THE FINITE DUAL COALGEBRA AS A QUANTIZATION OF THE MAXIMAL SPECTRUM

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Abstract. In pursuit of a noncommutative spectrum functor, we argue that the Heyneman-Sweedler finite dual coalgebra can be viewed as a quantization of the maximal spectrum of a commutative affine algebra, integrating prior perspectives of Takeuchi, Batchelor, Kontsevich-Soibelman, and Le Bruyn. We introduce *fully residually finite-dimensional* algebras A as those with enough finite-dimensional representations to let A° act as an appropriate depiction of the noncommutative maximal spectrum of A; importantly, this class includes affine noetherian PI algebras. We investigate cases where the finite dual coalgebra of a twisted tensor product is a crossed product coalgebra of the respective finite duals. This is achieved by interpreting the finite dual as a topological dual. Sufficient conditions for this result to be applied to Ore extensions, smash product algebras, and crossed product bialgebras are described. In the case of prime affine algebras that are module-finite over their center, we describe how the Azumaya locus is represented in the finite dual. Finally, we implement these techniques for quantum planes at roots of unity as an endeavor to visualize the noncommutative space on which these algebras act as functions.

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1. Introduction

1.1. **Prologue: noncommutative spectral theory.** This paper is intended as a contribution to *noncommutative spectral theory*, by which we mean the development and study of invariants of associative rings and algebras that extend the usual spectrum of a commutative ring or algebra, with the goal of gaining insight into the structure of noncommutative rings. Ideally one would wish for a noncommutative spectrum that can be equipped with enough extra structure to form a complete invariant, allowing us to recover the ring up to isomorphism. The basis for this hope is the Zariski spectrum of a commutative ring R, which can be equipped with its structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ that allows us to recover R up to isomorphism as the ring of global sections of the affine scheme $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$. Probably the best-known work on noncommutative spectral theory regards the development of structure sheaves on noncommutative prime spectra. Considerable efforts have been made in this direction over several decades; see [VO10] for a comprehensive survey and a large bibliography on this topic.

When considering extensions of the Zariski spectrum as an assignment from rings to topological spaces, we believe that there are two important properties that one should ask of such a spectrum:

(1) the assignment returns the usual Zariski spectrum of a commutative ring,
(2) the assignment extends to a functor $\mathsf{Ring}^{\mathrm{op}} \to \mathsf{Top}$.

Condition (2) would bring us closer to a duality between noncommutative rings and appropriate spatial objects. Aside from aesthetic considerations, it is also important because it guarantees that partial information about the spectrum of a ring can be obtained from the spectra of its subrings and quotient rings. It is also crucial if one wishes for such a spectrum to take into account algebraic quantum theory [Lan09], where commutative subalgebras of a noncommutative algebra are intimately tied to the relationship between classical and quantum information [Heu17].

Unfortunately, the results of [Rey12, BZMR17] indicate a major obstruction to this hope: any functor satisfying the above properties must assign the empty space to any ring with $n \times n$ matrix structure for $n \geq 3$. One might conclude from this obstruction that the problem somehow lies in the points themselves, particularly if one is familiar with point-free approaches to topology [Joh83] such as locales, toposes, and quantales. However, subsequent results [BH14, Rey14, BZMR17] indicate that similar obstructions persist at this level of generality.

In our view, these obstructions indicate that the usual building blocks of topology, whether from point sets or open covers without points, are too commutative in nature to serve as a deep spectral invariant for noncommutative rings. This highlights a fundamental question that has been minimized for too long:

> What objects should play the role of *noncommutative sets* within noncommutative geometry?

While many kinds of noncommutative spaces have been studied across the various branches of noncommutative geometry, the most basic case of noncommutative *discrete* spaces (i.e., sets) has received comparatively little attention. The obstructions discussed above suggest that we cannot make serious progress in noncommutative spectral theory until this gap in noncommutative mathematics has been filled. Fortunately, recent work such as [HR17, MRV18, Kor20, Kor21] has begun to address this issue, although it is limited to the setting of noncommutative geometry based on C*-algebras.

In light of the above, we will use the term *functorial spectral theory* to describe the pursuit of a functorial invariant of associative rings (or C*-algebras) that extends the Zariski (or Gelfand) spectrum from commutative to noncommutative rings. The major problem becomes locating a suitable category $\mathfrak{S}$ of *noncommutative sets*, with a full and faithful functor $\mathsf{Set} \hookrightarrow \mathfrak{S}$, such that there is a "quantum spectrum" functor $\Sigma\colon \mathsf{Ring}^{\mathrm{op}} \to \mathfrak{S}$ making the following diagram commute (up to natural isomorphism):

$$\begin{tikzcd}
\mathsf{cRing}^{\mathrm{op}} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook] & \mathsf{Top} \arrow[r, "U"] & \mathsf{Set} \arrow[d, hook] \\
\mathsf{Ring}^{\mathrm{op}} \arrow[rr, "\Sigma"] & & \mathfrak{S}
\end{tikzcd} \tag{1.1}$$

Ideally, one should be able to equip noncommutative sets with (some noncommutative version of) a topology in order to endow each quantum spectrum with a structure sheaf, from which one can recover the original ring up to isomorphism. While this is a daunting problem, there have been recent advances [HR14, HR17] in functorial spectral theory for C*-algebras that give us reason to maintain hope in the face of this challenge.

1.2. **Quantizing the maximal spectrum.** A comprehensive solution to the problems raised above remains out of reach for now. In this paper we pursue the more modest goal of extending the *maximal spectrum* functor from the category cAff of commutative affine algebras over a field k to a suitable class of algebras that are only "mildly noncommutative." Our intent is to catch a glimpse of the underlying discrete objects of noncommutative spaces, even if we cannot directly access the objects themselves.

Our point of departure lies in the algebraic approach to quantum mechanics [Lan09], where commutative and noncommutative algebras are respectively embodied as classical and quantum observables. We believe that there are deep reasons to attempt to build a functorial spectral theory that draws inspiration from quantum physics, in part because many of the obstructions mentioned above hinge upon the Kochen-Specker Theorem of quantum mechanics [BZMR17] and in part due to the important role that commutative subalgebras of noncommutative algebras play in each theory [Heu17, Rey12].

Let us imagine a commutative algebra as an algebra of observables in classical mechanics, so that its spectrum corresponds to the space of states of a classical system. In passing to quantum mechanics, the *superposition principle* forces physicists to allow for linear combinations (more precisely, convex combinations) of quantum states. Thus for a commutative k-algebra A we aim to replace the set $\mathrm{Max}(A)$ with k-linear combinations of its points. (We may imagine that k is algebraically closed for the moment, to minimize technicalities.) It turns out that we may recover any set from its k-linear span when endowed with just a bit of structure: a *coalgebra*. This yields a fully faithful embedding of the category Set into the category Coalg of k-coalgebras (Proposition 3.1 below). In this way we think of coalgebras as "linear spans of quantum states," with comultiplication functioning as a "quantum diagonal" map. We can imagine elements of the coalgebra as linear combinations of points, but it is not possible in general to isolate individual points. Note that Batchelor similarly viewed coalgebras as generalized sets in noncommutative geometry in [Bat91]. At the same time, we describe in Subsection 3.2 how each k-scheme has a cocommutative *coalgebra of distributions* that can be viewed as underlying discrete objects. This idea dates back to work of Takeuchi [Tak74, Tak77a], and we connect our point of view to the definitions used in that work. A similar perspective in differential geometry arose in work of Batchelor [Bat88].

The *finite dual coalgebra* construction of Heyneman and Sweedler [HS69, 1.3] associates to every k-algebra A a coalgebra A° (whose definition we recall in Subsection 2.1) in such a way that it forms a functor

$$(-)^\circ \colon \mathsf{Alg}^{\mathrm{op}} \to \mathsf{Coalg}\,.$$

For commutative affine algebras A, the coalgebra A° agrees with the coalgebra of distributions on the scheme $\mathrm{Spec}(A)$, and its simple subcoalgebras are in bijection with $\mathrm{Max}(A)$. Thus for k-algebras that are endowed with "many" finite-dimensional representations, we argue that the finite dual coalgebra serves as a suitable approximation to the quantized underlying discrete object of its noncommutative affine scheme. More precisely, it can be viewed as containing a linearization of the maximal spectrum of an affine scheme of finite type over k. We point out that dual coalgebras were similarly viewed as distributions on noncommutative spaces in the noncommutative thin schemes [KS09, Section 2] of Kontsevich and Soibelman, and the perspective of dual coalgebras as noncommutative spectral objects has been explored further by Le Bruyn in [LB08] and [LB12, Section 3.2].

Embedding the category of sets into the category of coalgebras as described above, we obtain a diagram

$$\begin{tikzcd} \mathsf{cAff}^{\mathrm{op}} \arrow[r, "\mathrm{Max}"] \arrow[d, hook] & \mathsf{Top} \arrow[r, "U"] & \mathsf{Set} \arrow[d, hook] \\ \mathsf{Alg}^{\mathrm{op}} \arrow[rr, "(-)^\circ"] & & \mathsf{Coalg} \end{tikzcd} \tag{1.2}$$

that is similar to (1.1), but which *does not commute* up to isomorphism. Thus the dual coalgebra only approximates the space of closed points in a noncommutative spectrum. This is a necessary evil if we wish to require functoriality and nontrivial behavior for matrix algebras, as explained precisely in Theorem 4.6. While we remain cautiously optimistic that diagram (1.1) might one day be realized, the situation in (1.2) can serve as a model for functorial invariants that do not strictly extend the Zariski spectrum, but rather extend some mild enhancement of it.

As alluded to above, it is only reasonable to expect that the dual coalgebra A° is a reasonable substitute for a maximal spectrum if A has sufficiently many finite-dimensional representations. In Subsection 4.2 we introduce a class of algebras A

for which we believe that A° captures a relatively full picture of the noncommutative geometry of A. These are called *fully residually finite-dimensional (RFD)* algebras, and their defining property roughly states that the finite-dimensional representation theory of A is rich enough to capture information about all finitely generated modules. Importantly, the fully RFD algebras contain a large class of familiar noncommutative algebras: the affine noetherian PI algebras.

It is certainly true that the fully RFD property is quite restrictive as a ring-theoretic condition. However, a common phenomenon within noncommutative algebraic geometry is that many q-deformed families of quantum algebras contain affine algebras that are module-finite over their center, typically when the deforming parameters are equal to roots of unity [BG02, Part III]. Such an algebra satisfies a polynomial identity and is noetherian (see [MR01, Corollary 13.1.13, Lemma 13.9.10]). Thus these quantum families of algebras have nontrivial intersection with the class of fully RFD algebras, and one can try to obtain a geometric understanding of algebras in this intersection by studying the finite dual as a quantized maximal spectrum.

If we are willing to accept the importance of the finite dual as a spectral invariant of an algebra, we are left with an important problem. How can one go about computing A° for a given algebra A? To our knowledge, this has been done for relatively few concrete algebras A. Happily, there are notable counterexamples to this trend in the literature on Hopf algebras, such as [Tak92, CM94, Jah15, Cou19, GL21, LL21, BCJ21]. For this reason, we provide the following new tools to assist with the computation of dual coalgebras.

