families) to study the one-sided structure of a ring. The diversity of concepts that interweave with the notion of completely prime right ideals (via right Oka families) in this section showcases the ubiquity of these objects. We remind the reader that when verifying that a set \mathcal{F} of right ideals in R is a right Oka family, we will often skip the step of checking that $R \in \mathcal{F}$.

Remark 5.1. An effective method of creating right Oka families is as follows. Consider a subclass $\mathcal{E} \subseteq \mathfrak{M}_R$ that is closed under extensions in the full class of right modules \mathfrak{M}_R . Then $\mathcal{C} = \mathcal{E} \cap \mathfrak{M}_R^c$ is a class of cyclic modules that is closed under extensions. Hence $\mathcal{F} := \mathcal{F}_{\mathcal{C}}$ is a right Oka family. (Notice that, according to Lemma 4.8, *every* such \mathcal{C} arises this way.)

When working relative to a ring homomorphism, a similar method applies. Recall that for a ring k , a k -ring R is a ring with a fixed homomorphism $k \rightarrow R$. Given a k -ring R , let \mathcal{E}_1 be any class of right k -modules that is closed under extensions in \mathfrak{M}_k , and let \mathcal{E} denote the subclass of \mathfrak{M}_R consisting of modules that lie in \mathcal{E}_1 when considered as k -modules under the map $k \rightarrow R$. Then \mathcal{E} is certainly closed under extensions in \mathfrak{M}_R , so $\mathcal{C} := \mathcal{E} \cap \mathfrak{M}_R^c$ is closed under extensions and $\mathcal{F} := \mathcal{F}_{\mathcal{C}}$ is a right Oka family.

5.A. Point annihilators and zero-divisors. Point annihilators are basic objects from commutative algebra that connect the modules over a commutative ring to the ideals of that ring. Prime ideals play an important role there in the form of associated primes of a module. Here we study these themes in the setting of noncommutative rings.

Definition 5.2. For a ring R and a module $M_R \neq 0$, a *point annihilator* of M is a right ideal of the form $\text{ann}(m)$ for some $0 \neq m \in M$.

A standard theorem of commutative algebra states that for a module M_R over a commutative ring R , a maximal point annihilator of M is a prime ideal. The next result is the direct generalization of this fact. This application takes advantage of the construction $\mathcal{E}[V]$ presented in (4.10).

Proposition 5.3. *Let R be a ring and $M_R \neq 0$ an R -module. The family \mathcal{F} of right ideals that are not point annihilators of M is a right Oka family. Thus, a maximal point annihilator of M is a completely prime right ideal.*

Proof. Following the notation of (4.10), let $\mathcal{C} = \mathcal{E}[M] \cap \mathfrak{M}_R^c$, which is a class of cyclic modules closed under extensions. Then $\mathcal{F}_{\mathcal{C}}$ is a right Oka family. But by definition of $\mathcal{E}[M]$, we see that

$$\begin{aligned} \mathcal{F}'_{\mathcal{C}} &= \{I_R \subseteq R : 0 \neq R/I \hookrightarrow M\} \\ &= \{\text{ann}(m) : 0 \neq m \in M\} \\ &= \mathcal{F}'. \end{aligned}$$

So $\mathcal{F} = \mathcal{F}_{\mathcal{C}}$ is a right Oka family. The last statement follows from the CPIP 3.4. \square

The proof that a maximal point annihilator of a module M_R is completely prime can also be achieved using the following family:

$$\mathcal{F} := \{I_R \subseteq R : \text{for } m \in M, mI = 0 \implies m = 0\}.$$

One can show that \mathcal{F} is a right Oka family. Moreover, it is readily checked that $\text{Max}(\mathcal{F}')$ consists of the maximal point annihilators of M . The CPIP again applies to show that the maximal point annihilators of M are completely prime. This was essentially the approach taken in the commutative case in [21, Prop. 3.5].

As in the theory of modules over commutative rings, one may wish to study “associated primes” of a module M over a noncommutative ring R . For a module M_R , let us say that a completely prime right ideal $P_R \subseteq R$ is *associated to* M if it is a point annihilator of M (equivalently, if $R/P \hookrightarrow M$). A famous fact from commutative algebra is that a noetherian module over a commutative ring has only finitely many associated primes; see [7, Thm. 3.1]. It is easy to show that the analogous statement for completely prime right ideals does not hold over noncommutative rings. For instance, Example 4.2 provided a ring R with infinitely many maximal right ideals $\{\mathfrak{m}_i\}$ such that the modules R/\mathfrak{m}_i were all isomorphic to the same simple module, say S_R . Then the \mathfrak{m}_i are infinitely many completely prime right ideals that are associated to the module S (which is simple and thus noetherian).

In response to this easy example, one may ask whether a noetherian module has finitely many associated completely prime right ideals *up to similarity*. Again, the answer is negative. We recall an example used by K. R. Goodearl in [11] to answer a question by Goldie. Let k be a field of characteristic zero and let D be the derivation on the power series ring $k[[y]]$ given by $D = y \frac{d}{dy}$. Define $R := k[[y]][x; D]$, a skew polynomial extension. Consider the right module $M_R = R/xR$. Notice that $M \cong k[[y]]$ as a module over $k[[y]]$. Goodearl showed that the nonzero submodules of M are precisely the $\bar{y}^i R \cong y^i k[[y]]$ (where $\bar{y}^i = y^i + xR \in M$) and that these submodules are pairwise nonisomorphic. From the fact that each of these submodules has infinite k -dimension and finite k -codimension in M , one can easily verify that M (and its nonzero submodules) are monoform (as in Definition 6.2). So the right ideals $\text{ann}(\bar{y}^i)$ are comonoform and thus are completely prime by Proposition 6.3 to be proved later. But they are pairwise nonsimilar because the factor modules $R/\text{ann}(\bar{y}^i) \cong \bar{y}^i R$ are pairwise nonisomorphic.

(In spite of this failure of finiteness, interested readers should note that O. Goldman developed a theory of associated primes of noncommutative rings in which every noetherian module has finitely many associated primes; see [10, Thm. 6.14]. We will not discuss Goldman’s prime torsion theories here but will simply remark that they are related to monoform modules and comonoform right ideals, which are discussed in the the next section. See, for instance, [32].)

The following is an application of Proposition 5.3. For a nonzero module M_R , we define the *zero-divisors of M* in R to be the set of all $z \in R$ such that $mz = 0$ for some $0 \neq m \in M$. A theorem from commutative algebra states that the set of zero-divisors of a module over a commutative ring R is equal to the union of some set of prime ideals. Here we generalize this fact to noetherian right modules over noncommutative rings.

Corollary 5.4. *Let M_R be a module over a ring R such that R satisfies the ACC on point annihilators of M (e.g., if M_R or R_R is noetherian). Then the set of zero-divisors of M is a union of completely prime right ideals.*

Proof. Let $z \in R$ be a zero-divisor of M_R . Then there exists $0 \neq m \in M$ such that $z \in \text{ann}(m)$. Because R satisfies ACC on point annihilators of M , there exists a maximal point annihilator $P_z \subseteq R$ of M containing $\text{ann}(m)$, so that $z \in \text{ann}(m) \subseteq P_z$. By Proposition 5.3, P_z is a completely prime right ideal. Choosing some such P_z for every zero-divisor z on M , we see that the set of zero-divisors of M is equal to $\bigcup_z P_z$.

If R is right noetherian, then the ACC hypothesis is certainly satisfied. Finally, let us assume that M_R is noetherian and prove that R satisfies ACC on point annihilators of M . Let $I := \text{ann}(m_0) \subseteq \text{ann}(m_1) \subseteq \cdots$ be an ascending chain of point annihilators of M (where $m_i \in M \setminus \{0\}$). Notice that $R/I \cong m_0 R \subseteq M$ is a noetherian module; thus R satisfies ACC on right ideals containing I . It follows that this ascending chain of point annihilators of M must stabilize. \square

Next we shall investigate conditions for a ring to be a domain. The following fact from commutative algebra was recovered in [21, Cor. 3.2]: a commutative ring R is a domain iff every nonzero prime ideal of R contains a regular element. We generalize this result through a natural progression of ideas, starting with another application of Proposition 5.3. Given a ring R , we will use the term *right principal annihilator* to mean a right ideal of the form $I = \text{ann}_r(x)$ for some $x \in R \setminus \{0\}$. This is just another name for a point annihilator of the module R_R , but we use this term below to evoke the idea of chain conditions on annihilators. Also, by a *left regular element* of R we mean an element $s \in R$ such that $\text{ann}_\ell(s) = 0$.

