

CONTEXTUALITY AND KOCHEN-SPECKER COLORINGS OF INTEGER VECTORS

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ABSTRACT. This note exhibits a new set of 85 three-dimensional integer vectors that has no Kochen-Specker coloring. These vectors represent rank-1 projection matrices with entries in the rational subring $\mathbb{Z}[1/462]$. Consequences are given for (non)contextuality in a purely algebraic sense for partial rings of symmetric matrices over finitely generated rational subrings and p -adic integers.

1. INTRODUCTION

Quantum contextuality [4] is a foundational feature of quantum theory, which has grown to a widely studied phenomenon since the work of Bell [1] and Kochen-Specker [8]. The work of Kochen and Specker famously reduced the proof of their no-hidden-variables result to the construction of a set of vectors in a Hilbert space that cannot be assigned a certain type of coloring, whose definition is recalled Section 2 below. Kochen and Specker's no-hidden-variables theorem was proved by demonstrating the existence of a set of 117 three-dimensional vectors for which there is no Kochen-Specker (KS) coloring. Since then, a wide variety of KS uncolorable vector sets of varying dimensions have been found.

In this we paper focus on Kochen-Specker uncolorable sets of *integer* vectors, i.e. vectors whose entries are all integers, with an eye toward understanding purely algebraic sources of noncontextuality. Although such examples have historically been relatively rare, a few of the best-known KS vector sets is are collections of vectors in dimension 4 with entries in $\{-1, 0, 1\} \subseteq \mathbb{Z}$, given by Peres [10], Kernaghan [7], and Cabello, Estebaranz, and García-Alcaine [5]. Another uncolorable set of 37 integer vectors in dimension 3 was produced by Bub [3]. This paper establishes the existence (Theorem 2) of a new set of 3-dimensional integer vectors that has no Kochen-Specker coloring, whose significance is discussed below. This line of investigation brings the study of quantum contextuality into direct contact with nontrivial number-theoretic issues, in a similar manner as has happened with other topics in quantum information theory ranging from mutually unbiased bases [6] to discrete quantum computation [9].

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Let $q: \mathbb{Z}^n \rightarrow \mathbb{Z}_{\geq 0}$ denote the sum-of-squares quadratic form, defined on a column vector $v = (v_1, \dots, v_n)^T$ by

$$q(v) = \|v\|^2 = v^T v = v_1^2 + \dots + v_n^2.$$

If $S \subseteq \mathbb{Z}^n$ is a Kochen-Specker uncolorable set of integer vectors, we will be interested in the numerical invariant

$$N(S) := \text{lcm}\{q(v) \mid v \in S\}.$$

If v is an integer column vector, then the orthogonal projection onto the line through v is given by the matrix

$$P_v = q(v)^{-1} v v^T,$$

whose entries are evidently *rational* and lie in the subring $\mathbb{Z}[1/q(v)] \subseteq \mathbb{Q}$. Thus for all $v \in S$, the projection matrix P_v has entries in $\mathbb{Z}[1/N(S)]$. For instance, if S_1 denotes each of the sets of 4-dimensional integer vectors in [5, 7, 10], then each $v \in S_1$ satisfies $q(v) \in \{1, 2, 4\}$ and thus $N(S_1) = 4$. Similarly, for the set $S_3 \subseteq \mathbb{Z}^3$ constructed in [3], each vector $v \in S_2$ satisfies $q(v) \in \{1, 2, 3, 5, 6, 30\}$ so that $N(S_2) = 30$.

Given a fixed positive integer N , we may take the reverse perspective and ask: *is there an uncolorable set of integer vectors whose corresponding projection matrices have entries in $\mathbb{Z}[1/N]$?* Given the prime factorization $N = \prod p_i^{e_i}$, recall that the *radical* is defined as $\text{rad } N = \prod p_i$. Noting that the rings $\mathbb{Z}[1/N] = \mathbb{Z}[1/\text{rad } N]$ coincide, to answer this question it suffices to consider only squarefree values of N .

For a positive squarefree integer N , we define

$$\begin{aligned} \mathcal{S}_n(N) &= \{v \in \mathbb{Z}^n : q(v) \text{ is a unit in } \mathbb{Z}[1/N]\} \\ &= \{v \in \mathbb{Z}^n : q(v) \text{ divides a power of } N\}. \end{aligned}$$

Note that $\mathcal{S}_n(N)$ is also the set of those $v \in \mathbb{Z}^n$ such that the primes occurring in the factorization of $q(v)$ form a subset of those occurring in N .