Many common constructions to build noncommutative algebras out of simpler (even commutative) algebras—such as skew polynomial rings, twisted group algebras, and smash product algebras—are instances of a *twisted tensor product* [ČSV95] (or *crossed product*) construction. Recent papers such as [GKM17, CG18, SW19, CG21] have studied a number of interesting aspects of this construction, and it seems that it could be useful to have spectral techniques to aid in this analysis. Thus we have focused our attention on finding a suitable method to compute the dual coalgebra of a twisted tensor product. Because there exists a formally dual construction of crossed product coalgebras [CIMZ00], one might naturally assume that the finite dual should interchange these operations in a formula such as

$$(A\#B)^\circ \cong A^\circ\#B^\circ,$$

but it is easy to construct examples where this fails (Example 5.13). Nevertheless, we are able to find a sufficient condition on the twisting map for an isomorphism of the above form to hold. We are able to prove this result by viewing the finite dual and its comultiplication $A^\circ \to A^\circ \otimes A^\circ$ formally as a *topological* dual to the multiplication $A \otimes A \to A$, when A is endowed with a suitable topology. The sufficient condition for the isomorphism above is then phrased in terms of continuity of the twisting map.

Another important source of affine noetherian PI algebras are the affine algebras A that are module-finite over the center $Z(A)$. Functoriality guarantees that the dual coalgebras are related by a morphism $A^\circ \to Z(A)^\circ$. In the case where A is prime, we give a clearer description of the relationship between A° and the geometry of $\operatorname{Spec} Z(A)$. Specifically, we describe a large direct summand of A°

that corresponds to the Azumaya locus [BG97, Section 3] tensored with a matrix coalgebra.

In order to illustrate the methods of the twisted tensor product and Azumaya locus in action, we close this paper with a description of the dual coalgebra of the quantum plane $\mathcal{O}_q(k^2)$ where k is algebraically closed and $q \in k^*$ is a root of unity. Our goal is to provide a glimpse of a concrete mathematical object that can be visualized as a quantum plane. This example also underscores the potential of the perspective described above to place geometric and quantum language on equal footing.

1.3. **Outline of the paper.** In Section 2 we recall the construction of the finite dual coalgebra. We also give careful attention to continuous duals of tensor products of topological vector spaces. The key result here is Theorem 2.12, which allows us to interpret the finite dual in terms of a formal topological duality.

In Section 3 we describe how coalgebras can be viewed both as quantized sets and as underlying discrete objects for (commutative) schemes of finite type over a field. This motivates the main thesis, described in Section 4, that the finite dual functor is a reasonable replacement for the maximal spectrum for algebras that are "not too far" from being commutative. Because A° is strictly larger than the linearized maximal spectrum for commutative affine algebras A, we take time to explain in Theorem 4.6 that it is, in a sense, the best choice for a coalgebra-valued spectrum functor whose behavior on matrix algebras agrees with quantum intuition. We then introduce *fully residually finite-dimensional* algebras as those algebras whose finite dual offers a useful substitute for the maximal spectrum. Proposition 4.12 shows that this class of algebras includes all affine noetherian PI algebras, an important class of algebras that has a nontrivial intersection with many interesting and important quantized families of noncommutative algebras.

In Section 5 we turn to the problem of computing the finite dual of a given algebra. Because many examples of interest arise as twisted tensor products [ČSV95], we focus on computing dual coalgebras at this level of generality. The key result is Theorem 5.14, which states that it is sufficient for the twisting map to be continuous in an appropriate topology. (This is the purpose for which we develop the tools for topological duality in Section 2.) We then explain conditions under which this result applies to Ore extensions, smash product algebras, and crossed product bialgebras.

Finally, in Section 6 we restrict to the case of prime affine algebras over an algebraically closed field that are module-finite over their center. For such algebras, we provide a direct sum decomposition of the dual coalgebra in which one of the summands corresponds to the Azumaya locus. The paper concludes with a study of the dual coalgebra of the quantum plane $\mathcal{O}_q(k^2)$ for k an algebraically closed field and $q \in k$ a root of unity.

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2. Duality between algebras and coalgebras

This section describes some elements of duality theory relating algebras and coalgebras. Many readers may wish to skip this material, proceeding directly to Section 3 and referring back to this section only as needed.

Let k be a field, which is completely arbitrary except where explicitly stated otherwise. Unadorned tensor symbols $-\otimes-$ denote tensor over k. In this paper, all algebras are unital and associative, all coalgebras are counital and coassociative, and morphisms of these objects preserve the (co)unit. We let $\mathsf{Alg} = \mathsf{Alg}_k$ denote the category of k-algebras and $\mathsf{Coalg} = \mathsf{Coalg}_k$ denote the category of k-coalgebras.

2.1. **Dual coalgebras.**

Our major source of coalgebras will be as suitable duals of algebras. Suppose that A is a finite-dimensional k-algebra. We consider its multiplication as a linear map $m\colon A \otimes A \to A$ and its unit as the linear map $u\colon k \to A$ given by $1 \mapsto A$, so that it forms a monoid object in $(\mathsf{Vect}, \otimes, k)$. There is a natural map

$$A^* \otimes A^* \to (A \otimes A)^*$$

defined by allowing pure tensors $\phi\otimes\psi \in A^*\otimes A^*$ to act as functionals on $a\otimes b \in A\otimes A$ via $(\phi\otimes\psi)(a\otimes b) = \phi(a)\psi(b)$; because A is finite-dimensional this is an isomorphism. Thus the multiplication $m\colon A\otimes A \to A$ and unit $\eta\colon k \to A$ maps respectively dualize to a comultiplication and counit

$$\begin{aligned}\Delta &= m^*\colon A^* \to (A \otimes A)^* \cong A^* \otimes A^*,\\ \epsilon &= \eta^*\colon A^* \to k^* = k.\end{aligned}$$

These satisfy coassociativity and counitality as duals of associativity and unitality, so that (A^*, Δ, ϵ) is a coalgebra.

For a finite-dimensional algebra A, we will occasionally wish to explicitly determine the structure of the coalgebra A^*. Let $b_1, \dots, b_n \in A$ be a basis for A, and let $b^1, \dots, b^n \in A^*$ denote the corresponding dual basis. Fix structure constants $c^r_{ij} \in k$ such that $b_i b_j = \sum c^r_{ij} b_r$, and write $1 = \sum u_r b_r$ for $u_r \in k$. Then the comultiplication Δ and counit ϵ of A^* are given by

$$\begin{aligned}\Delta(b^r) &= \sum_{ij} c^r_{ij}\, b^i \otimes b^j,\\ \epsilon(b^r) &= u_r.\end{aligned} \tag{2.1}$$

Indeed, recalling that Δ is the dual of the multiplication map $m\colon A \otimes A \to A$ via the isomorphism $A^* \otimes A^* \cong (A \otimes A)^*$, it suffices to compute that the expression for $\Delta(b^r)$ evaluated on each basis element $b_i \otimes b_j \in A \otimes A$ agrees with $\Delta(b^r)(b_i \otimes b_j) = b^r(b_i b_j) = c^r_{ij}$, which is clear. Similarly, one verifies that $\epsilon\colon A^* \to k$ has the expression above by evaluating each $\epsilon(b^r)$ on the basis elements $b_i \in A$.

For an infinite-dimensional algebra A, the natural embedding $A^* \otimes A^* \to (A \otimes A)^*$ is not an isomorphism, so the full linear dual A^* does not naturally inherit a coalgebra structure. Nevertheless, there is a subspace $A^\circ \subseteq A^*$ that *does* naturally form a coalgebra, called the *finite dual* (occasionally called the *Sweedler dual*). For details regarding the following discussion, see [Swe69, Chapter VI] or [DNR01, Section 1.5].

Letting $m\colon A \otimes A \to A$ denote the multiplication of A, the subspace $A^\circ \subseteq A^*$ is defined to be the set of those $\phi \in A^*$ that satisfy the following equivalent conditions:

(SD1) $m^*(\phi) \in (A \otimes A)^*$ lies in the subspace $A^* \otimes A^* \subseteq (A \otimes A)^*$;
(SD2) the kernel of ϕ contains an ideal $I \trianglelefteq A$ of finite codimension in A.

It turns out that the restriction of m^* to A° has image in $A^\circ \otimes A^\circ \subseteq A^* \otimes A^*$. In this way, $\Delta = m^*$ restricts to a comultiplication

$$\Delta\colon A^\circ \to A^\circ \otimes A^\circ$$

making the finite dual a coalgebra, whose counit is the dual of the unit of A.

The coalgebra structure on A° can be alternately described in terms of condition (SD2) above. Within the lattice of ideals of A, the family

$$\mathcal{F}(A) = \{I \trianglelefteq A \mid \dim_k(A/I) < \infty\} \tag{2.2}$$

of ideals having finite codimension forms a filter: it is upward-closed and closed under pairwise intersections. Thus the diagram of finite-dimensional algebras of the form A/I for $I \in \mathcal{F}(A)$ forms an inversely directed system. As each $\phi \in A^\circ$ is induced by some $\overline{\phi} \in (A/I)^*$ via the canonical homomorphism $A \to A/I$ for some $I \in \mathcal{F}(A)$, we have an isomorphism of vector spaces

$$A^\circ \cong \varinjlim_{I \in \mathcal{F}(A)} (A/I)^*. \tag{2.3}$$

Since the algebras A/I above are finite-dimensional, the above is a directed limit of finite-dimensional coalgebras, and the isomorphism (2.3) is in fact an isomorphism of coalgebras.

Let Alg denote the category of k-algebras and their homomorphisms. One can check using either condition (SD1) or (SD2) that any algebra homomorphism $\phi\colon A \to B$ dualizes to a morphism of coalgebras $B^\circ \to A^\circ$, so that the finite dual forms a functor

$$(-)^\circ\colon \mathsf{Alg}^{\mathrm{op}} \to \mathsf{Coalg}\,. \tag{2.4}$$

In fact, it enjoys the following adjoint relationship with the dual algebra functor $(-)^*\colon \mathsf{Coalg}^{\mathrm{op}} \to \mathsf{Alg}$: for any algebra A and coalgebra C, one has natural isomorphisms

$$\mathsf{Alg}(A, C^*) \cong \mathsf{Coalg}(C, A^\circ). \tag{2.5}$$

We remark that if A is affine, then every ideal in $\mathcal{F}(A)$ is finitely generated as a left (and right) ideal; see [Rey16, Theorem 4.17]. Furthermore, if every ideal of $\mathcal{F}(A)$ is finitely generated as a left ideal (e.g., if A is affine or left noetherian), then $\mathcal{F}$ is also closed under products; see [Rey16, Proposition 4.7(2)].

2.2. **Continuous duality and topological tensor products.** An alternate way to define Coalg_k is as the category of comonoid objects [ML98, VII.3] in the symmetric monoidal category $(\mathsf{Vect}, \otimes, k)$ of k-vector spaces equipped with the tensor product. It follows formally that the tensor product of coalgebras is a monoidal structure, making $(\mathsf{Coalg}, \otimes, k)$ into a monoidal category. While this may not be the most natural point of view for an algebraist who favors elements and equations, it has the advantage that certain dualities involving algebras and coalgebras can be proved in a straightforward and formal way. We will exploit this technique in Section 5, where we examine dual coalgebras of twisted tensor products.