Proposition 5.5. *For any nonzero ring R , the following are equivalent:*

- (1) R is a domain;
- (2) R satisfies ACC on right principal annihilators, and for every nonzero completely prime right ideal P of R , P is not a right principal annihilator;
- (3) R satisfies ACC on right principal annihilators, and every nonzero completely prime right ideal of R contains a left regular element.

Proof. Certainly (1) \implies (3) \implies (2), so it suffices to show (2) \implies (1). Let R be as in (2), and let \mathcal{F} be the family of right ideals of R which are not point annihilators of the module R_R . Then \mathcal{F} is a right Oka family by Proposition 5.3. Because every point annihilator of R_R is a right principal annihilator, the first hypothesis shows that \mathcal{F}' has the ascending chain condition. Furthermore, the second assumption shows that any nonzero completely prime right ideal of R lies in \mathcal{F} . By the CPIP Supplement 3.6(2), all nonzero right ideals lie in \mathcal{F} . It follows that every nonzero element of R has zero right annihilator, proving that R is a domain. \square

A simple example demonstrates that the chain condition is in fact necessary for (1) \iff (2) above. Indeed, let k be a field and let R be the commutative k -algebra generated by

$\{x_i : i \in \mathbb{N}\}$ with relations $x_i^2 = 0$. Clearly R is not a domain, but its unique prime ideal (x_0, x_1, x_2, \dots) is not a principal annihilator.

This leaves us with the following question: If every completely prime right ideal of a ring contains a left regular element, then is R a domain? Professor G. Bergman has answered this question in the affirmative. With his kind permission, we present a modified version of his argument below.

Lemma 5.6. *For a ring R and a module M_R , let \mathcal{F} be the family of right ideals I of R such that there exists a nonempty finite subset $X \subseteq I$ such that, for all $m \in M$, $mX = 0 \implies m = 0$. The family \mathcal{F} is a right Oka family.*

Proof. To see that $R \in \mathcal{F}$, simply take $X = \{1\} \subseteq R$. Now suppose that $I_R \subseteq R$ and $a \in R$ are such that $I + aR, a^{-1}I \in \mathcal{F}$. Choose nonempty subsets $X_0 = \{i_1 + ar_1, \dots, i_p + ar_p\} \subseteq I + aR$ (where each $i_k \in I$) and $X_1 = \{x_1, \dots, x_q\} \subseteq a^{-1}I$ such that, for $m \in M$, $mX_j = 0$ implies $m = 0$ (for $j = 0, 1$). Define

$$X := \{i_1, \dots, i_p, ax_1, \dots, ax_q\} \subseteq I.$$

Suppose that $mX = 0$ for some $m \in M$. Then $maX_1 \subseteq mX = 0$ implies that $ma = 0$. It follows that $mX_0 = 0$, from which we conclude $m = 0$. This proves that $I \in \mathcal{F}$, hence \mathcal{F} is right Oka. \square

Proposition 5.7. *For a module $M_R \neq 0$ over a ring R , the following are equivalent:*

- (1) M has no zero-divisors (i.e., $0 \neq m \in M$ and $0 \neq r \in R$ imply $mr \neq 0$);
- (2) Every nonzero completely prime right ideal of R contains a non zero-divisor for M ;
- (3) Every nonzero completely prime right ideal P of R has a nonempty finite subset $X \subseteq P$ such that, for all $m \in M$, $mX = 0 \implies m = 0$.

Proof. Clearly (1) \implies (2) \implies (3); we prove (3) \implies (1). Assume that (3) holds, and let \mathcal{F} be the Oka family of right ideals defined in Lemma 5.6. It is easy to check that the union of any chain of right ideals in \mathcal{F}' also lies in \mathcal{F}' . By (3), every nonzero completely prime right ideal of R lies in \mathcal{F} . Then the CPIP Supplement 3.6 implies that all nonzero right ideals of R lie in \mathcal{F} . It is clear that no right ideal in \mathcal{F} can be a point annihilator for M . It follows immediately that M has no zero-divisors. \square

Corollary 5.8. *For a ring $R \neq 0$, the following are equivalent:*

- (1) R is a domain;
- (2) Every nonzero completely prime right ideal of R contains a left regular element;
- (3) Every nonzero completely prime right ideal of R has a nonempty finite subset whose left annihilator is zero.

Here is another demonstration that completely prime right ideals control the structure of a ring better than the “extremely prime” right ideals (discussed in §2). Using Proposition 2.4 we constructed rings with no extremely prime right ideals that are not domains. But it is vacuously true that every extremely prime right ideal of such a ring contains a regular element. Thus there is no hope that the result above could be achieved using this more sparse collection of one-sided primes.

5.B. Homological properties. Module-theoretic properties that are preserved under extensions arise very naturally in homological algebra. This provides a rich supply of right Oka families, and consequently produces completely prime right ideals via the CPIP.

Example 5.9. For a ring k and a k -ring R , consider the following properties of a right ideal $I_R \subseteq R$ (which are known to be preserved by extensions of the factor module):

- (1) R/I is a projective right k -module;
- (2) R/I is an injective right k -module;
- (3) R/I is a flat right k -module.

For each property above, the family \mathcal{F} of all right ideals with that property is a right Oka family (by Remark 5.1); hence $\text{Max}(\mathcal{F}')$ consists of completely prime right ideals.

We have the following immediate application, which includes a criterion for a ring to be semisimple.

Proposition 5.10. *The family \mathcal{F} of right ideals that are direct summands of R_R is a right Oka family. A right ideal $I_R \subseteq R$ maximal with respect to not being a direct summand of R is a maximal right ideal. A ring R is semisimple iff every maximal right ideal of R is a direct summand of R_R .*

Proof. This family \mathcal{F} is readily seen to be equal to the family given in Example 5.9 (with $k = R$ and the identity map $k \rightarrow R$), and thus it is right Oka. Let $P \in \text{Max}(\mathcal{F}')$. Then P is completely prime, so R/P is indecomposable by Corollary 2.7. On the other hand, because every right ideal properly containing P splits in R_R , the module R/P is semisimple. It follows that R/P is simple, so P is maximal as claimed.

The nontrivial part of the last sentence is the “if” direction. Assume that every maximal right ideal of R is a direct summand. It suffices to show that every completely prime right ideal of R is maximal. (For if this is the case, then every completely prime right ideal will lie the right Oka family \mathcal{F} . Now \mathcal{F} consists of principal—hence f.g.—right ideals by the classical fact that $\mathcal{F} = \{eR : e^2 = e \in R\}$. Then the CPIP Supplement 3.6(3) will show that every right ideal of R is a direct summand, making R semisimple.) So suppose $P_R \subsetneq R$ is completely prime. Fix a maximal right ideal \mathfrak{m} of R with $\mathfrak{m} \supseteq P$. Because \mathfrak{m} is a proper direct summand of R_R , \mathfrak{m}/P must be a proper summand of R/P . But R/P is indecomposable by Proposition 2.7. Thus $\mathfrak{m}/P = 0$, so that $P = \mathfrak{m}$ is maximal. \square

Of course, we can also prove the “iff” statement above without any reference to right Oka families. (Suppose that every maximal right ideal of R is a direct summand. Assume for contradiction that the right socle $S_R := \text{soc}(R_R)$ is a proper right ideal. Then there is some maximal right ideal $\mathfrak{m} \subseteq R$ such that $S \subseteq \mathfrak{m}$. But by hypothesis there exists $V_R \subseteq R$ such that $R = V \oplus \mathfrak{m}$. Then $V \cong R/\mathfrak{m}$ is simple. So $V \subseteq S$, contradicting the fact that $V \cap S \subseteq V \cap \mathfrak{m} = 0$.) Although such ad hoc methods are able to recover this fact, our method involving the CPIP 3.4 has the desirable effect of fitting the result into a larger context. Also, the CPIP and right Oka families may point one to results that might not have otherwise been discovered without this viewpoint, even if these results could have been proven individually with other methods.

We can also use Example 5.9 to recover a bit of the structure theory of right PCI rings. A right module over a ring R is called a *proper cyclic module* if it is cyclic and not isomorphic to R_R . (Note that this is stronger than saying that the module is isomorphic to R/I for

some $0 \neq I_R \subseteq R$, though it is easy to confuse the two notions.) A ring R is a *right PCI ring* if every proper cyclic right R -module is injective, and such a ring R is called a *proper right PCI ring* if it is not right self-injective (by a theorem of Osofsky, this is equivalent to saying that R is not semisimple). C. Faith showed in [8] that any proper right PCI ring is a simple right semihereditary right Ore domain. In Faith’s own words [8, p. 98], “The reductions to the case R is a domain are long, and not entirely satisfactory inasmuch as they are quite intricate.” Our next application of the Completely Prime Ideal Principle shows how to easily deduce that a proper right PCI ring is a domain with the help of a later result on right PCI rings.