In this work we focus on the classical case of dimension $n = 3$. In this case we will simply denote $\mathcal{S}(N) := \mathcal{S}_3(N)$. We wish to address the following:

Question 1. *For which positive squarefree integers N is the set of vectors $\mathcal{S}(N)$ Kochen-Specker uncolorable?*

It is clear that if an integer M divides N , then $\mathcal{S}(M) \subseteq \mathcal{S}(N)$. It follows that for $M \mid N$,

$$\begin{aligned} \mathcal{S}(N) \text{ colorable} &\implies \mathcal{S}(M) \text{ colorable}, \\ \mathcal{S}(M) \text{ uncolorable} &\implies \mathcal{S}(N) \text{ uncolorable}. \end{aligned}$$

It was shown in [2, Theorem 3.4] that if N is not divisible by 2 or not divisible by 3, then $\mathcal{S}(N)$ has a KS coloring. As an immediate consequence,

$$\text{if } \mathcal{S}(N) \text{ is KS uncolorable, then } 6 \text{ divides } N.$$

On the other hand, Bub's uncolorable set $S \subseteq \mathbb{Z}^3$ with $N(S) = 30$ implies that

$$\text{if } 30 \text{ divides } N, \text{ then set } \mathcal{S}(N) \text{ is KS uncolorable.}$$

In light of the above facts, it seems natural to guess that $\mathcal{S}(6)$ has no Kochen-Specker coloring. However, to date we have not managed to find a proof or refutation of this statement. Lacking such an answer, we wondered whether there exists

an integer N divisible by 6 but not 5 that is KS uncolorable. In this paper we exhibit just such an example, for the specific value

$$N = 2 \cdot 3 \cdot 7 \cdot 11 = 462.$$

Theorem 1. *There is no Kochen-Specker coloring of the set $\mathcal{S}(462)$.*

The original discussion of hidden variable theories by Kochen and Specker was framed in terms of partial algebras, partial Boolean algebras, and their homomorphisms; see [8, Section 2]. Their goal was to prove that the non-existence of morphisms as partial rings from the observables of “most” quantum systems to any commutative ring. This was achieved by showing the non-existence of KS colorings on the partial Boolean algebra of projections, which in turn was deduced from the KS uncolorability of vectors (representing rank-1 projections).

This suggests a purely algebraic treatment of contextuality in the setting of noncommutative algebra via partial rings, which was explored in [11] and more thoroughly in [2]. As a consequence of Theorem 1 we are able to characterize exactly which partial rings of symmetric matrices with entries in $\mathbb{Z}[1/N]$ forbid the algebraic analogue of a contextual hidden variable theory.

Corollary 1. *The partial ring $\mathbb{M}_3(\mathbb{Z}[1/N])_{\text{sym}}$ admits no homomorphism to any commutative ring if and only if N is divisible by 6.*

Section 2 is devoted to the proof of Theorem 1, by the explicit construction of a KS uncolorable subset $Q \subseteq \mathcal{S}(462)$ (Theorem 2). In section 3 we discuss some consequences of this result for contextuality of partial rings, including a proof of Corollary 1.

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2. A NOVEL UNCOLORABLE SET OF INTEGER VECTORS

Let S be a set of noncollinear vectors in an n -dimensional Hilbert space $H \cong \mathbb{C}^n$. Recall that a *Kochen-Specker coloring* of a set of S is $\{0, 1\}$ -coloring (i.e., a function $S \rightarrow \{0, 1\}$) such that, for any orthogonal set of vectors $v_1, \dots, v_m \in S$, at most one of the v_i is colored 1 and furthermore, if $m = n$ then exactly one of the v_i is colored 1. Given a subset $S_0 \subseteq S$, a Kochen-Specker coloring of S restricts to a coloring of S_0 ; thus if S_0 has no Kochen-Specker coloring then also S is uncolorable.

When considering Kochen-Specker colorings of sets of vectors, each vector is intended to represent the rank-one projection onto the line through that vector. For this reason it suffices to consider sets of noncollinear vectors. When restricting attention to integer vectors, this means that we may consider only those vectors whose entries have greatest common divisor equal to 1. We will call such integer vectors *primitive*.