Let R be a topological ring. A topological left R-module M is *linearly topologized* if it has a neighborhood basis of zero consisting of open submodules. We say R is *left linearly topologized* if ${}_R R$ is a linearly topologized module, and *right linearly topologized rings* are similarly defined. We say that R is *linearly topologized* if it is both left and right linearly topologized, or equivalently [IMR16, Remark 2.19], if it has a neighborhood basis of zero consisting of open ideals.

We consider our field k as a topological field with the discrete topology. Let R be a topological k-algebra. Recall that a topological left R-module M is said to be *pseudocompact* if it satisfies the following equivalent conditions:

- M is separated (i.e., Hausdorff), complete, and has a neighborhood basis of zero consisting of open submodules of finite codimension;
- M is an inverse limit (in the category of topological algebras) of discrete, finite-dimensional modules;
- The natural homomorphism $M \to \varprojlim M/N$, where N ranges over the open ideals of finite codimension in M, is an isomorphism of topological modules.

The topology on such a module M is induced by viewing it as a subspace $M \cong \varprojlim M/N \subseteq \prod M/N$ of the product space $\prod M/N$ where again N ranges over the open submodules of finite codimension, and each finite-dimensional M/N is endowed with the discrete topology [IMR16, Lemma 2.17].

Let R-PC denote the category of pseudocompact left R-modules with continuous homomorphisms. In particular, taking $R = k$, we have the category $\mathsf{PC}_k := k$-PC of *pseudocompact vector spaces.* These are topological vector spaces that are inverse limits of finite-dimensional discrete vector spaces.

We let Top_k denote the category of linearly topologized k-vector spaces with continuous k-linear maps. Then PC_k forms a full subcategory of Top_k. We will identify Vect_k with the full subcategory of discrete topological vector spaces within Top_k. We say that a linearly topologized vector space E is *cofinite* if its open subspaces all have finite codimension in E. We let CF_k denote the full subcategory of Top_k consisting of cofinite spaces. Then we have the following inclusions of full subcategories:

$$\mathsf{PC}_k \subseteq \mathsf{CF}_k \subseteq \mathsf{Top}_k \, .$$

The completion of a separated topological vector space is a well-known construction [War93, Section 7], but we will require a straightforward extension to spaces that are not necessarily separated. For a linearly topologized space E, we let $\mathcal{F}(E)$ denote the family of open subspaces of E. The *separation* of E,

$$E_s := E \big/ \overline{\{0\}} = E \Big/ \left(\bigcap \mathcal{F}(E) \right) ,$$

is universal among all separated spaces with a continuous surjection from E. Then for such E, we can define the *completion* equivalently as the usual completion of its separation, or as a colimit in Top_k,

$$\widehat{E} := \varprojlim_{W \in \mathcal{F}(E)} E/W = \widehat{E_s}.$$

It is straightforward to verify that the assignments $E \mapsto E_s$ and $E \mapsto \widehat{E}$ both naturally extend to endofunctors of Top_k. See also [Pos21, Section 1] for many more details.

Let E and F be linearly topologized vector spaces. Following [Pos21, Section 12], let $E \otimes^! F$ denote the vector space $E \otimes F$ equipped with the linear topology whose open subspaces are the subspaces $W \subseteq E \otimes F$ for which there exist open subspaces $E_0 \in \mathcal{F}(E)$ and $F_0 \in \mathcal{F}(F)$ such that

$$E_0 \otimes F + F \otimes E_0 \subseteq W.$$

Note that the basic open sets described above are precisely the kernels of the natural surjections $E \otimes F \twoheadrightarrow (E/E_0) \otimes (F/F_0)$. It follows immediately that if E and F both have cofinite topologies, then the same is true for $E \otimes^! F$. In this way we obtain a monidal category structure $(\mathsf{CF}_k, \otimes^!, k)$ on the category of cofinite linearly topologized spaces.

Now let $E \widehat{\otimes} F$ denote the completion of $E \otimes^! F$. While this construction is defined for general objects of Top_k that are not necessary separated or complete, one can readily verify that

$$E \widehat{\otimes} F \cong \varprojlim E/E' \otimes F/F' \cong \widehat{E} \widehat{\otimes} \widehat{F},$$

where E' and F' range over the open subspaces of E and F, respectively. Furthermore, if E and F are pseudocompact, the equation above makes it clear that the same is true for $E \widehat{\otimes} F$, and we obtain the *completed tensor product* defined in [Bru66]. There it was characterized by the following universal property: the pseudocompact space $E \widehat{\otimes} F$ is equipped with a continuous bilinear map $B \colon E \times F \to E \widehat{\otimes} F$ (where $B(v, w) = v \widehat{\otimes} w$) such that every continuous bilinear map $E \times F \to X$ to a pseudocompact k-vector space X factors uniquely through a continuous linear map ϕ as follows:

$$E \times F \xrightarrow{B} E \widehat{\otimes} F \xrightarrow{\phi} X.$$

We now turn to the relationship between the tensor products discussed above and dual space functors. Because Top_k is a k-linear category, we have a k-linear *continuous dual* functor

$$(-)^\circ := \mathsf{Top}_k(-, k) \colon \mathsf{Top}_k^{\mathrm{op}} \to \mathsf{Vect}_k.$$

The following key result describes how the continuous dual plays well with the topological tensor $- \otimes^! -$, especially when restricted to the category CF_k of spaces with cofinite topology. The definitions of lax and strong monoidal functors can be found in [ML98, XI.2].

Theorem 2.6. *The continuous dual forms a lax monoidal functor*

$$(-)^\circ \colon (\mathsf{Top}_k, \otimes^!, k)^{\mathrm{op}} \to (\mathsf{Vect}_k, \otimes, k),$$

which restricts to a strong monoidal functor

$$(-)^\circ \colon (\mathsf{CF}_k, \otimes^!, k)^{\mathrm{op}} \to (\mathsf{Vect}_k, \otimes, k).$$

Proof. First note that duality preserves the monoidal unit as $k^\circ = k^* \cong k$. Let E and F be linearly topologized vector spaces. Consider the natural embedding for the discrete dual spaces $E^* \otimes F^* \hookrightarrow (E \otimes F)^*$, where a pure tensor $\phi \otimes \psi \in E^* \otimes F^*$ acts on a pure tensor $e \otimes f \in E \otimes F$ by

$$(\phi \otimes \psi)(e \otimes f) = \phi(e)\psi(f).$$

If $\phi \in E^\circ$ and $\psi \in F^\circ$ both have open kernels, note that

$$\ker \phi \otimes F + E \otimes \ker \psi \subseteq \ker(\phi \otimes \psi)$$

implies that $\phi \otimes \psi \in (E \otimes^! F)^\circ$. Thus the natural embedding for discrete dual spaces restricts to an analogous embedding for continuous dual spaces:

$$\begin{tikzcd} E^* \otimes F^* \arrow[r, hook] & (E \otimes F)^* \\ E^\circ \otimes F^\circ \arrow[r, hook] \arrow[u, hook] & (E \otimes^! F)^\circ \arrow[u, hook] \end{tikzcd}$$

To show that this natural embedding

$$\Phi_{E,F} \colon E^\circ \otimes F^\circ \hookrightarrow (E \otimes^! F)^\circ, \tag{2.7}$$

makes F into a lax monoidal functor, we must check the coherence axioms [ML98, XI.2(3,4)]. The left and right unitor diagrams are easily seen to commute:

$$\begin{tikzcd}
k \otimes E^\circ \arrow[r, "\sim"] \arrow[d, "\wr"] & E^\circ \\
k^\circ \otimes E^\circ \arrow[r, hook] & (k \otimes^! E)^\circ \arrow[u, "\wr"']
\end{tikzcd}
\qquad
\begin{tikzcd}
E^\circ \otimes k \arrow[r, "\sim"] \arrow[d, "\wr"] & E^\circ \\
E^\circ \otimes k^\circ \arrow[r, hook] & (E \otimes^! k)^\circ \arrow[u, "\wr"']
\end{tikzcd}$$

The coherence diagram for the associator takes the following form:

$$\begin{tikzcd}
E^\circ \otimes (F^\circ \otimes G^\circ) \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{id} \otimes \Phi_{F,G}"'] & (E^\circ \otimes F^\circ) \otimes G^\circ \arrow[d, "\Phi_{E,F}, G"] \\
E^\circ \otimes (F \otimes^! G)^\circ \arrow[d, "\Phi_{E, F \otimes^! G}"'] & (E \otimes^! F)^\circ \otimes G^\circ \arrow[d, "\Phi_{E \otimes^! F, G}"] \\
(E \otimes^! (F \otimes^! G))^\circ \arrow[r, "\alpha_2^\circ"] & ((E \otimes^! F) \otimes^! G)^\circ
\end{tikzcd} \tag{2.8}$$

where the morphisms α_1 in Vect_k and α_2 in Top_k are the appropriate associators. To verify that this diagram commutes, we may take $(\phi_1, \phi_2, \phi_3) \in E^\circ \times F^\circ \times G^\circ$ and chase the pure tensor $\phi_1 \otimes (\phi_2 \otimes \phi_3) \in E^\circ \otimes (F^\circ \otimes G^\circ)$ around the diagram. The composite morphism down the left column interprets $\phi_1 \otimes (\phi_2 \otimes \phi_3)$ as a functional on $E \otimes^! (F \otimes^! G)$ by the action

$$(\phi_1 \otimes (\phi_2 \otimes \phi_3))(x \otimes (y \otimes z)) = \phi_1(x)\phi_2(y)\phi_3(z).$$

The actions of α_1 and α_2 simply regroup the tensor factors, and the composite down the right column of the diagram acts in a completely analogous way. In this way one can check that the diagram commutes when restricted to pure tensors, so by linearity it commutes in Vect_k. Thus $(-)^\circ$ is a lax monoidal functor $(\mathsf{Top}_k, \otimes^!, k)^{\mathrm{op}} \to (\mathsf{Vect}_k, \otimes, k)$.

To show that $(-)^\circ$ forms a strict monoidal functor on $(\mathsf{CF}_k, \otimes^!, k)$, we assume that E and F have cofinite topologies, and we show that the embedding is surjective. Thus let $\varphi \in (E \otimes F)^\circ$. Then there exist open subspaces $E_0 \subseteq E$ and $F_0 \subseteq F$ such that $E^0 \otimes F + E \otimes F_0 \subseteq \ker \varphi$. Thus φ factors as

$$E \otimes F \twoheadrightarrow E/E_0 \otimes F/F_0 \xrightarrow{\overline{\varphi}} k.$$

Because E and F are cofinite, both E_0 and F_0 both have finite codimension, and we have

$$\overline{\varphi} \in (E/E_0 \otimes F/F_0)^* \cong (E/E_0)^* \otimes (F/F_0)^*.$$

So we may in fact write

$$\varphi = \sum_{i=1}^{n} \phi_i \otimes \psi_i \in E^* \otimes F^*$$

where each $\ker \phi_i \supseteq E_0$ and $\ker \psi_i \supseteq F_0$, making each $\phi_i \in E^\circ$ and $\psi_i \in F^\circ$. So in this case the map (2.7) is an isomorphism as desired. $\square$

In the opposite direction, the discrete dual space gives a k-linear functor

$$(-)^* := \mathrm{Hom}_k(-,k)\colon \mathsf{Vect}_k^{\mathrm{op}} \to \mathsf{PC}_k$$

where the dual $V^* = \mathrm{Hom}(V,k)$ of a vector space V is equipped with the *finite topology*: the linear topology whose basic open subspaces are the annihilators $X^\perp = \{\phi \in V^* \mid \phi(X) = 0\}$ of all finite-dimensional subspaces $X \subseteq V$. Restricting the continuous dual to the subcategory PC_k of Top_k, it is well known that these two functors are mutually quasi-inverse, yielding a duality between Vect_k and PC_k; see [Gab62, IV.4] or [Sim01, Proposition 2.6]. We will show in Corollary 2.9 that this can be enhanced to a duality of monoidal categories with appropriate tensor products.