Proposition 5.11 (Faith). *A proper right PCI ring is a domain.*

Proof. A theorem of R. F. Damiano [6] states that any right PCI ring is right noetherian. (Another proof of this result, due to B. L. Osofsky and P. F. Smith, appears in [28, Cor. 7].) In particular, any right PCI ring is Dedekind-finite.

Now let R be a proper right PCI ring. Because R is Dedekind-finite, for every nonzero right ideal I , $R/I \not\cong R_R$ is a proper cyclic module. Letting \mathcal{F} denote the family of right ideals I such that R/I is injective, we have $0 \in \text{Max}(\mathcal{F})$. But \mathcal{F} is a right Oka family by Example 5.9, so the CPIP 3.4 and Proposition 2.2 together show that R is a domain. \square

The astute reader may worry that the above proof is nothing more than circular reasoning, because the proof of Damiano’s theorem in [6] seems to rely on Faith’s result! This would indeed be the case if Damiano’s were the only proof available for his theorem. (Specifically, Damiano cites another result of Faith—basically [8, Prop. 16A]—to conclude that over a right PCI ring, every finitely presented proper cyclic module has a von Neumann regular endomorphism ring. But Faith’s result is stated only for cyclic *singular* finitely presented modules. So Damiano seems to be implicitly applying the fact that a proper right PCI ring is a right Ore domain.) Thankfully, we are saved by the fact that Osofsky and Smith’s (considerably shorter) proof [28, Cor. 7] of Damiano’s result does not require any of Faith’s structure theory.

It is worth noting that A. K. Boyle had already provided a proof [3, Cor. 9] that a right noetherian proper right PCI ring is a domain. (This was before Damiano’s theorem had been proved.) One difference between our approach and that of [3] is that we do not use any facts about direct sum decompositions of injective modules over right noetherian rings. Of course, the proof using the CPIP is also desirable because we are able to fit the result into a larger context in which it becomes “natural” that such a ring should be a domain.

As in [21], we can generalize Example 5.9 with items (1)–(3) below. One may think of the following examples as being defined by the existence of certain (co)resolutions of the modules. Recall that a module M_R is said to be *finitely presented* if there exists an exact sequence of the form $R^m \rightarrow R^n \rightarrow M \rightarrow 0$, and that a *finite free resolution* of M is an exact sequence of the form

$$0 \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

where the F_i are finitely generated free modules.

Example 5.12. Let R be a k -ring, and fix any one of the following properties of a right ideal I in R (known to be closed under extensions), where n is a nonnegative integer:

- (1) R/I has k -projective dimension $\leq n$ (or $< \infty$);

- (2) R/I has k -injective dimension $\leq n$ (or $< \infty$);
- (3) R/I has k -flat dimension $\leq n$ (or $< \infty$);
- (4) R/I is a finitely presented right R -modules;
- (5) R/I has a finite free resolution as a right R -module.

Then the family \mathcal{F} of right ideals satisfying that property is right Oka (as in Remark 5.1), and $\text{Max}(\mathcal{F}')$ consists of completely prime right ideals.

The families in Example 5.9 are just (1)–(3) above with $n = 0$. Restricting to the case $n = 1$ and $k = R$, the family obtained from part (1) (resp. part (3)) of Example 5.12 is the family of projective (resp. flat) right ideals of R (for the latter, see [17, (4.86)(2)]). In particular, the CPIP 3.4 implies that *a right ideal of R maximal with respect to not being projective (resp. flat) over R is completely prime*.

The family \mathcal{F} in part (4) above is actually equal to the family of *finitely generated* right ideals. Indeed, if $I_R \subseteq R_R$ is finitely generated, then the module R/I is certainly finitely presented. Conversely, if R/I is a finitely presented module, [17, (4.26)(b)] implies that I_R is f.g. This recovers the last sentence of Proposition 3.7 from a module-theoretic perspective.

We can also use the family in (2) above to generalize Proposition 5.11 about proper right PCI rings. Given a nonnegative integer n , let us say that a ring R is a *right n -PCI ring* if the supremum of the injective dimensions of all proper cyclic right R -modules is equal to n . (Thus a right 0-PCI ring is simply a right PCI ring.) Also, call a right n -PCI ring R *proper* if R_R has injective dimension greater than n (possibly infinite). Then the following is proved as in 5.11, using the family from Example 5.12(2).

Proposition 5.13. *If a proper right n -PCI ring is Dedekind-finite, then it is a domain.*

Unlike the above result, Proposition 5.11 is not a conditional statement because Damiano’s theorem guarantees that a right 0-PCI ring is right noetherian, hence Dedekind-finite. Also, it is known that right PCI rings are right hereditary, which implies that the injective dimension of R_R is 1 if R is proper right 0-PCI. Thus we pose the following questions.

Question 5.14. *What aspects of the Faith-Damiano structure theory for PCI rings carry over to n -PCI rings? In particular, for a proper right n -PCI ring R , we ask:*

- (1) *Must R_R have finite injective dimension? If so, is this dimension necessarily equal to $n + 1$?*
- (2) *Must R be Dedekind-finite, or even possibly right noetherian? What if we assume that R_R has finite injective dimension, say equal to $n + 1$ (if (1) above fails)?*

Further generalizing Examples 5.9 and 5.12, we have the following.

Example 5.15. Given a k -ring R , fix a right module M_k and a left module ${}_k N$, and let n be a nonnegative integer. Fix one of the following properties of a right ideal $I \subseteq R$:

- (1) R/I satisfies $\text{Ext}_k^n(R/I, M) = 0$;
- (2) R/I satisfies $\text{Ext}_k^n(M, R/I) = 0$;
- (3) R/I satisfies $\text{Tor}_n^k(R/I, N) = 0$.

Applying Remark 5.1, the family \mathcal{F} of right ideals satisfying that fixed property is right Oka. Thus $\text{Max}(\mathcal{F}')$ consists of completely prime right ideals.

(The fact that the corresponding classes of cyclic modules are closed under extensions follows from a simple analysis of the long exact sequences for Ext and Tor derived from a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathfrak{M}_R .)

We can actually use these to recover the families (1)–(3) of Example 5.12 as follows. It is known that a module B has k -projective dimension n iff $\text{Ext}_k^n(B, M) = 0$ for all right modules M_k . Then intersecting the families in Example 5.15(1) over all modules M_k gives the class in Example 5.12(1). A similar process works for (2) and (3) of Example 5.12.

As an application of case (1) above, we present the following interesting family of right ideals in any ring associated to an arbitrary module M_R .

Proposition 5.16. *For any module M_R , the family \mathcal{F} of all right ideals $I_R \subseteq R$ such that any homomorphism $f: I \rightarrow M$ extends to some $\tilde{f}: R \rightarrow M$ is a right Oka family. A right ideal maximal with respect to $I \notin \mathcal{F}$ is completely prime.*

Proof. Let \mathcal{G} be the family in Example 5.15(1) with $k = R$ and $n = 1$. We claim that $\mathcal{F} = \mathcal{G}$, from which the proposition will certainly follow. Given $I_R \subseteq R$, consider the long exact sequence in Ext associated to the short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$:

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(R/I, M) &\rightarrow \text{Hom}_R(R, M) \rightarrow \text{Hom}_R(I, M) \\ &\rightarrow \text{Ext}_R^1(R/I, M) \rightarrow \text{Ext}_R^1(R, M) = 0 \end{aligned}$$

($\text{Ext}_R^1(R, M) = 0$ because R_R is projective). Thus $I \in \mathcal{F}$ iff the natural map $\text{Hom}_R(R, M) \rightarrow \text{Hom}_R(I, M)$ is surjective, iff its cokernel $\text{Ext}_R^1(R/I, M)$ is zero, iff $I \in \mathcal{G}$. \square

It is an interesting exercise to “check by hand” that the family \mathcal{F} above satisfies the Oka property (3.2). When R is a right self-injective ring, one can dualize the above proof of Proposition 5.16, (R_R must be injective to ensure that $\text{Ext}_R^1(M, R) = 0$), and a similar argument works using the functor Tor_1 in place of Ext^1 . We obtain the following.