Furthermore, for each nonzero vector v , our vector sets need to only contain one of the two vectors $\{v, -v\}$. For convenience, we will say that a vector $v = (v_1, v_2, v_3) \in \mathbb{Z}^3 \setminus \{0\}$ is *well-signed* if either:

- v has only one nonzero entry which is positive,
- v has two nonzero entries and its first nonzero entry is positive, or
- v has three nonzero entries, two of which are positive.

For instance, the vectors $(1, 0, 0)$, $(0, 1, -1)$, and $(1, -1, 1)$ are well-signed while $(-1, 0, 0)$, $(0, -1, 1)$, and $(-1, 1, -1)$ are not. It is clear that for each $v \in \mathbb{Z}^3 \setminus \{0\}$, exactly one of v or $-v$ is well-signed. It follows that if a set $S \subseteq \mathbb{Z}^3$ consists of primitive, well-signed vectors, then the vectors in S must be noncollinear.

For $n = 1, 2, 3, 6, 21, 33, 77$, we will define subsets $Q_n \subseteq \mathcal{S}(462)$ such that every $v \in Q_n$ satisfies $q(v) = n$. For the values $n = 1, 2, 3, 6, 21$, we let Q_n denote the set of all well-signed primitive integer vectors v such that $q(v) = n$. It follows that:

- $Q_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
- $Q_2 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, -1, 0), (1, 0, -1), (0, 1, -1)\}$.
- $Q_3 = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$.
- Q_6 contains 12 such vectors, and they are the well-signed vectors whose entries up to sign are $1, 1, 2$
- Q_{21} contains 24 vectors, and they are the well-signed vectors whose entries up to sign are $1, 2, 4$.

For the remaining values $n = 33, 77$, we define the sets as follows.

- Q_{33} is the set of all well-signed vectors whose entries up to sign are $2, 2, 5$. There are 12 vectors in this set.
- Q_{77} is the set of all well-signed vectors whose entries up to sign are $2, 3, 8$. There are 24 vectors in this set.

As in the previous cases, the vectors $v \in Q_n$ are all well-signed and primitive, satisfying $q(v) = n$. However, there are certain vectors with the ‘‘correct’’ value of $q(v)$ that are excluded from these sets, such as $(1, 4, 4) \notin Q_{33}$ and $(4, 5, 6) \notin Q_{77}$.

Finally, we will define $Q \subseteq \mathcal{S}(462)$ to be the disjoint union

$$Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_6 \cup Q_{21} \cup Q_{33} \cup Q_{77}. \quad (\star)$$

This is a set of 85 integer vectors that are primitive, well-signed, and thus non-collinear. Having constructed the relevant set of vectors, we are now ready to prove the main result of the paper.

Theorem 2. *There is no Kochen-Specker coloring of Q .*

Proof. Assume toward a contradiction that Q has a KS coloring. Because Q is invariant under permutation of coordinates, we may assume without loss of generality that in the orthogonal triple

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

the vector $(1, 0, 0)$ is colored 1 while the other two are colored 0. It follows that in the triple

$$\{(1, 0, 0), (0, 1, 1), (0, 1, -1)\}$$

the second and third vectors must be colored 0. We also see that in the orthogonal triple

$$\{(0, 1, 0), (1, 0, 1), (1, 0, -1)\}$$

either the second or third vector must be colored 1. Because multiplication by the diagonal reflection matrix $R_z = \text{diag}(1, 1, -1)$ leaves the coordinate axes invariant while interchanging the vectors $(1, 0, \pm 1)$, we may again assume without loss of generality that $(1, 0, 1)$ is colored 0 and $(1, 0, -1)$ is colored 1. Finally, in the orthogonal triple

$$\{(0, 0, 1), (1, 1, 0), (1, -1, 0)\}$$

either the second or third vector must be colored 1. Since multiplication by the reflection matrix $R_y = \text{diag}(1, -1, 1)$ again leaves the coordinate axes and the lines through $(1, 0, \pm 1)$ invariant while interchanging the vectors $(1, \pm 1, 0)$, we may assume without loss of generality that $(1, 1, 0)$ is assigned 0 and $(1, -1, 0)$ is 1.