This will provide an alternative explanation for another well-known duality between coalgebras and pseudocompact algebras [Sim01, Theorem 3.6(d)]. Recall that a *pseudocompact algebra* A is a linearly topologized algebra that is separated, complete, and has a neighood basis of zero consisting of open ideals having finite codimension, or equivalently, if it is the topological inverse limit of finite-dimensional discrete algebras. We let PCAlg denote the category of pseudocompact k-algebras with continuous algebra homomorphisms.

Let (C, Δ, ϵ) be a coalgebra. As is well known [Swe69, Section 4.0], the dual vector space C^* is endowed with the structure of a k-algebra, with the *convolution* product induced by restricting the dual of the comultiplication $\Delta^*\colon (C \otimes C)^* \to C^*$ to the subspace $C^* \otimes C^* \hookrightarrow (C \otimes C)^*$, and having unit $\epsilon \in C^*$. Explicitly, the convolution product is given as follows: given $f, g \in C^*$ and an element $q \in C$ with $\Delta(q) = \sum q^{(1)} \otimes q^{(2)}$ written in Sweedler notation, their convolution $fg \in C^*$ is the element that acts via

$$fg(q) = \sum f(q^{(1)})g(q^{(2)}).$$

Thanks to the fundamental theorem of coalgebras [Rad12, Theorem 2.2.3], C is the directed union of its finite-dimensional subcoalgebras. Thus the dual algebra is an inverse limit $C^* \cong \varprojlim S^*$, where S ranges over the finite-dimensional subcoalgebras $S \subseteq C$. By endowing each of the finite-dimensional algebras S^* with the discrete topology, we may view the inverse limit above in the category of topological algebras. The resulting topology is the *finite topology* on C^*. It is evident from the construction that C^* is a pseudocompact k-algebra.

Corollary 2.9. *There is a strong monoidal duality between* $(\mathsf{Vect}_k, \otimes, k)$ *and* $(\mathsf{PC}_k, \widehat{\otimes}, k)$ *given by the dual space functors*

$$(-)^\circ\colon\ \mathsf{PC}_k^{\mathrm{op}} \to \mathsf{Vect}_k,$$
$$(-)^*\colon \mathsf{Vect}_k^{\mathrm{op}} \to \mathsf{PC}\,.$$

These functors induce further dualities between:

1. Coalg *and* PCAlg,
2. C-Comod *and* C^*-PC *for any coalgebra* C.

Proof. As mentioned before, it is well-established [Gab62, IV.4] that the continuous and discrete dual functors provide a duality between PC_k and Vect_k. These functors clearly interchange the monoidal units. Thus we only need to examine the effect of these functors on the tensor structures.

First let E and F be objects of PC_k. Recalling that $E\widehat{\otimes}F$ is the completion of the linearly topologized space $E \otimes^! F$, we obtain a natural isomorphism

$$(E\widehat{\otimes}F)^\circ = \mathsf{PC}_k(E\widehat{\otimes}F, k) \cong \mathsf{Top}_k(E \otimes^! F, k) = (E \otimes^! F)^\circ. \tag{2.10}$$

Because the pseudocompact spaces E and F have cofinite topologies, Theorem 2.6 yields a natural isomorphism

$$(E \otimes^! F)^\circ \cong E^\circ \otimes F^\circ,$$

which composes with (2.10) to provide natural isomorphisms

$$\Psi_{E,F} \colon E^\circ \otimes F^\circ \xrightarrow{\sim} (E\widehat{\otimes}F)^\circ.$$

The unitor coherence axioms are as easily verified as before. The associator coherence diagram can be derived from (2.8) as follows. For pseudocompact spaces E, F, and G, one verifies that the induced diagrams

$$\begin{tikzcd}
E^\circ \otimes (F\widehat{\otimes}G)^\circ \arrow[d, "\Psi_{E,F\widehat{\otimes}G}"'] & E^\circ \otimes (F \otimes^! G)^\circ \arrow[l, "\sim"'] \arrow[d, "\phi_{E,F\otimes^! G}"] \\
(E\widehat{\otimes}(F\widehat{\otimes}G))^\circ & (E \otimes^! (F \otimes^! G))^\circ \arrow[l, "\sim"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(E \otimes^! F)^\circ \otimes G^\circ \arrow[r, "\sim"] \arrow[d, "\phi_{E\otimes^! F,G}"'] & (E\widehat{\otimes}F)^\circ \otimes G^\circ \arrow[d, "\Psi_{E\widehat{\otimes}F,G}"] \\
((E \otimes^! F) \otimes^! G)^\circ \arrow[r, "\sim"] & ((E\widehat{\otimes}F)\widehat{\otimes}G)^\circ
\end{tikzcd}$$

commute, where the horizontal isomorphisms are induced as in (2.10). Then "pasting" these to (2.8) results in a larger commuting diagram, whose outer paths will be the desired coherence diagram for the isomorphisms of Ψ. Thus the continuous dual $(-)^\circ$ is a strong monoidal functor.

Since the quasi-inverse $(-)^*$ must be adjoint to $(-)^\circ$, we can deduce [SS03, 3.3] that it is also a strong monoidal functor. (It is also possible to verify this directly. While we will not check the coherence axioms here, we will at least demonstrate how to locate the relevant natural isomorphisms. Fix objects V and W of Vect_k. Let $\{V_i\}$ be an indexing of the finite-dimensional subspaces of V, so that $V \cong \varinjlim V_i$ in Vect_k. Then we obtain natural isomorphisms

$$\begin{aligned}
(V \otimes W)^* &= \mathrm{Hom}_k(V \otimes W, k) \\
&\cong \mathrm{Hom}_k(V, \mathrm{Hom}(W, k)) \\
&\cong \mathrm{Hom}_k(\varinjlim V_i, W^*) \\
&\cong \varprojlim \mathrm{Hom}_k(V_i, W^*) \\
&\cong \varprojlim V_i^* \otimes W^* \\
&\cong V^*\widehat{\otimes}W^*,
\end{aligned}$$

for which it would remain to verify coherence.)

Finally, we derive the further dualities (1) and (2). Note that Coalg is the category of comonoid objects in Vect_k, while we verify in Lemma 2.11 below that PCAlg is the category of monoid objects in PC_k. Since the strong monoidal duality interchanges comonoids and monoids, we obtain the duality (1) between coalgebras and pseudocompact algebras. Finally, fix a k-coalgebra C. Then C-Comod is the category of left C-comodule objects over the comonoid C in Vect, while C^*-PC is the category of monoid objects over the monoid C^* in PC_k. So the duality (2) follows similarly. $\square$

Lemma 2.11. *Pseudocompact algebras are precisely the monoid objects in the monoidal category* $(\mathsf{PC}_k, \widehat{\otimes}, k)$.

Proof. If A is a pseudocompact k-algebra, then its multiplication induces a continuous bilinear map $m\colon A \times A \to A$, which factors through the completed tensor product $\widehat{m}\colon A\widehat{\otimes}A \to A$ thanks to its universal property [Bru66]. It is then clear that this map and the usual unit map $u\colon k \to A$ form a monoid object $(A, \widehat{m}, u)$ in $(\mathsf{PC}_k, \widehat{\otimes}, k)$.

Conversely, suppose that (A, m, u) is a monoid object of $(\mathsf{PC}_k, \widehat{\otimes}, k)$. If we view the multiplication as a map

$$m^!\colon A \otimes^! A \hookrightarrow A\widehat{\otimes}A \xrightarrow{m} A,$$

then in fact $(A, m^!, u)$ is a monoid object in $(\mathsf{CF}_k, \otimes^!, k)$. By Theorem 2.6, the continuous dual functor sends this to a comonoid, whose dual is then a pseudocompact algebra $(A^\circ)^*$. The natural isomorphism of pseudocompact spaces $A \cong (A^\circ)^*$ is then readily verified to be an isomorphism of topological algebras. Thus A is in fact a pseudocompact algebra as desired. $\square$

We close with the main result of this preliminary section, showing that the finite dual of an algebra can be viewed as the continuous dual with respect to an appropriate topology. If A is a k-algebra, then the *cofinite* topology on A is the linear topology whose open ideals are exactly the ideals of finite codimension—those ideals in the family $\mathcal{F}(A)$ defined in (2.2).

Theorem 2.12. *Let A be a k-algebra, viewed as a linearly topologized algebra with its cofinite topology. Then the continuous dual functor $(-)^\circ = \mathsf{Top}_k(-, k)$ applied to the multiplication*

$$m\colon A \otimes A \to A$$

yields the finite dual coalgebra A° with its comultiplication

$$m^\circ\colon A^\circ \to (A \otimes^! A)^\circ \xrightarrow{\sim} A^\circ \otimes A^\circ.$$

Proof. First we verify that multiplication yields a continuous map $m\colon A \otimes^! A \to A$. If $I \in \mathcal{F}(A)$ is an ideal of finite codimension, then $I \otimes A + A \otimes I$ is an open subspace of $A \otimes^! A$ such that $m(I \otimes A + A \otimes I) = IA + AI \subseteq I$. So m is indeed continuous.

It follows that A with its unit $\eta\colon k \to A$ can be viewed as a monoid object (A, m, η) in the monoidal category $(\mathsf{CF}_k, \otimes^!, k)$, or equivalently, as a comonoid object in the opposite tensor category. Because oplax monoidal functors send comonids to comonoids, it follows from Theorem 2.6 that $(A^\circ, m^\circ, \eta^\circ)$ is a coalgebra. It is clear that A° coincides with the underlying vector space of the finite dual coalgebra. Let $\phi \in A^\circ$. The counit $\epsilon = \eta^\circ$ is easily seen to be given by $\epsilon(\phi) = \phi(1)$. Finally, the comultiplication $\Delta = m^\circ$ given by

$$\Delta(\phi) = m(\phi) \in (A \otimes^! A)^\circ \cong A^\circ \otimes A^\circ \subseteq A^* \otimes A^*$$

agrees with the comultiplication of the finite dual (see [Swe69, Section 6] or [Rad12, Section 2.5]). This completes the proof. $\square$

Note that any algebra homomorphism $f\colon A \to B$ is continuous with respect to the cofinite topologies on A and B. Thus we see that the functoriality of finite dual $(-)^\circ\colon \mathsf{Alg}^{\mathrm{op}} \to \mathsf{Coalg}$ can be explained in light of the duality of Theorem 2.6 quite simply: the continuous dual functor sends monoid objects in $(\mathsf{CF}_k, \otimes^!, k)$ to comonoid objects in $(\mathsf{Vect}_k, \otimes, k)$.