Proposition 5.17. (A) *Let R be a right self-injective ring and let M_R be any module. The family \mathcal{F} of right ideals $I \subseteq R$ such that every homomorphism $f: M \rightarrow R/I$ lifts to some $f': M \rightarrow R$ is a right Oka family. Hence, any $I \in \text{Max}(\mathcal{F}')$ is completely prime.*

(B) *For a R and a module ${}_R N$, let \mathcal{F} be the family of right ideals $I_R \subseteq R$ such that the natural map $I \otimes_R N \rightarrow R \otimes_R N \cong N$ is injective. Then \mathcal{F} is an Oka family of right ideals. Hence, any $I \in \text{Max}(\mathcal{F}')$ is completely prime.*

For us, what is most interesting about Propositions 5.16 and 5.17 is that they provide multiple ways to define right Oka families starting with *any given module* M_R . Thanks to the Completely Prime Ideal Principle 3.4, each of these families \mathcal{F} gives rise to completely prime right ideals in $\text{Max}(\mathcal{F}')$ whenever this set is nonempty.

5.C. Finiteness conditions, multiplicative sets, and invertibility. The final few applications of the Completely Prime Ideal Principle given here come from finiteness conditions on modules, multiplicatively closed subsets of a ring, and invertible right ideals.

We first turn our attention to finiteness conditions. We remind the reader that a module M_R is said to be *finitely cogenerated* provided that, for every set $\{N_i : i \in I\}$ of submodules of M such that $\bigcap_{i \in I} N_i = 0$, there exists a finite subset $J \subseteq I$ such that $\bigcap_{j \in J} N_j = 0$. This is equivalent to saying that the socle of M is finitely generated and is an essential submodule of M . See [17, §19A] for further details.

Example 5.18. Let R be a k -ring. Fix any one of the following properties of a right ideal I of R :

- (1A) R/I is a finitely generated right k -module;
- (1B) R/I is a finitely cogenerated right k -module;
- (2) R/I has cardinality $< \alpha$ for some infinite cardinal α ;
- (3A) R/I is a noetherian right k -module;
- (3B) R/I is an artinian right k -module;
- (4) R/I is a right k -module of finite length;
- (5) R/I is a right k -module of finite uniform dimension.

The family \mathcal{F} of right ideals satisfying that fixed property is right Oka by Remark 5.1; hence $\text{Max}(\mathcal{F}')$ consists of completely prime right ideals.

As a refinement of (4) above, notice that the right k -modules of finite length whose composition factors have certain prescribed isomorphism types is closed under extensions. The same is true for the right k -modules whose length is a multiple of a fixed integer d . Thus these classes give rise to two other Oka families of right ideals.

Right Oka families and completely prime right ideals also arise in connection with multiplicatively closed subsets of a ring.

Example 5.19. Consider a multiplicative subset S of a ring R (i.e., S is a submonoid of the multiplicative monoid of R). A module M_R is said to be S -torsion if, for every $m \in M$ there exists $s \in S$ such that $ms = 0$. It is easy to see that the class of S -torsion modules is closed under extensions. Thus the family \mathcal{F} of right ideals $I_R \subseteq R$ such that R/I is S -torsion is a right Oka family, and $\text{Max}(\mathcal{F}')$ consists of completely prime right ideals.

Recall that a multiplicative set S in a ring R is called a *right Ore set* if, for all $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$. (For example, it is easy to see that any multiplicative set in a commutative ring is right Ore.) One can show that a multiplicative set S is right Ore iff for every module M_R the set

$$t_S(M) := \{m \in M : ms = 0 \text{ for some } s \in S\}$$

of S -torsion elements of M is a submodule of M . This makes it easy to verify that for such S , R/I is S -torsion iff $I \cap S \neq \emptyset$. So for a right Ore set $S \subseteq R$, the family of all right ideals I of R such that $I \cap S \neq \emptyset$ is equal to the family \mathcal{F} above and thus is a right Oka family. In particular, a right ideal maximal with respect to being disjoint from S is completely prime. We will be able to strengthen these statements later—see Example 6.24.

One further right Oka family comes from the notion of invertibility of right ideals. Fix a ring Q with a subring $R \subseteq Q$. For any submodule $I_R \subseteq Q_R$ we write $I^* := \{q \in Q : qI \subseteq R\}$, which is a *left* R -submodule of Q . We will say that a right R -submodule $I \subseteq Q$ is *right invertible (in Q)* if there exist $x_1, \dots, x_n \in I$ and $q_1, \dots, q_n \in I^*$ such that $\sum x_i q_i = 1$. (This definition is inspired by [31, §II.4].) Notice that if I is right invertible as above, then I is necessarily finitely generated, with generating set x_1, \dots, x_n . The concept of a right invertible right ideal certainly generalizes the notion of an invertible ideal in a commutative ring, and it gives rise to a new right Oka family.

Proposition 5.20. *Let R be a subring of a ring Q . The family \mathcal{F} of right ideals of R that are right invertible in Q is a right Oka family. The set $\text{Max}(\mathcal{F}')$ consists of completely prime right ideals.*

Proof. Let $I_R \subseteq R$ and $a \in R$ be such that $I + aR$ and $a^{-1}I$ are right invertible. We want to show that I is also right invertible. There exist $i_1, \dots, i_m \in I$ and $q_k, q \in (I + aR)^*$ such that $\sum_{i=1}^m i_k q_k + aq = 1$. Similarly, there exist $x_1, \dots, x_n \in a^{-1}I$ and $p_j \in (a^{-1}I)^*$ such that $\sum x_j p_j = 1$. Combining these equations, we have

$$\begin{aligned} 1 &= \sum i_k q_k + aq \\ &= \sum i_k q_k + a \left(\sum x_j p_j \right) q \\ &= \sum i_k q_k + \sum (ax_j)(p_j q). \end{aligned}$$

In this equation we have $i_k \in I$, $q_k \in (I + aR)^* \subseteq I^*$, and $ax_j \in a(a^{-1}I) \subseteq I$. Thus we will be done if we can show that every $p_j q \in I^*$.

We claim that $qI \subseteq a^{-1}I$. This follows from the fact that, for any $i \in I$, $q_k i \in R$ so that

$$aqi = \left(1 - \sum i_k q_k \right) i = i - \sum i_k (q_k i) \in I.$$

Thus we find

$$(p_j q)I = p_j(qI) \subseteq (a^{-1}I)^*(a^{-1}I) \subseteq R.$$

It follows that $p_j q \in I^*$, completing the proof. \square

In the case that R is a right Ore ring, it is known (see [31, II.4.3]) that the right ideals of R that are right invertible in its classical right ring of quotients Q are precisely the projective right ideals that intersect the right Ore set S of regular elements of R . (Recall that a ring R is right Ore if the multiplicatively closed set of regular elements in R is right Ore. This is equivalent to the statement that R has a classical right ring of quotients Q ; see [17, §10B].) We can use this to give a second proof that the family \mathcal{F} of right invertible right ideals of R is a right Oka family in this case. The alternative characterization of right invertibility in this setting means that \mathcal{F} is the intersection of the family \mathcal{F}_1 of projective right ideals (which was shown to be a right Oka family as an application of Example 5.12) with the family \mathcal{F}_2 of right ideals that intersect the right Ore set S (which was shown to be a right Oka family in Example 5.19). Recalling Remark 3.3, we conclude that $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ is a right Oka family.

Using this notion of invertibility, we can generalize Theorem 1.2, due to Cohen, which states that a commutative ring R is a Dedekind domain iff every nonzero prime ideal of R is invertible.

Proposition 5.21. *For a subring R of a ring Q , every nonzero right ideal of R is right invertible in Q iff every nonzero completely prime right ideal of R is right invertible in Q . If R is a right Ore ring with classical right ring of quotients Q , then R is a right hereditary right noetherian domain iff every nonzero completely prime right ideal of R is right invertible in Q .*

Proof. First suppose that R is right Ore. According to [31, Prop. II.4.3], a right ideal of R is right invertible in Q iff it is projective and contains a regular element. Thus the right Ore ring R is a right hereditary right noetherian domain iff every nonzero right ideal of R

is right invertible in the classical right quotient ring of R . So it suffices to prove the first statement.

Now for any R and Q , the family \mathcal{F} of right ideals of R that are right invertible in Q is a right Oka family by Proposition 5.20. Once we recall that a right invertible right ideal is finitely generated, the claim follows from the CPIP Supplement 3.6(2). \square

6. COMONOFORM RIGHT IDEALS AND DIVISIBLE RIGHT OKA FAMILIES

We devote the final section of this paper to study a particularly well-behaved subset of the completely prime right ideals of a general ring, the *comonoform right ideals* (Definition 6.2). Our purpose is to provide a richer understanding of the completely prime right ideals of a general ring. There is a special type of Prime Ideal Principle that accompanies this new set of right ideals, as well as new applications to the one-sided structure of rings.