To summarize, we are assuming without loss of generality the following assignments of colors:

- **Color 1:** $(1, 0, 0), (1, 0, -1), (1, -1, 0)$
- **Color 0:** $(0, 1, 0), (0, 0, 1), (0, 1, 1), (0, 1, -1), (1, 0, 1), (1, 1, 0)$.

We now deduce the following sequence of colorings from the listed orthogonal triples:

$$\begin{aligned} \{(1, 0, -1), (1, -1, 1)\} \text{ orthogonal} &\implies (1, -1, 1) \mapsto 0, \\ \{(0, 1, 1), (1, -1, 1), (2, 1, -1)\} \text{ orthogonal} &\implies (2, 1, -1) \mapsto 1, \\ \{(2, 1, -1), (-3, 8, 2)\} \text{ orthogonal} &\implies (-3, 8, 2) \mapsto 0. \end{aligned}$$

In a similar manner, we have the following colorings:

$$\begin{aligned} \{(1, -1, 0), (1, 1, -1)\} \text{ orthogonal} &\implies (1, 1, -1) \mapsto 0, \\ \{(1, 0, 1), (1, 1, -1), (-1, 2, 1)\} \text{ orthogonal} &\implies (-1, 2, 1) \mapsto 1, \\ \{(-1, 2, 1), (4, 1, 2)\} \text{ orthogonal} &\implies (4, 1, 2) \mapsto 0, \\ \{(1, -1, 0), (2, 2, -5)\} \text{ orthogonal} &\implies (2, 2, -5) \mapsto 0, \\ \{(4, 1, 2), (2, 2, -5), (-3, 8, 2)\} \text{ orthogonal} &\implies (-3, 8, 2) \mapsto 1. \end{aligned}$$

But this contradicts the previous deduction that $(-3, 8, 2)$ must be colored 0, completing the proof. \square

3. CONSEQUENCES FOR PROJECTION MATRICES OVER COMMUTATIVE RINGS

In this section we describe consequences of Theorem 1 for the study of contextuality in this purely algebraic setting. To do so we briefly provide several definitions from [8] and [2], to which readers are referred for more details.

A *partial Boolean algebra* B is set equipped with a reflexive, symmetric binary operation \odot of *commensurability*, distinguished elements $0, 1 \in B$, a unary operation \neg of negation, and partially defined operations of meet \wedge and join \vee for commensurable pairs of elements, such that every pairwise commensurable set $S \subseteq B$ is contained in a pairwise commensurable set $C \subseteq B$ containing 0 and 1 for which the restricted negation, meet, and join make C into an ordinary (“total”) Boolean algebra. The colorings of Kochen and Specker correspond to *homomorphisms* of partial Boolean algebras $B \rightarrow \mathbf{2}$, where $\mathbf{2} = \{0, 1\}$ is the ordinary two-element Boolean algebra. We simply refer to such morphisms as Kochen-Specker colorings of a partial Boolean algebra as in [2, Theorem 2.13].

If A is a commutative ring, then the symmetric (under the transpose operation) idempotent elements of the matrix ring $\mathbb{M}_n(A)$ will be called *projections* by analogy with the case of real matrices. The set of all projections $\text{Proj}(\mathbb{M}_n(A))$ forms a partial Boolean algebra, where commensurability is given by commutativity in $\mathbb{M}_n(A)$, negation is the complement $\neg e = I - e$, and meet and join are given by $e \wedge f = ef$ and $e \vee f = e + f - ef$. For $A = \mathbb{R}$, this yields the usual partial Boolean algebra of projection matrices. (A suitable modification to the case where A is equipped with an involution allows one to recover the partial Boolean algebra of complex projection matrices, as well.)

This algebraic formulation allows us to investigate KS colorability of projections over some rings of number-theoretic interest. This was explored to some extent in [2], including for finite fields of prime order. Now thanks to the uncolorable set of Theorem 2, we are able to extend this treatment to p -adic numbers and localizations of the ring of integers in a unified way. For a prime p , let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denote the field with p elements, let \mathbb{Z}_p denote the ring of p -adic integers, and let $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z}, p \nmid b\}$ denote the localization of the integers at the prime ideal (p) .