3. Coalgebras as noncommutative discrete spaces

In this section we will describe how coalgebras can be viewed as discrete geometric objects. We begin with a discussion of coalgebras as quantized sets in Subsection 3.1, where we attempt to provide some more intuition behind this viewpoint. Then in Subsection 3.2 we explain how cocommutative coalgebras arise in algebraic geometry as collections of distributions on k-schemes. Taken together, these suggest how coalgebras serve as discrete objects in noncommutative geometry.

3.1. **Coalgebras as quantized sets.** As stated in the introduction, we will view coalgebras as the underlying discrete objects of noncommutative spaces. To motivate this perspective, we will explain in detail how sets can be "linearized" in order to provide a full and faithful embedding into the category of k-coalgebras.

The category of sets equipped with the Cartesian product becomes a monoidal category $(\mathsf{Set}, \times, 1)$, where $1 = \{*\}$ is a terminal object. Every set X carries a comonoid structure (X, Δ, ϵ) with comultiplication being the diagonal $\Delta\colon X \to X \times X$ and counit being the terminal morphism $\epsilon\colon X \to 1$. Moreover, this structure is unique. (Similarly, in any cartesian monoidal category one can verify that each object carries a unique comonoid structure, defined analogously.)

For a set X, we let kX denote the k-vector space with basis X, which we call the *linearization* of X. Since any set function $f\colon X \to Y$ extends uniquely to a linear map $kf\colon kX \to kY$, we obtain a linearization functor $k-\colon \mathsf{Set} \to \mathsf{Vect}$. For sets X and Y, it is straightforward to verify that the map $k(X \times Y) \to kX \otimes kY$ determined by $(x, y) \mapsto x \otimes y$ for $x \in X$ and $y \in Y$ is a k-linear isomorphism that is natural in both X and Y, and that $k1 \cong k$. These isomorphisms make the linearization into a strong monoidal functor $(\mathsf{Set}, \times, 1) \to (\mathsf{Vect}, \otimes, k)$.

Now since every set is uniquely a comonoid in $(\mathsf{Set}, \times, 1)$, it follows immediately that the linearization kX carries the structure of a comonoid in $(\mathsf{Vect}, \otimes, k)$. As one can readily compute, its comulitplication and counit

$$\begin{aligned} \Delta &\colon kX \to kX \otimes kX, \\ \epsilon &\colon kX \to k, \end{aligned}$$

are defined on each basis element $x \in X$ by $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$. Thus (kX, Δ, ϵ) is a k-coalgebra. (This construction is well known, as in [DNR01, Example 1.1.4] and [Kas95, III.1, Example 3].) Since the monoidal structure of Vect restricts to a monoidal structure on the subcategory Coalg, we may in fact view the linearization as a strong monoidal functor

$$k-\colon (\mathsf{Set}, \times, 1) \to (\mathsf{Coalg}, \otimes, k).$$

This functor also preserves terminal objects, since the trivial coalgebra k is terminal in Coalg. (The last observation follows directly from the fact that the counit of any coalgebra (Q, Δ, ϵ) is a morphism in Coalg; this is verified in Sweedler notation by

$$\sum \epsilon(q_{(1)})\epsilon(q_{(2)}) = \epsilon\left(\sum q_{(1)}\epsilon(q_{(2)})\right) = \epsilon(q)$$

for any $q \in Q$.)

This linearization functor has a right adjoint, constructed as follows. For a k-coalgebra (Q, Δ, ϵ), we say that a nonzero element $x \in Q$ is *pointlike*[1] (or is a *k-point*) if $\Delta(x) = x \otimes x$; the counit axiom yields $x = \epsilon(x)x$, so that $x \neq 0$ ensures $\epsilon(x) = 1$. We let $\mathrm{pts}(Q)$ denote the set of k-points of any coalgebra Q. As any coalgebra morphism preserves pointlike elements, the assignment of k-point-sets forms a functor $\mathrm{pts}\colon \mathsf{Coalg} \to \mathsf{Set}$, which is easily seen to be naturally isomorphic to the functor $\mathsf{Coalg}(k, -)$. Now for a set X and a coalgebra Q, a routine verification yields the following adjoint isomorphism:

$$\mathsf{Coalg}(kX, Q) \cong \mathsf{Set}(X, \mathrm{pts}(Q)).$$

So $\mathrm{pts} \cong \mathsf{Coalg}(k, -)$ is a strong monoidal functor because it is right adjoint to a strong monoidal functor [SS03, 3.3].

As the terminology suggests, a coalgebra kX defined from a set X satisfies $\mathrm{pts}(kX) = X$. Then for sets X and Y, the adjunction above restricts to

$$\mathsf{Coalg}(kX, kY) \cong \mathsf{Set}(X, \mathrm{pts}(kY)) = \mathsf{Set}(X, Y).$$

Thus we obtain the following.

Proposition 3.1. *The functor $k-\colon \mathsf{Set} \to \mathsf{Coalg}$ yields a full and faithful strong monoidal embedding of the category of sets into the category of k-coalgebras, with strong monoidal right adjoint given by the k-points functor $\mathrm{pts}\colon \mathsf{Coalg} \to \mathsf{Set}$.*

In this way we can view sets as a full subcategory of Coalg after linearization, allowing us to treat coalgebras as generalizations of (linearized) sets. Indeed, as discussed in Section 1, we in fact see coalgebras as a *quantization* of sets, where k-linear combinations of points are imagined as a *superposition* (over k) of states. Proposition 3.1 and the preceding discussion illustrate that the added structure of a coalgebra allows us to recover the individual points of X from kX. There is a strong resonance between this perspective and the idea that a quantum system cannot be cloned [WZ82] and quantum information cannot be copied [CP08, Section 16.3]. We view the k-points of a coalgebra (Q, Δ, ϵ) as "classical points" in a quantum set precisely because they are *copyable* via the quantum diagonal map Δ.

Below we recall a few examples of well-known coalgebras that are not cocommutative. We will view all of these as "non-classical quantum sets."

Example 3.2. The *comatrix coalgebra* $\mathbb{M}^d = \mathbb{M}^d(k)$ has basis $\{E^{ij} \mid 1 \leq i, j \leq d\}$ with

$$\Delta(E^{ij}) = \sum_r E^{ir} \otimes E^{rj}$$
$$\epsilon(E^{ij}) = \delta_{ij}.$$

If we identify the E^{ij} with the dual basis of the matrix units $E_{ij} \in \mathbb{M}_d(k)$, then computing as in (2.1) we see that $\mathbb{M}^d \cong (\mathbb{M}_d(k))^*$.

[1]In the literature, such an element is typically called *grouplike*. We use this alternate terminology in order to emphasize a geometric perspective on coalgebras, especially ones that do not arise from any connection to a Hopf algebra or bialgebra. Fortunately, this slightly modified term fits well with the established notion of *pointed* coalgebras as well as *points* of an object in a category as global elements.

Example 3.3. Let Γ be a (possibly infinite) quiver. The *path coalgebra* $k\Gamma$ is the k-span of the paths in Γ, with comultiplication and counit defined on a path p in Γ of length $|p|$ by

$$\begin{aligned}\Delta(p) &= \sum_{p=p_1p_2} p_1 \otimes p_2,\\ \epsilon(p) &= \delta_{0,|p|}.\end{aligned}$$

(Note that if Γ is a finite, acyclic quiver, so that its path algebra $k[\Gamma]$ is finite-dimensional, then the dual coalgebra is isomorphic to the path coalgebra: $k[\Gamma]^* \cong k\Gamma$.) The vertices of Γ then become the k-points of this coalgebra. Thus we have a quantum set with "classical points" corresponding to the vertices, but whose comultiplication introduces quantum relationships (governed by the arrows) between the points. Probably the best picture for this quantum set is the quiver Γ itself! In the case where Γ has no arrows, the path coalgebra $k\Gamma$ coincides with the linearization of Γ considered as a set whose elements are its vertices.

Example 3.4. Let $\mathbb{T}_n = \bigoplus_{i\leq j} kE_{ij} \subseteq \mathbb{M}_n(k)$ denote the algebra of upper-triangular $n \times n$ matrices, and denote its dual coalgebra as $\mathbb{T}^n = \bigoplus_{i\leq j} E^{ij}$, so that we have a surjective coalgebra morphism $\mathbb{M}^n \twoheadrightarrow \mathbb{T}^n$. Thus its coalgebra structure is inherited from the structure in Example 3.2 along this morphism, so that

$$\begin{aligned}\Delta(E^{ij}) &= \sum_{i\leq r\leq j} E^{ir} \otimes E^{rj}\\ \epsilon(E^{ij}) &= \delta_{ij}.\end{aligned}$$

In particular, notice that each E^{ii} is pointlike, so that $\mathbb{T}^n$ has n distinct k-points. If we let Γ be a quiver of type A_n with linear orientation:

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \quad \cdots \quad \longleftarrow \overset{n}{\bullet}$$

then $\mathbb{T}_n$ is isomorphic to the path algebra on Γ, so that its dual is isomorphic to the path coalgebra $\mathbb{T}^n \cong k\Gamma$.

Next we discuss how the interpretation of coalgebras as quantized sets can be interpreted at the level of dual algebras. If X is a set, one can easily verify that the pseudocompact algebra dual to kX is isomorphic to the product algebra

$$(kX)^* \cong k^X \tag{3.5}$$

(recalling that k is discrete). To interpret this in a manner that is consistent both with geometry and quantum mechanics, we equivalently view k^X as the algebra of k-valued functions on X. The evaluation of an element $f \in k^X \cong (kX)^*$ on an element of the linearized set kX can be viewed as analogous to pairing a physical observable with a state to obtain a measurement outcome. Furthermore, the topology on k^X can be interpreted as the topology of pointwise convergence of functions. Informed by this (non-standard) perspective, for a general coalgebra Q we choose to refer to the dual pseudocompact algebra

$$\mathrm{Obs}(Q) = Q^*$$

as the *algebra of (k-valued) observables* on Q, and we can imagine its pseudocompact topology as a topology of "quantum pointwise convergence."

Viewing $\mathrm{Obs}(Q)$ as an algebra of functions on a quantum set Q also suggests that there should be a correspondence between quantum discrete spaces (= quantum sets) and their function algebras, which is seen as follows. For "classical" sets, it is shown in [IMR16, Theorem 4.7] that the functor $X \mapsto k^X$ yields a duality between Set and a certain full subcategory $\mathcal{C} \subseteq \mathsf{PCAlg}_k$. As described in Corollary 2.9, dual algebra yields a dual equivalence

$$\mathrm{Obs} = (-)^* \colon \mathsf{Coalg}_k^{\mathrm{op}} \to \mathsf{PCAlg}_k$$

whose quasi-inverse is the continuous k-dual $(-)^\circ$. In physical terms, we view this as a duality between state spaces and observables. The classical and quantum dualities thus fit into a diagram that commutes up to isomorphism

$$\begin{tikzcd} \mathsf{Set} \arrow[r, hook] \arrow[d, "\wr"] & \mathsf{Coalg}_k \arrow[d, "\wr"] \\ \mathcal{C}^{\mathrm{op}} \arrow[r, hook] & \mathsf{PCAlg}_k^{\mathrm{op}} \end{tikzcd}$$

where the horizontal arrows are fully faithful embeddings.