These special right ideals $I_R \subseteq R$ are defined by imposing a certain condition on the factor module R/I . First we must describe the many equivalent ways to phrase this module-theoretic condition. Given an R -module M_R , a submodule $N \subseteq M$ is said to be *dense* if, for all $x, y \in M$ with $x \neq 0$, $x \cdot (y^{-1}N) \neq 0$ (recall the definition of $y^{-1}M$ from §2). We write $N \subseteq_d M$ to mean that N is a dense submodule of M , and we let $E(M)$ denote the injective hull of M . It is known that $N \subseteq_d M$ iff $\text{Hom}_R(M/N, E(M)) = 0$, iff for every submodule U with $N \subseteq U \subseteq M$ we have $\text{Hom}_R(U/N, M) = 0$. In addition, for any submodules $N \subseteq U \subseteq M$, it turns out that $N \subseteq_d M$ iff $N \subseteq_d U$ and $U \subseteq_d M$. (See [17, (8.6) & (8.7)] for details.) Finally, any dense submodule of M is *essential* in M , meaning that it has nonzero intersection with every nonzero submodule of M .

Proposition 6.1. *For a module $M_R \neq 0$, the following are equivalent:*

- (1) *Every nonzero submodule of M is dense in M ;*
- (2) *Every nonzero cyclic submodule of M is dense in M ;*
- (3) *For any $x, y, z \in M$ with $x, z \neq 0$, $x \cdot y^{-1}(zR) \neq 0$;*
- (4) *Any nonzero $f \in \text{Hom}_R(M, E(M))$ is injective;*
- (5) *For any submodule $C \subseteq M$, any nonzero $f \in \text{Hom}_R(C, M)$ (resp. any nonzero $f \in \text{Hom}_R(C, E(M))$) is injective;*
- (5') *M is uniform and for any cyclic submodule $C \subseteq M$, any nonzero $f \in \text{Hom}_R(C, M)$ is injective;*
- (6) *There is no nonzero R -homomorphism from any submodule of any proper factor of M to M (resp. to $E(M)$).*

Proof. Clearly (1) \implies (2). For (2) \implies (1), let P be any nonzero submodule of M . Then, taking some cyclic submodule $0 \neq C \subseteq P$, we have $C \subseteq_d M \implies P \subseteq_d M$.

Now (2) \iff (3) is clear from the definition of density. Also, (1) \iff (4) and (1) \iff (5) follow from the various reformulations of density stated above. The equivalence of (5), (6), and their parenthetical formulations is straightforward.

Finally we prove (5) \iff (5'). Assume (5) holds; to verify (5'), we only need to show that M is uniform. By the equivalence of (1) and (5), we see that every nonzero submodule of M is dense and is therefore essential. This proves that M is uniform. Now suppose that (5') holds, and let $0 \neq f \in \text{Hom}(C, M)$ where C is any submodule of M . Fix some cyclic submodule $0 \neq C_0 \subseteq C$ such that $C_0 \not\subseteq \ker f$, and let g denote the restriction of f to

C_0 . By hypothesis, $0 = \ker g = \ker f \cap C_0$. Because M is uniform this implies that $\ker f = 0$, proving that (5) is true. \square

An easy example shows that the requirement in (5') that M be uniform is in fact necessary. If V_k is a vector space over a division ring k then it is certainly true that every nonzero homomorphism from a cyclic submodule of V into V is injective. However, if $\dim_k V > 1$, then V has nontrivial direct summands and cannot be uniform.

Definition 6.2. A nonzero module M_R is said to be *monoform* (following [12]) if it satisfies the equivalent conditions of Proposition 6.1. A right ideal $P_R \subsetneq R$ is *comonoform* if the factor module R/P is monoform.

As a basic example, notice that simple modules are monoform and hence maximal right ideals are comonoform. We can easily verify that the comonoform right ideals of a ring form a subset of the set of completely prime right ideals, as mentioned earlier.

Proposition 6.3. *If M_R is monoform, then every nonzero endomorphism of M is injective. In particular, every comonoform right ideal of R is completely prime.*

Proof. The first claim follows from Proposition 6.1(5) by taking $C = M$ there. Now the second statement is true by Proposition 2.5. \square

Some clarifying remarks about terminology are appropriate. Monoform modules have been given several other names in the literature. They seem to have been first investigated by O. Goldman in [10, §6]. Each monoform module is associated to a certain *prime right Gabriel filter* \mathcal{F} (a term which we will not define here), and Goldman referred to such a module as a *supporting module for \mathcal{F}* . They have also been referred to as *cocritical modules*, *\mathcal{F} -cocritical modules*, and *strongly uniform modules*. The latter term is justified because, as shown in (5') above, any monoform module is uniform. Also, comonoform right ideals have been referred to as *critical right ideals* [23] (which explains the term ‘‘cocritical module’’) and *super-prime right ideals* [29]. We have chosen to use the term ‘‘monoform’’ because we feel that it best describes the properties of these modules, and we are using the term ‘‘comonoform’’ rather than ‘‘critical’’ for right ideals in order to avoid confusion with the modules that are critical in the sense of the Gabriel-Rentschler Krull dimension.

Comonoform right ideals enjoy special properties that distinguish them from the more general completely prime right ideals. For instance, if P is a comonoform right ideal of R , then R/P is uniform by Proposition 6.1(5'). On the other hand, Example 2.8 showed that the more general completely prime right ideals do not always have this property. A second desirable property of comonoform right ideals is given in the following lemma. It is easy to verify (from several of the characterizations in Proposition 6.1) that a nonzero submodule of a monoform module is again monoform. Applying Lemma 4.5(A) yields the following result.

Lemma 6.4. *For any comonoform right ideal $P_R \subseteq R$ and any element $x \in R \setminus P$, the right ideal $x^{-1}P$ is also comonoform.*

It is readily verified that the lemma above does not hold if we replace the word ‘‘comonoform’’ with ‘‘completely prime.’’ For instance, consider again Example 2.8. For the completely prime right ideal P of the ring R described there and the element $x = E_{12} + E_{13} \in R$,

it is readily verified that

$$x^{-1}P = \begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not a completely prime right ideal (because the module $R/x^{-1}P$ is decomposable).

Without going into details, we mention that the lemma above suggests that comonoform right ideals P can be naturally grouped into equivalence classes corresponding to the isomorphism classes of the injective hulls $E(R/P)$. (This is directly related to Goldman's notion of primes in [10], and is also investigated in [23].)

Let us consider a few ways one might find comonoform right ideals in a given ring R . First, we have already seen that *every maximal right ideal in R is comonoform*. Second, Remark 2.12 shows that if $I \triangleleft R$ is an ideal contained in a right ideal J , then J is a comonoform right ideal of R iff J/I is a comonoform right ideal of R/I . Next, let us examine which (two-sided) ideals of R are comonoform as right ideals. The following result seems to have been first recorded (without proof) in [13, Prop. 4].

Proposition 6.5. *An ideal $P \triangleleft R$ is comonoform as a right ideal iff R/P is a right Ore domain.*

Proof. First suppose that P_R is comonoform. Then P is completely prime by Proposition 6.3, so that R/P is a domain by Proposition 2.2. Also R/P is right uniform (see Proposition 6.1(5')). Now because the domain R/P is right uniform, it is right Ore.

Conversely, suppose that $S := R/P$ is a right Ore domain, with right division ring of quotients Q . Then because $E(S_R) = Q_R$, it is easy to see that every nonzero map

$$f \in \text{Hom}(S_R, E(S_R)) = \text{Hom}(S_S, Q_S)$$

must be injective. Thus S_R is monoform, completing the proof. \square

Remark 6.6. This result makes it easy to construct an example of a ring with a completely prime right ideal that is not comonoform. Let R be any domain that is not right Ore (such as the free algebra generated by two elements over a field), and let $P = 0 \triangleleft R$. Then P_R is completely prime (recall Proposition 2.2), but it cannot be right comonoform by the above.

Incidentally, because every monoform module is uniform, the ring R constructed in Example 2.8 is an example of an *artinian* (hence noetherian) ring with a completely prime right ideal that is not comonoform. This is to be contrasted with the non-Ore domain example, which is necessarily non-noetherian.

Another consequence of this result is that the comonoform right ideals directly generalize the concept of a prime ideal in a commutative ring, just like the completely prime right ideals.

Corollary 6.7. *In a commutative ring R , an ideal $P \triangleleft R$ is comonoform iff it is a prime ideal.*

The Completely Prime Ideal Principle 3.4 gives us a method for exploring the existence of completely prime right ideals. We will provide a similar tool for studying the existence of the more special comonoform right ideals in Theorem 6.10. The idea is that comonoform right

ideals occur as the right ideals that are maximal in the complement of right Oka families that satisfy an extra condition, defined below.

Definition 6.8. A family \mathcal{F} of right ideals in a ring R is *divisible* if, for all $a \in R$,

$$I \in \mathcal{F} \implies a^{-1}I \in \mathcal{F}.$$

The next lemma is required to prove the “stronger PIP” for divisible right Oka families.