Corollary 2. *Let p be a prime, and let R denote any of the rings \mathbb{F}_p , $\mathbb{Z}_{(p)}$, or \mathbb{Z}_p . Then $\text{Proj}(\mathbb{M}_3(R))$ has a Kochen-Specker coloring if and only if $p = 2, 3$.*

Proof. For each prime p we have injective and surjective ring homomorphisms $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p$ which induce homomorphisms between their corresponding matrix rings

$$\mathbb{M}_3(\mathbb{Z}_{(p)}) \hookrightarrow \mathbb{M}_3(\mathbb{Z}_p) \twoheadrightarrow \mathbb{M}_3(\mathbb{F}_p).$$

For $p = 2, 3$ it follows from [2, Theorem 3.4] that the projections of $\mathbb{M}_3(\mathbb{F}_p)$ have a Kochen-Specker coloring. By lifting along the homomorphism $\mathbb{M}_3(\mathbb{Z}_p) \rightarrow \mathbb{M}_3(\mathbb{F}_p)$ (see also [2, Theorem 2.13]), we obtain a coloring of the 3×3 projection matrices over \mathbb{Z}_p and $\mathbb{Z}_{(p)}$.

For $p = 5$, we have $\mathbb{Z}[1/462] \subseteq \mathbb{Z}_{(5)}$ and for $p > 5$ we have $\mathbb{Z}[1/30] \subseteq \mathbb{Z}_{(p)}$. The 3×3 projection matrices over $\mathbb{Z}[1/462]$ are uncolorable by Theorem 1, while those over $\mathbb{Z}[1/30]$ are uncolorable by [3]. In either case, we can embed an uncolorable set of projection matrices into $\mathbb{M}_3(\mathbb{Z}_{(p)})$, and the image of this set remains uncolorable over \mathbb{Z}_p and \mathbb{F}_p . \square

We close with one further application of Theorem 1 to the study of algebraic contextuality by determining which of the rings $\mathbb{Z}[1/N]$ give rise to prime partial ideals of symmetric matrices. We state the requisite definitions from [2].

A *partial ring* is a set R with distinguished elements $0, 1 \in R$, a unary operation of negation, a binary relation \odot of *commensurability*, and partial binary operations of addition and multiplication that are defined on commensurable pairs of elements, such that any pairwise commensurable set $S \subseteq R$ is contained in a pairwise commensurable set $C \subseteq R$ such that the restriction of the (partial) addition and multiplication make C into a commutative ring with identity 1 and zero element 0. Given a partial ring R , a subset $P \subseteq R$ is a (*prime*) *partial ideal* if, for every commensurable total subring $C \subseteq R$, the intersection $C \cap P$ is a (prime) ideal of C . The *partial spectrum* $p\text{-Spec}(R)$ is the set of all prime partial ideals of R .

There is a link [2, (2.15)] between the partial spectrum of R and Kochen-Specker colorings as follows. If there exists a prime partial ideal $P \in p\text{-Spec}(R)$, then we obtain a Kochen-Specker coloring on the partial Boolean algebra $B = \text{Idpt}(R)$ of idempotents in R , given by the following assignment for $e \in B$:

$$e \mapsto \begin{cases} 0, & e \in B \cap P, \\ 1, & e \in B \setminus P. \end{cases}$$

On the other hand, if $p\text{-Spec}(R) = \emptyset$ then there is no morphism of partial rings $R \rightarrow C$ for any commutative (total) ring C by [2, Lemma 2.4]. In this way, the non-existence of prime partial ideals can be viewed as an algebraic manifestation of noncontextuality.

Note that if A is a commutative ring, then the set $R = \mathbb{M}_n(A)_{\text{sym}}$ of symmetric $n \times n$ matrices over A forms a partial ring, where commensurability is taken to be commutativity as elements in the matrix ring $\mathbb{M}_n(A)$ and the partial binary operations are simply the restriction of the total binary operations of $\mathbb{M}_n(A)$. Note that the partial Boolean algebra of idempotents in this case recovers the partial Boolean algebra of projections discussed above: $\text{Idpt}(R) = \text{Proj}(\mathbb{M}_n(A))$.

We can now characterize exactly which of the rings of the form $A = \mathbb{Z}[1/N]$ (i.e., finitely generated subrings of \mathbb{Q}) have $p\text{-Spec}(\mathbb{M}_3(A)_{\text{sym}}) = \emptyset$.