In order to build intuition for coalgebras as quantum collections, we end this subsection by using observable-state duality to motivate a physical interpretation of comatrix coalgebras. Recall that a qu-$d$it (or qu$d$it) is a d-level (or d-state) quantum system, with the special case $d = 2$ famously called a qubit [NC00, 2.2.1]. The physical observables on a qu-$d$it are described by the self-adjoint elements of the C*-algebra $\mathbb{M}_d(\mathbb{C})$. For a general field k, the comatrix coalgebra $\mathbb{M}^d(k)$ of Example 3.2 has algebra of observables

$$\mathrm{Obs}(\mathbb{M}^d(k)) \cong \mathbb{M}_d(k).$$

Thus by analogy with the quantum situation, we imagine $\mathbb{M}^d(k)$ as quantum set that represents a *qu-dit defined over the field k*. Note that while the matrix algebra indirectly represents the qu$d$it, our intention is to view the comatrix coalgebra as the underlying object that assumes the role of the qu$d$it itself.

Recall [Rad12, Section 3.4] that the *coradical* $\operatorname{corad} Q$ of a coalgebra Q is defined to be the (direct) sum of all simple subcoalgebras, so that it is the largest cosemisimple subcoalgebra of Q. The span of the k-points of Q forms a subcoalgebra of the coradical, so that we have coalgebra embeddings

$$k\,\mathrm{pts}(Q) \hookrightarrow \operatorname{corad} Q \subseteq Q.$$

(Note that this embedding is in fact the counit of the adjunction of Proposition 3.1 above.) If k is algebraically closed, then the only cosemisimple coalgebras over k are the comatrix coalgebras; in this case we have that $\operatorname{corad} Q$ consists of all qu$d$its in Q while $k\,\mathrm{pts}(Q)$ is the span of all classical points. Thus if we wish to study coalgebras over more general fields, we can view the coradical as the collection of all "generlized qu$d$its" inside of a quantum set. We will return to this interpretation in Subsection 4.1 when discussing the coradical of a finite dual coalgebra.

3.2. **Coalgebras of distributions on commutative schemes.** To close this section, we explain how a certain coalgebra can be viewed as an "underlying discrete object" of a scheme that is locally of finite type over k. This idea dates back to work of Takeuchi [Tak74, Tak77a], where the *underlying coalgebra* of a k-scheme is described in terms of representable functors. By contrast, we construct these coalgebras in a more concrete manner via the language of *distributions*, and we explicitly connect our presentation with that of Takeuchi in Proposition 3.16. Note that a similar perspective in differential geometry is given by Batchelor in [Bat88].

We motivate our approach with the following observation about underlying sets of topological spaces.

Remark 3.6. Let X be a topological space satisfying the T_1 separation axiom, such as a Hausdorff space. Since points are closed in X, every finite subset $S \subseteq X$ is closed, and the subspace topology on S is discrete. The family $\mathcal{F}(X)$ of finite subsets of X is directed by inclusion. Taking the directed limit of this family in the category of topological spaces yields a discrete space, which is the underlying set of X:

$$\varinjlim_{S \in \mathcal{F}(X)} S = |X|.$$

We will similarly consider "underlying discrete objects" of certain k-schemes X by first associating a suitable object to every closed subscheme of X that is finite over k, and then taking the direct limit of these objects. By contrast with the remark above, these "discrete objects" will be coalgebras that both "linearize" the set of closed points and include extra information about the formal neighborhood of every closed point.

Let Sch_k denote the category of schemes over k, and let fSch_k denote the full subcategory of schemes that are finite over $\operatorname{Spec}(k)$, which we abbreviate in the typical way to *finite over k*. Let cAlg denote the category of commutative k-algebras, and let fdcAlg denote the full subcategory of finite-dimensional commutative algebras. Similarly, let fdAlg, $\mathsf{fdCoalg}$, and $\mathsf{fdcCoalg}$ respectively denote the categories of algebras, coalgebras, or cocommutative coalgebras that are finite-dimensional.

Let S be a scheme finite over k, so that $S \cong \operatorname{Spec}(A)$ is affine, where A is a finite-dimensional commutative k-algebra. We define the *coalgebra of distributions* on S to be the dual coalgebra

$$\operatorname{Dist}(S) = \Gamma(S, \mathcal{O}_S)^*.$$

As this is the composite of the global sections functor $\mathsf{fSch}_k^{\mathrm{op}} \to \mathsf{fdAlg}$ with the contravariant dual functor $\mathsf{fdAlg}^{\mathrm{op}} \to \mathsf{fdCoalg}$, this assignment yields a (covariant) functor

$$\operatorname{Dist}\colon \mathsf{fSch}_k \to \mathsf{fdCoalg}\,.$$

Further, these functors respectively provide dualities $\mathsf{fSch}_k^{\mathrm{op}} \xrightarrow{\sim} \mathsf{fdcAlg}$ and $\mathsf{fdAlg}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{fdCoalg}$. Combining these dualities yields the following elementary fact, which is essentially contained in [Tak77a, Theorem 1.1] as well.

Lemma 3.7. *The functor* $\operatorname{Dist}\colon \mathsf{fSch}_k \to \mathsf{fdcCoalg}$ *gives an equivalence between the category of schemes finite over k and the category of finite-dimensional cocommutative k-coalgebras.*

In this sense, we imagine that finite-dimensional cocommutative coalgebras are "the same as" schemes finite over k. The covariance of the functor Dist suggests that we may view these coalgebras as the "underlying (discrete) object" of such k-schemes.

Example 3.8. The algebra $A = k[\varepsilon]$ with $\varepsilon^n = 0$ has basis $1, \varepsilon, \ldots, \varepsilon^{n-1}$. One may verify as in (2.1) that $\operatorname{Dist}(\operatorname{Spec}(A)) \cong A^*$ is the coalgebra on the dual basis

$\varepsilon_r = \widehat{\varepsilon}^r$ having comultiplication and counit

$$\begin{aligned}\Delta(\varepsilon_r) &= \sum_{i+j=r} \varepsilon_i \otimes \varepsilon_j, \\ \epsilon(\varepsilon_r) &= \delta_{r0}.\end{aligned}$$

We view this coalgebra as the "underlying discrete object" for the closed subscheme $\operatorname{Spec}(k[\varepsilon]) \subseteq \mathbb{A}^1_k$, an infinitesmial neighborhood of order $n-1$ of the origin in the affine line.

In extending this construction to arbitrary k-schemes, we will repeatedly use of the following elementary fact. We recall [Gro60, Chapter I, (2.2.6)] that a morphism of schemes is *surjective* if it is surjective on the underlying topological spaces.

Lemma 3.9. *Let $g\colon S \to Y$ be a morphism in Sch_k, and let Z denote the scheme-theoretic image of g in Y. If S is finite over k, then the closed subscheme Z of Y is also finite over k.*

Proof. Because S has a finite underlying space, g is certainly a quasi-compact morphism. Let $\mathcal{I}$ denote the kernel of the morphism of sheaves $g^\#\colon \mathcal{O}_Y \to g_*\mathcal{O}_S$. It follows [Sta21, Tag 01R8] that $\mathcal{I}$ is quasi-coherent, that Z is the closed subscheme of Y corresponding to $\mathcal{I}$, and that the image of the underlying space of S is dense in Z. The density of S in Z implies that Z is quasi-compact. As in [Har77, Proposition II.5.9], the underlying space of Z is the support of the quotient sheaf $\mathcal{O}_Y/\mathcal{I}$, and this quotient sheaf is the structure sheaf of Z. From the induced sheaf monomorphism $\mathcal{O}_Z = \mathcal{O}_Y/\mathcal{I} \to g_*\mathcal{O}_S$, we see that the algebra of sections of $\mathcal{O}_Z$ on any open subset of Z must be finite-dimensional, as S is finite over k. In particular, every affine open subscheme of Z is finite over k, whence Z is zero-dimensional. Since Z is quasi-compact, we now see that it has a finite open cover by affine schemes finite over k. Thus Z is noetherian, and since Z is zero-dimensional we find that the underlying space $|Z|$ is finite and discrete.

Now for each $z \in |Z|$, the stalk $B_z = \mathcal{O}_{Z,z}$ is a finite-dimensional algebra, and from the discreteness of Z we have

$$Z \cong \coprod_{z \in |Z|} \operatorname{Spec}(B_z) \cong \operatorname{Spec}(B)$$

for the finite-dimensional algebra $B = \bigoplus_{z \in |Z|} B_z$. Thus Z is finite over k. $\square$

Now let X denote an arbitrary k-scheme, and let $\mathcal{F}(X)$ denote the diagram of closed subschemes of X that are finite over k with the naturally induced closed immersionsn between them. We remark that $\mathcal{F}(X)$ is directed. Indeed, given $S \cong \operatorname{Spec}(B_1)$ and $T \cong \operatorname{Spec}(B_2)$ in $\mathcal{F}(X)$, one has the union of closed subschemes $S \cup T \hookrightarrow X$ given by the ideal sheaf $\mathcal{I}_{S\cup T} = \mathcal{I}_S \cap \mathcal{I}_T$. The immersions of S, T, and $S \cup T$ into X factor through the coproduct $S \coprod T$ as

$$\begin{tikzcd} S \arrow[dr] & & & \\ & S \coprod T \arrow[r, two heads] & S \cup T \arrow[r, two heads] & X \\ T \arrow[ur] & & & \end{tikzcd}$$

where $S \cup T$ is the scheme-theoretic image of the coproduct. Because $S \coprod T \cong \operatorname{Spec}(B_1 \oplus B_2)$ is finite over k, it follows from Lemma 3.9 that $S \cup T$ is also finite over k.

Applying the global sections functor to the diagram $\mathcal{F}(X)$ yields an inversely directed system of commuative algebras $\Gamma_S(S,\mathcal{O}_S)$ in fdAlg, whose duals in turn form a directed system of cocommutative coalgebras in $\mathsf{fdCoalg}$. Motivated by Remark 3.6, we arrive at the following definition of distributions for a general scheme.

Definition 3.10. For a k-scheme X, the *coalgebra of distributions (of finite support)* on X is the directed limit of dual coalgebras

$$\mathrm{Dist}(X) = \varinjlim_{S\in\mathcal{F}(X)} \mathrm{Dist}(S) = \varinjlim_{S\in\mathcal{F}(X)} \Gamma(S,\mathcal{O}_S)^*.$$

The assignment $X \mapsto \mathrm{Dist}(X)$ can be extended to a functor as follows. Let $f\colon X \to Y$ be a morphism of k-schemes. Given $S \in \mathcal{F}(X)$, we obtain a composite morphism of k-schemes

$$S \hookrightarrow X \xrightarrow{f} Y.$$

By Lemma 3.9, the scheme-theoretic image S' of S in Y is finite over k. Thus (co)restriction of f induces a morphism of k-schemes $S \to S'$, which induces a morphism of coalgebras $\mathrm{Dist}(S) \to \mathrm{Dist}(S')$ as in Lemma 3.7. As $\mathrm{Dist}(X)$ is the colimit of the finite-dimensional subcoalgebras of the form $\mathrm{Dist}(S)$, and $\mathrm{Dist}(S') \hookrightarrow \mathrm{Dist}(Y)$ is a subcoalgebra, we may define

$$\mathrm{Dist}(f)\colon \mathrm{Dist}(X) \to \mathrm{Dist}(Y)$$

to be the directed limit of the morphisms induced from each $S \in \mathcal{F}(X)$. In this way, distributions of finite support form a functor

$$\mathrm{Dist}\colon \mathsf{Sch}_k \to \mathsf{Coalg}\,. \tag{3.11}$$

In the case of an affine scheme over k, this functor amounts to the finite dual of the coordinate ring.