Lemma 6.9. *In a ring R , suppose that I and K are right ideals and that $a \in R$. Then*

$$K \supseteq a^{-1}I \iff K = a^{-1}J \text{ for some right ideal } J_R \supseteq I.$$

Proof. If $K = a^{-1}J$ for some $J \supseteq I$, then clearly $K = a^{-1}J \supseteq a^{-1}I$. Conversely, suppose that $K \supseteq a^{-1}I$. Then for $J := I + aK$, we claim that $K = a^{-1}J$. Certainly $K \subseteq a^{-1}J$. So suppose that $x \in a^{-1}J$. Then $ax \in J = I + aK$ implies that $ax = i + ak$ for some $i \in I$, $k \in K$. Because $a(x - k) = i$, we see that $x - k \in a^{-1}I$. Hence $x = (x - k) + k \in a^{-1}I + K = K$. \square

Theorem 6.10. *Let \mathcal{F} be a divisible right Oka family. Then every $P \in \text{Max}(\mathcal{F}')$ is a comonoform right ideal.*

Proof. Let $P \in \text{Max}(\mathcal{F}')$. To show that $R/P \neq 0$ is monoform, it is sufficient by Proposition 6.1(1) to show that every nonzero submodule $I/P \subseteq R/P$ is dense. That is, for any $0 \neq x + P \in R/P$ and any $y + P \in R/P$, we wish to show that $(x + P) \cdot (y + P)^{-1}(I/P) \neq 0$. It is straightforward to see that $(y + P)^{-1}(I/P) = y^{-1}I$. Thus it is enough to show, for any right ideal $I \supseteq P$ and elements $x \in R \setminus P$ and $y \in R$, that $x \cdot y^{-1}I \not\subseteq P$.

Assume for contradiction that $x \cdot y^{-1}I \subseteq P$ for such x , y , and I . Then $y^{-1}I \subseteq x^{-1}P$, and Lemma 6.9 shows that $x^{-1}P = y^{-1}J$ for some right ideal $J \supseteq I$. Since $P \in \text{Max}(\mathcal{F}')$, the fact that $J \supseteq I \supseteq P$ implies that $J \in \mathcal{F}$. Because \mathcal{F} is divisible, $x^{-1}P = y^{-1}J \in \mathcal{F}$. Also $x \notin P$ and maximality of P give $P + xR \in \mathcal{F}$. Since \mathcal{F} is right Oka we conclude that $P \in \mathcal{F}$, a contradiction. \square

As with the more general completely prime right ideals, there is a “Supplement” that accompanies this “stronger PIP.” We omit its proof, which parallels that of Theorem 3.6.

Theorem 6.11. *Let \mathcal{F} be a divisible right Oka family in a ring R such that every nonempty chain of right ideals in \mathcal{F}' (with respect to inclusion) has an upper bound in \mathcal{F}' . Let \mathcal{S} denote the set of comonoform right ideals of R .*

- (1) *Let \mathcal{F}_0 be a semifilter of right ideals in R . If $\mathcal{S} \cap \mathcal{F}_0 \subseteq \mathcal{F}$, then $\mathcal{F}_0 \subseteq \mathcal{F}$.*
- (2) *For $J_R \subseteq R$, if all right ideals in \mathcal{S} containing J (resp. properly containing J) belong to \mathcal{F} , then all right ideals containing J (resp. properly containing J) belong to \mathcal{F} .*
- (3) *If $\mathcal{S} \subseteq \mathcal{F}$, then \mathcal{F} consists of all right ideals of R .*

To apply the last two theorems, we must provide some ways to construct divisible right Oka families. The first method is extremely straightforward.

Remark 6.12. Let $\mathcal{E} \subseteq \mathfrak{M}_R$ be a class of right R -modules that is closed under extensions and closed under passing to submodules. Then for $\mathcal{C} := \mathcal{E} \cap \mathfrak{M}_R^c$, the right Oka family $\mathcal{F}_{\mathcal{C}}$ is divisible. For if $I \in \mathcal{F}_{\mathcal{C}}$ and $x \in R$, then $R/x^{-1}I$ is isomorphic to the submodule $(I + xR)/I$ of $R/I \in \mathcal{C} \subseteq \mathcal{E}$. Then by hypothesis, $R/x^{-1}I \in \mathcal{E} \cap \mathfrak{M}_R^c = \mathcal{C}$, proving that $x^{-1}I \in \mathcal{F}_{\mathcal{C}}$.

The above method applies immediately to many of the families \mathcal{F}_c which we have already investigated. To begin with, for a k -ring R , all of the finiteness properties listed in Example 5.18 pass to submodules, with the exception of finite generation (1A). Thus a right ideal I maximal with respect to R/I not having one of those properties is comonoform.

We can apply this specifically to rings with the so-called *right restricted minimum condition*; these are the rings R such that R/I is an artinian right R -module for all right ideals $I \neq 0$. If such a ring R is not right artinian, we see that the zero ideal is in $\text{Max}(\mathcal{F}')$ where \mathcal{F} is the divisible Oka family of right ideals $I \subseteq R$ such that R/I is an artinian R -module. Thus the zero ideal is right comonoform by Theorem 6.10. Hence R is a right Ore domain by Proposition 6.5. The fact that such a ring is a right Ore domain was proved by A. J. Ornstein in [27, Thm. 13] as a generalization of a theorem of Cohen [4, Cor. 2].

Corollary 6.13 (Ornstein). *If a ring R satisfies the right restricted minimum condition and is not right artinian, then R is a right Ore domain.*

In addition, for a multiplicative set $S \subseteq R$ the class of S -torsion modules (see Example 5.19) is closed under extensions and submodules. So a right ideal I maximal with respect to R/I not being S -torsion is comonoform. Notice that this is true *whether or not the set S is right Ore*. (However, if S is not right Ore then we do not have the characterization that R/I is S -torsion iff $I \cap S \neq \emptyset$.)

For another example, fix a multiplicative set $S \subseteq R$, which again need not be right Ore. A module M_R is said to be *S -torsionfree* if, for any $m \in M$ and $s \in S$, $ms = 0$ implies $m = 0$. The class of S -torsionfree modules is easily shown to be closed under extensions. Hence the family \mathcal{F} of right ideals in R such that R/I is S -torsionfree is a right Oka family. Notice that \mathcal{F} can alternatively be described as

$$\mathcal{F} = \{I_R \subseteq R : \text{for } r \in R \text{ and } s \in S, rs \in I \implies r \in I\}.$$

Furthermore, \mathcal{F} is divisible because any submodule of a torsionfree module is torsionfree. So every right ideal $P \subseteq R$ with $P \in \text{Max}(\mathcal{F}')$ is comonoform.

A second effective method of constructing a divisible right Oka family is by defining it in terms of certain families of two-sided ideals. This is achieved in Proposition 6.15 below. Given a right ideal I of R , recall that the largest ideal of R contained in I is called the *core of I* , denoted $\text{core}(I)$. It is straightforward to check that $\text{core}(I) = \text{ann}(R/I)$ for any $I_R \subseteq R$.

Lemma 6.14. *Let \mathcal{F} be a semifilter of right ideals in a ring R that is generated as a semifilter by two-sided ideals—that is to say, there exists a set \mathcal{G} of two-sided ideals of R such that*

$$\begin{aligned} \mathcal{F} &= \{I_R \subseteq R : I \supseteq J \text{ for some } J \in \mathcal{G}\} \\ &= \{I_R \subseteq R : \text{core}(I) \in \mathcal{G}\}. \end{aligned}$$

Then \mathcal{F} is divisible.

Proof. The equality of the two descriptions of \mathcal{F} above follows from the fact that \mathcal{G} is a semifilter. Suppose that $I \in \mathcal{F}$, so that there exists $J \in \mathcal{G}$ such that $I \supseteq J$. Then for any $a \in R$, $aJ \subseteq J \subseteq I$ implies that $J \subseteq a^{-1}I$. It follows that $a^{-1}I \in \mathcal{F}$, and \mathcal{F} is divisible. \square

By analogy with Definition 3.5, we define a *semifilter of (two-sided) ideals* in a ring R to be a family \mathcal{G} of ideals of R such that, for $I, J \triangleleft R$, $I \in \mathcal{G}$ and $J \supseteq I$ imply $J \in \mathcal{G}$. As in [21] we define the following property of a family \mathcal{G} of two-sided ideals in R :

(P_1): \mathcal{G} is a semifilter of ideals that is closed under pairwise products and that contains the ideal R (equivalently, is nonempty).