Corollary 3. *For a squarefree positive integer N , the partial ring $\mathbb{M}_3(\mathbb{Z}[1/N])_{\text{sym}}$ has a prime partial ideal if and only if N is not divisible by 6.*

Proof. If N is not divisible by 6, then it is relatively prime to 2 or 3 (possibly both). This means that N is a unit modulo 2 or 3, so that we have a ring homomorphism from $\mathbb{Z}[1/N] \rightarrow \mathbb{F}_p$, and consequently a morphism $\mathbb{M}_3(\mathbb{Z}[1/N])_{\text{sym}} \rightarrow \mathbb{M}_3(\mathbb{F}_p)_{\text{sym}}$, for either $p = 2, 3$. By [2, Theorem 3.4] there exists a prime partial ideal of $\mathbb{M}_3(\mathbb{F}_p)_{\text{sym}}$ for each of these values of p , and its preimage gives a prime partial ideal of $\mathbb{M}_3(\mathbb{Z}[1/N])_{\text{sym}}$.

Now suppose that N is divisible by 6, and assume toward a contradiction that there exists a prime partial ideal $P \in p\text{-Spec}(\mathbb{M}_3(\mathbb{Z}[1/N])_{\text{sym}})$. First we claim that

$$5 \cdot I \in P.$$

Denote $Q = \{\frac{1}{5^n} \cdot x \mid x \in P, n \geq 0\} \subseteq \mathbb{M}_3(\mathbb{Z}[1/5N])_{\text{sym}}$. It is straightforward to verify that Q is also a partial ideal of the partial ring $R = \mathbb{M}_3(\mathbb{Z}[1/5N])_{\text{sym}}$, and that it still satisfies the condition that if $r, s \in R$ are commensurable and $rs \in Q$, then r or s lies in Q . But it follows from the uncolorability [3] of the 3×3 projections over $\mathbb{Z}[1/30] \subseteq \mathbb{Z}[1/5N]$ that $p\text{-Spec}(R) = \emptyset$. This means that Q cannot be a prime partial ideal, which is only possible if $Q = R$. Then $I \in Q$, so that $5^n I \in P$ for some $n \geq 1$. Since P is prime, we must have $5I \in P$.

By a similar argument using the embedding $\mathbb{Z}[1/462] \subseteq \mathbb{Z}[1/77N]$ and Theorem 1, we can deduce that $77I \in P$. But then because $5I$ and $77I$ are commensurable, we must have

$$I = 31 \cdot 5I - 2 \cdot 77I \in P,$$

contradicting that P is prime. □

It follows from the previous result and [2, Lemma 2.17] that for $n \geq 3$ and N divisible by 6,

$$p\text{-Spec}(\mathbb{M}_n(\mathbb{Z}[1/N])_{\text{sym}}) = \emptyset.$$

We remark that by [2, Corollary 2.16], Corollary 3 would follow from uncolorability of $\mathcal{S}(6)$. Thus in reference to Question 1, it remains an intriguing problem to determine whether or not $\mathcal{S}(6)$ has a Kochen-Specker coloring.

We close with a proof of Corollary 1 from Section 1.

Proof of Corollary 1. First suppose that $6 \mid N$. By the previous result and [2, Lemma 2.4], there is no morphism of partial rings $\mathbb{M}_n(\mathbb{Z}[1/N])_{\text{sym}} \rightarrow A$ for any commutative ring A .

Now suppose that $p \nmid N$ for either $p = 2, 3$. Then the ring homomorphism $\mathbb{Z}[1/N] \rightarrow \mathbb{F}_p$ induces a morphism of partial rings $\mathbb{M}_3(\mathbb{Z}[1/N])_{\text{sym}} \rightarrow \mathbb{M}_3(\mathbb{F}_p)_{\text{sym}}$.

There exists a morphism of partial rings $\mathbb{M}_3(\mathbb{F}_p)_{\text{sym}} \rightarrow \mathbb{F}_{p^2}$ by [2, Theorem 3.4, Lemma A.1]. Composing these maps gives a morphism of partial rings

$$\mathbb{M}_3(\mathbb{Z}[1/N])_{\text{sym}} \rightarrow \mathbb{F}_{p^2}$$

to a commutative ring, proving the claim. \square

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