Proposition 3.12. *The restriction of the functor* $\mathrm{Dist}\colon \mathsf{Sch}_k \to \mathsf{Coalg}$ *to the full subcategory of affine k-schemes is naturally isomorphic to the functor $X \mapsto \Gamma(X,\mathcal{O}_X)^\circ$.*

Proof. Writing $X \cong \mathrm{Spec}(A)$, it is clear that each $S \in \mathcal{F}(X)$ is of the form $S \cong \mathrm{Spec}(A/I)$ for some ideal $I \in \mathcal{F}(A)$ of finite codimension in A. Then

$$\mathrm{Dist}(X) = \varinjlim_{S\in\mathcal{F}(X)} \mathrm{Dist}(S) \cong \varinjlim_{I\in\mathcal{F}(A)} (A/I)^* = A^\circ \cong \Gamma(X,\mathcal{O}_X)^\circ.$$

Naturality in X is easily verified. □

A coalgebra of distributions supported at a point can also be defined in the following way. For a k-scheme X and a point x of X, the *coalgebra of distributions supported at x* is the finite dual of the stalk at x of the structure sheaf:

$$\mathrm{Dist}(X,x) = (\mathcal{O}_{X,x})^\circ. \tag{3.13}$$

Note that if x is not a closed point, then $\mathcal{O}_{X,x}$ has no ideals of finite codimension and $\mathrm{Dist}(X,x) = 0$; for this reason, distributions are typically only examined at closed points.

Let X_0 denote the set of closed points of X. For $x \in X_0$, Takeuchi [Tak74, 2.1] called (3.13) the *tangent coalgebra to X at x* and referred to the direct sum $\bigoplus_{x\in X_0} \mathrm{Dist}(X,x)$ as the *underlying coalgebra of X*. (See also [Jan03, Chapter 7] and [DG80, II, §4, no. 5–6] for details on coalgebras of distributions at a point.)

Our next goal is to show that this underlying coalgebra is in fact isomorhpic to the coalgebra $\mathrm{Dist}(X)$ defined above.

Example 3.14. We compute the coalgebra of distributions supported at the origin on the affine line over k. It is the directed union $\mathrm{Dist}(\mathbb{A}^1_k, 0) = \varinjlim (k[t]/(t^n))^*$, with the latter coalgebras being defined as in Example 3.8. Thus we have

$$\mathrm{Dist}(\mathbb{A}^1, 0) = \bigoplus_{i=0}^{\infty} k\varepsilon_i$$

with the same comultiplication and counit formulas from Example 3.8; this is known as the *divided power coalgebra*. (Note that this is isomorphic to the coalgebra associated to the monoid $\mathbb{N}$; see [Rad12, p. 26].) Its dual algebra of observables is isomorphic to the formal power series ring $k[[t]] \cong \varprojlim k[t]/(t^n)$ with the t-adic topology.

Let x be a closed point of a k-scheme X. The canonical map $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ induces a map on distributions

$$\mathrm{Dist}(X, x) = (\mathcal{O}_{X,x})^\circ \cong \mathrm{Dist}(\mathrm{Spec}\,\mathcal{O}_{X,x}) \hookrightarrow \mathrm{Dist}(X),$$

and the assignment $(X, x) \mapsto \mathrm{Dist}(X, x)$ is evidently functorial (see also [Tak74, Introduction]). In this way we obtain a naturally induced injection

$$\bigoplus_{x \in X_0} \mathrm{Dist}(X, x) \hookrightarrow \mathrm{Dist}(X). \tag{3.15}$$

Note that the only nonzero summands above are from those points in the set

$$\mathrm{fin}(X) = \{x \in X \mid [\kappa(x) : k] < \infty\} \subseteq X_0.$$

If X is locally of finite type over k, then we have $\mathrm{fin}(X) = X_0$ by the general Nullstellensatz.

We now verify our claim that the coalgebra of distributions on a k-scheme coincides with Takeuchi's underlying coalgebra.

Proposition 3.16. *For a k-scheme X, the map* (3.15) *induces a natural isomorphism of coalgebras*

$$\mathrm{Dist}(X) \cong \bigoplus_{x \in \mathrm{fin}(X)} \mathrm{Dist}(X, x) = \bigoplus_{x \in \mathrm{fin}(X)} (\mathcal{O}_{X,x})^\circ.$$

Proof. Let S be a closed subscheme of X that is finite over k. Let $\{x_1, \ldots, x_n\}$ denote the underlying set of S. Consider each singleton $\{x_i\}$ as both a closed and an open subscheme of S (since S carries the discrete topology). This subscheme is isomorphic to $\mathrm{Spec}(B_i)$ for the finite-dimensional local algebra $B_i = \mathcal{O}_S(\{x_i\}) = \mathcal{O}_{S,x_i}$. Passing to an open affine neighborhood of x_i, we see that $B_i \cong \mathcal{O}_{X,x_i}/J_i$ for an ideal J_i of finite codimension in the local ring $\mathcal{O}_{X,x_i}$. Notice that there exists a proper ideal of finite codimension in $\mathcal{O}_{X,x_i}$ if and only if $x_i \in \mathrm{fin}(X)$.

Setting $B = B_1 \times \cdots \times B_n$, we have the following commutative diagram

$$\begin{tikzcd}
\coprod \{x_i\} \arrow[r, "\sim"] \arrow[d, "\wr"] & S \arrow[d, "\wr"] \\
\coprod \mathrm{Spec}(B_i) \arrow[r, "\sim"] & \mathrm{Spec}(B)
\end{tikzcd}$$

of isomorphisms. On the level of distributions, we find that

$$\mathrm{Dist}(S) \cong \bigoplus_{i=1}^{n} (B_i)^* \cong \bigoplus (\mathcal{O}_{X,x_i}/J_i)^*.$$

If we now pass to the directed limit of all closed subschemes S of X that are finite over k, it is straightforward to see that

$$\mathrm{Dist}(X) = \varinjlim_{S \in \mathcal{F}(X)} \mathrm{Dist}(S) \cong \bigoplus_{x \in \mathrm{fin}(X)} (\mathcal{O}_{X,x})^\circ$$

as desired. □

The closed points of a k-scheme are embodied in the structure of the coradical of $\mathrm{Dist}(X)$ in the following way.

Corollary 3.17. *Let X be a k-scheme.*

(1) $\mathrm{corad}(\mathrm{Dist}(X)) = \bigoplus_{x \in X_0} \kappa(x)^\circ$*, where $\kappa(x)$ denotes the residue field at x.*
(2) [Tak74, 2.1.8] *There is a bijection* $\mathrm{pts}(\mathrm{Dist}(X)) \cong X(k)$ *between the k-points of the distributions on X and the k-rational points of X.*
(3) *If k is algebraically closed and X is locally of finite type over k, then* $\mathrm{corad}(\mathrm{Dist}(X)) \cong kX_0$ *(the coradical of* $\mathrm{Dist}(X)$ *is the linearization of the closed points of X).*

Proof. Claim (1) follows directly from the structure given in Proposition 3.16, and (2) follows from (1) because a closed point x is k-rational iff $\kappa(x) = k$. Now (3) follows from (2) since every closed point is k-rational under the hypothesis. □

Assume that the scheme X is locally of finite type over k, so that $\mathrm{fin}(X) = X_0$. From Proposition 3.16 and the corresponding isomorphism of pseudocompact algebras

$$\mathrm{Obs}(\mathrm{Dist}(X)) \cong \prod_{x \in X_0} \widehat{\mathcal{O}}_{X,x}, \tag{3.18}$$

we see that $\mathrm{Dist}(X)$ contains information about the closed points of X along with their formal neighborhoods. While the coradical $\mathrm{corad}(\mathrm{Dist}(X))$ is a good choice of coalgebra corresponding to the closed points of X (by Corollary 3.17) and has good functorial properties for schemes, we will see in Theorem 4.6 that it is necessary to retain the additional data of these formal neighborhoods in order for functoriality to persist in noncommutative geometry.

We conclude this section by describing the distributions on the affine line over an algebraically closed field. This example will play an important role when examining the quantum plane in Subection 6.2.

Example 3.19. Suppose k is algebraically closed, so that the closed points of $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ are in bijection with k and are all k-rational. From Proposition 3.16 we obtain

$$\mathrm{Dist}(\mathbb{A}^1_k) \cong \bigoplus_{\lambda \in k} \mathrm{Dist}(\mathbb{A}^1_k, \lambda). \tag{3.20}$$

By translation invariance we have each $\mathrm{Dist}(\mathbb{A}^1_k, \lambda) \cong \mathrm{Dist}(\mathbb{A}^1, 0)$, which was already described in Example 3.14. However, it will be important later for us to describe this coalgebra with a more careful consideration of the pointwise structure. At each point $\lambda \in k$, we have

$$\operatorname{Dist}(\mathbb{A}^1_k, \lambda) = \varinjlim_n (k[t]/(t-\lambda)^n)^* = \bigoplus_{i=0}^{\infty} k\varepsilon_\lambda^{(i)}, \tag{3.21}$$

and its coalgebra structure is given by

$$\Delta(\varepsilon_\lambda^{(r)}) = \sum_{i=0}^{r} \varepsilon_\lambda^{(i)} \otimes \varepsilon_\lambda^{(r-i)},$$
$$\epsilon(\varepsilon_\lambda^{(r)}) = \delta_{r0}.$$

Viewing these $\varepsilon_\lambda^{(i)}$ as distributions using (3.20) and Proposition 3.12, one can check that $\varepsilon_\lambda^{(i)} \in k[t]^\circ$ is described on the basis $\{(t-\lambda)^j\}_{j=0}^\infty$ by $\varepsilon_\lambda^{(i)}((t-\lambda)^j) = \delta_{ij}$. In order to find a describe $\operatorname{Dist}(\mathbb{A}^1_k)$ in terms of distributions that are independent of a choice of basis of $k[t]$, it is instructive to rewrite this structure in terms of Dirac distributions [Hör83, Example 2.1.2] in the characteristic zero case. The Dirac distribution $\delta_\lambda \in \operatorname{Dist}(\mathbb{A}^1_k)$ supported at $\lambda \in k$ is the functional that evaluates at $t = \lambda$:

$$\delta_\lambda(f(t)) = f(\lambda).$$

In particular, $\delta_\lambda = \varepsilon_\lambda^{(0)}$. The distributional derivatives [Hör83, Definition 3.1.1] of the Dirac distributions $\delta = \delta_\lambda$ are defined by

$$\delta'(f) = -\delta(f'), \text{ and more generally}$$
$$\delta^{(i)}(f) = (-1)^i \delta(f^{(i)}).$$