In [21, Thm. 2.7], it was shown that any (P_1) family of ideals in a commutative ring is an Oka family. The following shows how to define a right Oka family from a (P_1) family of ideals in a noncommutative ring.

Proposition 6.15. *Let \mathcal{G} be a family of ideals in a ring R satisfying (P_1). Then the semifilter \mathcal{F} of right ideals generated by \mathcal{G} (as in Lemma 6.14) is a divisible right Oka family. Thus, every right ideal in $\text{Max}(\mathcal{F})$ is comonoform.*

Proof. Let \mathcal{E} be the class of right R -modules M such that $\text{ann}(M) \in \mathcal{G}$. We claim that \mathcal{E} is closed under extensions in \mathfrak{M}_R . Indeed, let $L, N \in \mathcal{E}$ and suppose $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of right R -modules. We want to conclude that $M \in \mathcal{E}$. Because $\text{ann}(L)$ and $\text{ann}(N)$ belong to \mathcal{G} , the fact that \mathcal{G} is (P_1) means that $\text{ann}(M) \supseteq \text{ann}(N) \cdot \text{ann}(L)$ must also lie in \mathcal{G} . Thus $M \in \mathcal{E}$ as desired.

Now any cyclic module R/I has annihilator $\text{ann}(R/I) = \text{core}(I)$. So for $\mathcal{C} := \mathcal{E} \cap \mathfrak{M}_R^c$ we see that our family is $\mathcal{F} = \mathcal{F}_{\mathcal{C}}$. Hence \mathcal{F} is a right Oka family. Lemma 6.14 implies that \mathcal{F} is divisible. The last sentence of the proposition now follows from Theorem 6.10. \square

We will apply the result above to a special example of such a family \mathcal{G} of ideals. For a ring R , recall that a subset $S \subseteq R$ is called an *m-system* if $1 \in S$ and for any $s, t \in S$ there exists $r \in R$ such that $srt \in S$. It is well-known that an ideal $P \triangleleft R$ is prime iff $R \setminus P$ is an *m-system*.

Corollary 6.16. (1) *For an m-system S in a ring R , the family \mathcal{F} of right ideals I such that $\text{core}(I) \cap S \neq \emptyset$ is a divisible right Oka family. A right ideal maximal with respect to having its core disjoint from S is comonoform.*

(2) *For a prime ideal P of a ring R , the family of all right ideals I_R such that $\text{core}(I) \not\subseteq P$ is a divisible right Oka family. A right ideal I maximal with respect to $\text{core}(I) \subseteq P$ is comonoform. In particular, if R is a prime ring, a right ideal maximal with respect to $\text{core}(I) \neq 0$ is comonoform.*

Proof. For (1), we can apply Proposition 6.15 to the family \mathcal{G} of ideals having nonempty intersection with the *m-system* S , which is certainly a (P_1) family of ideals. Then (2) follows from (1) if we let $S = R \setminus P$, which is an *m-system* when P is a prime ideal. \square

Another application of Proposition 6.15 involves the notion of boundedness. Recall that a ring R is said to be *right bounded* if every essential right ideal contains a two-sided ideal that is right essential. (Another way to say this is that if $I_R \subseteq R$ is essential, then $\text{core}(I)$ is right essential.) Then one can characterize whether certain types of rings are right bounded in terms of their comonoform right ideals. Given a module M_R , we write $N \subseteq_e M$ to mean that N is an essential submodule of M .

Proposition 6.17. *Let R be a ring in which the set of ideals $\{J \triangleleft R : J_R \subseteq_e R_R\}$ is closed under squaring (e.g. a semiprime ring or a right nonsingular ring), and suppose that every ideal of R that is right essential is finitely generated as a right ideal (this holds, for instance,*

if R is right noetherian). Then R is right bounded iff every essential comoniform right ideal of R has right essential core.

Proof. Assume that R satisfies the two stated hypotheses. We claim that the ideal family $\{J \triangleleft R : J_R \subseteq_e R\}$ is in fact closed under pairwise products. Indeed, if $I, J \triangleleft R$ are essential as right ideals, then their product IJ contains the essential right ideal $(I \cap J)^2$ and thus is right essential. This allows us to apply Proposition 6.15 to say that the family \mathcal{F} of right ideals with right essential core is a divisible right Oka family. Next, the assumption that every ideal that is right essential is right finitely generated implies that the union of any nonempty chain of right ideals in \mathcal{F}' lies in \mathcal{F}' . Also, the set \mathcal{F}_0 of essential right ideals is a semifilter. Thus the statement of the proposition, excluding the first parenthetical remark, follows from Theorem 6.11(1).

It remains to verify that a semiprime or right nonsingular ring R satisfies the first hypothesis. Suppose that $J \triangleleft R$ is right essential, and let I_R be a right ideal such that $I \cap J^2 = 0$. Then

$$(I \cap J)^2 \subseteq I \cap J^2 = 0 \quad \text{and} \quad (I \cap J)J \subseteq I \cap J^2 = 0$$

respectively imply that $I \cap J$ squares to zero and has essential right annihilator. Thus if R is either semiprime or right nonsingular, then $I \cap J = 0$. Because J is right essential, we conclude that $I = 0$. Hence J^2 is right essential as desired. \square

Corollary 6.18. *A prime right noetherian ring R is right bounded iff every essential comoniform right ideal of R has nonzero core.*

Proof. It is a well-known (and easy to verify) fact that every nonzero ideal of a prime ring is right essential. Thus a right ideal of R has right essential core iff its core is nonzero. Because R is prime and right noetherian, we can directly apply Proposition 6.17. \square

In fact, the last result can be directly deduced from Corollary 6.16(2). We chose to include Proposition 6.17 because it seems to apply rather broadly.

Next we will show that certain well-studied families of right ideals are actually examples of divisible right Oka families, providing a third method of constructing the latter. The concept of a *Gabriel filter* of right ideals arises naturally in the study of torsion theories and the related subject of localization in noncommutative rings. The definition of these families is recalled below.

Definition 6.19. A *right Gabriel filter* (or *right Gabriel topology*) in a ring R is a nonempty family \mathcal{F} of right ideals of R satisfying the following four axioms (where $I_R, J_R \subseteq R$):

- (1) If $I \in \mathcal{F}$ and $J \supseteq I$ then $J \in \mathcal{F}$;
- (2) If $I, J \in \mathcal{F}$ then $I \cap J \in \mathcal{F}$;
- (3) If $I \in \mathcal{F}$ and $x \in R$ then $x^{-1}I \in \mathcal{F}$;
- (4) If $I \in \mathcal{F}$ and $J_R \subseteq R$ is such that $x^{-1}J \in \mathcal{F}$ for all $x \in I$, then $J \in \mathcal{F}$.

Notice that axiom (3) above simply states that a *right Gabriel filter is divisible*. For the reader's convenience, we outline some basic facts regarding right Gabriel filters and torsion theories that will be used here. Refer to [31, VI.1-5] for further details.

Given any right Gabriel filter \mathcal{F} and any module M_R , we define a subset of M :

$$t_{\mathcal{F}}(M) := \{m \in M : \text{ann}(m) \in \mathcal{F}\}.$$

Axioms (1), (2), and (3) of Definition 6.19 guarantee that this is a submodule of M , and it is called the \mathcal{F} -torsion submodule of M . A module M_R is defined to be \mathcal{F} -torsion if $t_{\mathcal{F}}(M) = M$ or \mathcal{F} -torsionfree if $t_{\mathcal{F}}(M) = 0$. One can easily verify that for a right Gabriel filter \mathcal{F} , a right ideal $I \subseteq R$ lies in \mathcal{F} iff R/I is \mathcal{F} -torsion.

For any Gabriel filter \mathcal{F} , it turns out that the class

$$\mathcal{T}_{\mathcal{F}} := \{M_R : M \text{ is } \mathcal{F}\text{-torsion, i.e. } M = t_{\mathcal{F}}(M)\}$$

of all \mathcal{F} -torsion right R -modules satisfies the axioms of a *hereditary torsion class*. While we shall not define this term here, it is equivalent to saying that the class $\mathcal{T}_{\mathcal{F}}$ is closed under factor modules, direct sums of arbitrary families, and extensions (in \mathfrak{M}_R). (Thus the reader may simply take this to be the definition of a hereditary torsion class.)

With the information provided above we will prove that right Gabriel filters are examples of divisible right Oka families.

Proposition 6.20. *Over a ring R , any right Gabriel filter \mathcal{F} is a divisible right Oka family. Any right ideal $P \in \text{Max}(\mathcal{F}')$ is comonoform.*

Proof. Any right Gabriel filter is tautologically a divisible family of right ideals. The torsion class $\mathcal{T}_{\mathcal{F}}$ is closed under extensions in \mathfrak{M}_R , so the class $\mathcal{C} := \mathcal{T}_{\mathcal{F}} \cap \mathfrak{M}_R^c$ of cyclic \mathcal{F} -torsion modules is closed under extensions. A right ideal $I \subseteq R$ lies in \mathcal{F} iff $R/I \in \mathcal{T}_{\mathcal{F}}$ (as mentioned above), iff $R/I \in \mathcal{C}$ (since $R/I \in \mathfrak{M}_R^c$), iff $I \in \mathcal{F}_{\mathcal{C}}$. It follows from Theorem 4.7 that $\mathcal{F} = \mathcal{F}_{\mathcal{C}}$ is a right Oka family. The last sentence is true by Theorem 6.10. \square

We pause for a moment to give a sort of “converse” to this result, in the spirit of Proposition 4.11. Given any injective module E_R , the class $\{M_R : \text{Hom}(M, E) = 0\}$ is a hereditary torsion class. This is called the *torsion class cogenerated by E* . We will also say that the corresponding right Gabriel filter is the *right Gabriel filter cogenerated by E* . As stated in [31, VI.5.6], this is the largest right Gabriel filter with respect to which E is torsionfree. Let I be a right ideal in R . In the following, we let \mathcal{F}_I denote the right Gabriel filter cogenerated by $E(R/I)$; that is, \mathcal{F}_I is the set of all right ideals $J \subseteq R$ such that $\text{Hom}_R(R/J, E(R/I)) = 0$. We are now ready for the promised result.

Proposition 6.21. *For any right ideal $P \subsetneq R$, the following are equivalent:*

- (1) $P \in \text{Max}(\mathcal{F}')$ for some right Gabriel filter \mathcal{F} ;
- (2) $P \in \text{Max}(\mathcal{F}'_P)$;
- (3) P is a comonoform right ideal.

Proof. (2) \implies (1) is clear, and (1) \implies (3) follows from Theorem 6.20. For (3) \implies (2), assume that R/P is monoform. Proposition 6.1 implies that for every right ideal $I \supsetneq P$ we have $\text{Hom}_R(R/I, E(R/P)) = 0$. Then every such right ideal I tautologically lies in \mathcal{F}'_P , proving that $P \in \text{Max}(\mathcal{F}'_P)$. \square

We mention in passing that this result is similar to [12, Thm. 2.9], though it is not stated in quite the same way. This proposition actually provides a second, though perhaps less satisfying, proof that any comonoform right ideal is completely prime. Given a comonoform right ideal $P \subseteq R$, Proposition 6.21 provides a right Gabriel filter \mathcal{F} with $P \in \text{Max}(\mathcal{F}')$. Then because \mathcal{F} is a right Oka family (by Theorem 6.20), the CPIP implies that P is a completely prime right ideal.

As a first application of Theorem 6.20 we explore the maximal point annihilators of an *injective* module, recovering a result of Lambek and Michler in [23, Prop. 2.7]. This should be compared with Proposition 5.3.

Proposition 6.22 (Lambek and Michler). *For any injective module E_R , a maximal point annihilator of E is comonoform.*

Proof. Let $\mathcal{F} = \{I_R \subseteq R : \text{Hom}_R(R/I, E) = 0\}$ be the right Gabriel filter cogenerated by E . Then the set of maximal point annihilators of E is clearly equal to $\text{Max}(\mathcal{F}')$. By Theorem 6.20, any $P \in \text{Max}(\mathcal{F}')$ is comonoform. \square

Example 6.23. As shown in [31, VI.6], the set \mathcal{F} of all dense right ideals in any ring R is a right Gabriel filter. (In fact, it is the right Gabriel filter cogenerated by the injective module $E(R_R)$.) Therefore \mathcal{F} is a right Oka family, and a right ideal maximal with respect to not being dense in R is comonoform. Furthermore, in a right nonsingular ring, this family \mathcal{F} coincides with the set of all essential right ideals (see [17, (8.7)] or [31, VI.6.8]). Thus in a right nonsingular ring, the family \mathcal{F} of essential right ideals is a right Gabriel filter, and a right ideal maximal with respect to not being essential is comonoform.

Example 6.24. Let S be a right Ore set in a ring R , and let \mathcal{F} denote the family of all right ideals $I_R \subseteq R$ such that $I \cap S \neq \emptyset$. It is shown in the proof of [31, Prop. VI.6.1] that \mathcal{F} is a right Gabriel filter. It follows from Theorem 6.20 that a right ideal maximal with respect to being disjoint from S is comonoform.

We offer an application of the example above. Let us say that a multiplicative set S in a ring R is *right saturated* if $ab \in S$ implies $a \in S$ for all $a, b \in R$.

Corollary 6.25. *For every right saturated right Ore set $S \subseteq R$, there exists a set $\{P_i\}$ of comonoform right ideals such that $R \setminus S = \bigcup P_i$.*

Proof. Indeed, for all $x \in R \setminus S$, we must have $xR \subseteq R \setminus S$ because S is right saturated. By a Zorn's Lemma argument, there is a right ideal P_x containing x maximal with respect to being disjoint from S . Example 6.24 implies that P_x is comonoform. Choosing such P_x for all $x \in R \setminus S$, we have $R \setminus S = \bigcup P_x$. \square

Next we apply Example 6.24 to show that a “nice enough” prime (two-sided) ideal must be “close to” some comonoform right ideal.

Corollary 6.26. *Let $P_0 \in \text{Spec}(R)$ be such that R/P_0 is right Goldie. Then there exists a comonoform right ideal $P_R \supseteq P_0$ such that $\text{core}(P) = P_0$. In particular, if R is right noetherian then every prime ideal occurs as the core of some comonoform right ideal.*

Proof. Remark 2.12 shows that, for any right ideal $L_R \subseteq R$ and any two-sided ideal $I \subseteq L$, L is comonoform in R iff L/I is comonoform in R/I . Then passing to the factor ring R/P_0 , it clearly suffices to show that *in a prime right Goldie ring R there exists a comonoform right ideal P of R with zero core*. Indeed, let $S \subseteq R$ be the set of regular elements, and let $P_R \subseteq R$ be maximal with respect to $P \cap S = \emptyset$. Because R is prime right Goldie it is a right Ore ring by Goldie's Theorem, making S a right Ore set. Then P is comonoform by Example 6.24. We claim that $\text{core}(P) = 0$. Indeed, suppose that $I \neq 0$ is a nonzero ideal of R . Then since R is prime, I is essential as a right ideal in R . It follows from the theory of semiprime right Goldie rings that $I \cap S \neq \emptyset$ (see, for instance, [17, (11.13)]). This means that we cannot have $I \subseteq P$, verifying that $\text{core}(P) = 0$. \square

Notice that the above condition on P_0 is satisfied if R/P_0 is right noetherian. Conversely, it is not true that the core of every comonoform right ideal is prime, even in an artinian ring. For example, let R be the ring of $n \times n$ upper-triangular matrices over a division ring k for $n \geq 2$, and let $P \subsetneq R$ be the right ideal consisting of matrices in R whose first row is zero. Then one can verify that R/P is monoform (for example, using a composition series argument), so that P is comonoform. But the ideal $\text{core}(P) = 0$ is not (semi)prime.

We also provide a slight variation of Corollary 5.8, which tested whether or not a ring R is a domain. The version below applies when R is a right Ore ring.

Proposition 6.27. *A right Ore ring R is a domain iff every nonzero comonoform right ideal of R contains a regular element.*

Proof. (“If” direction) Let $S \subseteq R$ be the set of regular elements of R . Then S is a right Ore set, so the family $\mathcal{F} := \{I_R \subseteq R : I \cap S \neq \emptyset\}$ is a right Gabriel filter (in particular, a divisible right Oka family) by Example 6.24. Clearly the union of a chain of right ideals in \mathcal{F} also lies in \mathcal{F} . By Theorem 3.6, if every nonzero comonoform right ideal of R contains a regular element, then so does every nonzero right ideal. It follows easily that R is a domain. \square

As a closing observation, we note that there is a second way (aside from Theorem 6.20) that right Gabriel filters give rise to comonoform right ideals. Given a right Gabriel filter \mathcal{G} in a ring R , the class of \mathcal{G} -torsionfree modules is closed under extensions and submodules (just as the class of \mathcal{G} -torsion modules was). Thus a right ideal I of R maximal with respect to the property that R/I is not \mathcal{G} -torsionfree must be comonoform by Theorem 6.10. A similar statement was shown to be true for the S -torsionfree property, where S is a multiplicative set. However, there is a logical relation between these facts only in the case that S is right Ore, when the family \mathcal{G} of right ideals intersecting S is a right Gabriel filter.

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