This means that when k has characteristic 0,

$$\varepsilon_\lambda^{(i)} = \frac{(-1)^i}{i!}\,\delta_\lambda^{(i)}.$$

So the comultiplication $\Delta(\varepsilon_\lambda^{(r)}) = \sum \epsilon_\lambda^{(i)} \otimes \epsilon_\lambda^{(r-i)}$ translates to

$$\Delta(\delta_\lambda^{(r)}) = \frac{r!}{(-1)^r} \sum_{i=0}^{r} \frac{(-1)^i}{i!} \delta_\lambda^{(i)} \otimes \frac{(-1)^{r-i}}{(r-i)!} \delta_\lambda^{(j)}$$
$$= \sum_{i=0}^{r} \binom{r}{i} \delta_\lambda^{(i)} \otimes \delta_\lambda^{(r-i)}$$

4. The finite dual as a quantized maximal spectrum

Let A be a k-algebra. If A happens to be a commutative affine k-algebra, then the corresponding scheme $X = \operatorname{Spec} A$ is of finite type over k, and Propositions 3.12 and 3.16 combine to give the following isomorphisms that are natural in A:

$$\operatorname{Dist}(\operatorname{Spec}(A)) \cong A^\circ \cong \bigoplus_{\mathfrak{m} \in \operatorname{Max}(A)} (A_\mathfrak{m})^\circ. \tag{4.1}$$

(The pseudocompact dual of this isomorphism was noted in [KS09, Example 2.9] for algebraically closed fields, and more generally in [LB08, Section 2].) This isomorphism leads directly to the main tenet of this paper:

> *For affine k-algebras that are not necessarily commutative but have enough finite-dimensional k-representations, the finite dual coalgebra functor $(-)^\circ$ is a meaningful substitute for the maximal spectrum functor.*

Another way to motivate this is by viewing measuring coalgebras $P(A,B)$ as a quantization [Bat91] of the set of maps between two k-algebras A and B. This provides an enrichment of the category Alg of k-algebras over the category Coalg; see [Vas12, HLFV17]. If k is algebraically closed and A is a commutative affine algebra, then the Nullstellensatz yields a bijection

$$\mathsf{Alg}(A,k) \cong \mathrm{Max}(A).$$

If we upgrade the above Hom-set using this internal Hom structure, it is known [Vas12, Remark 3.1] that

$$P(A,k) \cong A^\circ.$$

This provides a second perspective on the finite dual as a quantized maximal spectrum. Note that the adjoint relationship (2.5) allows us to recover the former Hom-set from the enriched one using k-points, as

$$\mathrm{pts}(A^\circ) = \mathsf{Coalg}(k, A^\circ) \cong \mathsf{Alg}(A,k).$$

The algebra of observables on the dual coalgebra can be described as follows. Recalling the definition of $A^\circ = \varinjlim_{I\in\mathcal{F}(A)} (A/I)^*$ with notation as in Section 2, we have

$$\mathrm{Obs}(A^\circ) = (A^\circ)^* \cong \varprojlim_{I\in\mathcal{F}(A)} A/I =: \widehat{A}.$$

Recall that $\mathcal{F}(A)$ is the basis of open sets for the cofinite topology on A. Thus $\widehat{A}$ is the completion of A in this topology, which we can view as a *pseudocompactification* of A. In the case where A is commutative and affine, then as in (3.18) we have $\widehat{A} \cong \prod_{\mathfrak{m}\in\mathrm{Max}(A)} \widehat{A_\mathfrak{m}}$. This underscores the fact that A° contains information not only about closed points, but also about their infinitesimal neighborhoods. We will see in Theorem 4.6 that this infinitesimal information is unavoidable if we desire a useful noncommutative spectrum functor.

The remainder of this section is divided as follows. First we will discuss the failure of functoriality of the coradical of A° to clarify our insistence on retaining the full finite dual when quantizing the maximal spectrum functor; we also examine a few cases where the coradical happens to be functorial. We then discuss those noncommutative algebras A for which we believe A° is an appropriate quantization of the maximal spectrum, the *fully residually finite-dimensional* algebras. We end by examining the relationship between Morita equivalence of algebras and Takeuchi equivalence of their dual coalgebras, treating it as an object lesson on the role of Morita equivalence in functorial spectral theory.

4.1. **Functoriality and the coradical.** In light of Corollary 3.17, it seems natural to infer from the isomorphism 4.1 that the coradical of A° is a more appropriate quantization of the maximal spectrum of a commutative affine algebra. It is well known (see also Corollary 4.8(1) below) that the coradical of the finite dual forms a functor on commutative k-algebras:

$$\begin{aligned} \mathsf{cAlg}^{\mathrm{op}} &\to \mathsf{Coalg} \\ A &\mapsto \operatorname{corad} A^\circ. \end{aligned} \tag{4.2}$$

Indeed, we agree that for suitable algebras A (as discussed in Subsection 4.2 below) the coradical of A° is an appropriate substitute for the set of points of the noncommutative affine variety associated to A. What we illustrate in this subsection is rather that, in order to preserve functoriality of the maximal spectrum in these cases, one is forced to include the full finite dual. So while points of affine varieties may behave well in commutative algebraic geometry, we are forced to include infinitesimal neighborhoods in order to retain functoriality in noncommutative geometry.

First let us describe the structure of $\operatorname{corad} A^\circ$. It is clear from the isomorphism (2.3) that the simple subcoalgebras of A° correspond to the quotient algebras A/I that are simple and finite-dimensional. Let

$$\{\mathfrak{m}_\alpha\} = \operatorname{Max}(A) \cap \mathcal{F}(A)$$

be an indexing of the set of maximal ideals of finite codimension in A, and let $S_\alpha = A/\mathfrak{m}_\alpha$ denote the simple quotient algebras. Then we have

$$\operatorname{corad} A^\circ = \bigoplus_\alpha S_\alpha^*. \tag{4.3}$$

If $S_\alpha \cong \mathbb{M}_d(k)$, then as discussed in Subsection 3.1 we may view $S_\alpha^* \cong \mathbb{M}^d$ as a qu$d$it over k. So S_α^* might generally be imagined as representing a "generalized qu$d$it" over k (although this is probably most appropriate in the case where S_α is central simple over k).

Thus in light of (4.3), we may interpret $\operatorname{corad} A^\circ$ as a "disjoint union of generalized qu$d$its" in the noncommutative spectrum of A. (Note that if k is algebraically closed, then all S_α are isomorphic to matrix algebras, so that the coradical truly does represent a disjoint union of qu$d$its over k.) Furthermore, there is a bijection between the $\mathfrak{m}_\alpha$ and the finite-dimensional simple left A-modules ${}_AV_\alpha$, where $\mathfrak{m}_\alpha$ is the annihilator of V_α. So $\operatorname{corad} A^\circ$ essentially encodes the irreducible finite-dimensional representations of A.

We now turn to the issue of non-functoriality of the coradical. To begin, very simple examples can be constructed to show that the coradical does not form a subfunctor of the finite dual, even when restricting to the full subcategory of affine noetherian PI algebras that are module-finite over their centers. We find the following particularly helpful as an instance that can be visualized.

Example 4.4. Consider the algebra homomorphism $\phi\colon k[t] \to \mathbb{M}_2(k)$ given by $\phi(t) = E_{12}$. Its image is isomorphic to the ring of dual numbers $k[\varepsilon]$ (with $\varepsilon^2 = 0$), so that ϕ factors as $k[t] \twoheadrightarrow k[\varepsilon] \hookrightarrow \mathbb{M}_2(k)$. This dualizes to a factorization of coalgebra morphisms $\mathbb{M}^2 = \mathbb{M}_2(k)^* \twoheadrightarrow k[\varepsilon]^* \hookrightarrow k[t]^\circ$. In particular, we see that the image of $\operatorname{corad}(\mathbb{M}^2) = \mathbb{M}^2$ under ϕ° is not cosemisimple and thus does not lie in $\operatorname{corad} k[t]^\circ$.

To paint a more suggestive picture, let us first note that the image of ϕ in lies in the subalgebra $\mathbb{T}_2(k) \subseteq \mathbb{M}_2(k)$ of upper-triangular matrices, so that ϕ factors as

$$k[t] \twoheadrightarrow k[\varepsilon] \hookrightarrow \mathbb{T}_2(k) \hookrightarrow \mathbb{M}_2(k).$$

The finite duals of the algebras were respectively described in Examples 3.19, 3.8, 3.4, and 3.2. So the above factorization of ϕ dualizes to a factorization of coalgebra

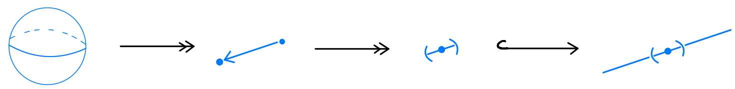

FIGURE 1. Illustration of the maps between quantum sets in (4.5)

morphisms

$$\mathbb{M}^2 \twoheadrightarrow \mathbb{T}^2 \twoheadrightarrow k[\varepsilon]^\circ \hookrightarrow \operatorname{Dist}(\mathbb{A}^1_k). \tag{4.5}$$

Recall from Example 3.4 that if we let Γ be a quiver of type A_2, then we may view $\mathbb{T}^2 = k\Gamma$ as a path coalgebra. Figure 1 is a visualization the sequence of morphisms above. We use the traditional Bloch sphere depiction [NC00, p. 15] of the qubit to represent $\mathbb{M}^2$, the quiver Γ to depict $\mathbb{T}^2 = k\Gamma$, and an infinitesimal neighborhood of the origin to represent $k[\varepsilon]^*$. In total, we have a map from the qubit to the affine line (both defined over k) whose image is not a reduced subscheme of $\mathbb{A}^1_k$.

In the spirit of functorial noncommutative spectral theory as described in Subsection 1.1, we prefer stronger evidence than a single example in order to conclude that the full finite dual functor should not be replaced by a functor

$$F\colon \mathsf{Alg}^{\mathrm{op}} \to \mathsf{Coalg}$$

that is a better approximation to the coradical. Theorem 4.6 below gives a more precise restriction.

To motivate the statement, we first note that there exist functors F as above whose restriction to $\mathsf{cAlg}^{\mathrm{op}}$ is the functor (4.2). Given a k-algebra A, let $F(A) \subseteq \operatorname{corad} A^\circ$ be the sum of all simple subcoalgebras dual to a a finite-dimensional division algebra, which is the largest basic [CM97, p. 43] subcoalgebra. The argument given in the proof of Corollary 4.8(1) below shows that this choice of F is a subfunctor of the finite dual, and it is routine to check that $F(A) = \operatorname{corad} A^\circ$ for commutative algebras A. However, this functor is easily seen to satisfy $F(\mathbb{M}_d(k)) = 0$ for $d \geq 2$, so that matrix algebras yield familiar obstructions as in [Rey12, BH14, Rey14, BZMR17].

So from the point of view of Subsection 1.1, we wish to impose a nondegeneracy condition which guarantees that, at the very least, matrix algebras are assigned nontrivial coalgebras. If we resolve to view matrix algebras as observables on qu*d*its as discussed at the end of Subsection 3.1, then it is more natural to seek a subfunctor F of $(-)^\circ$ that satisfies $F(\mathbb{M}_d(k)) = \mathbb{M}_d(k)^* = \mathbb{M}^d$ for all matrix algebras. This leads us uniquely to the functor $F = (-)^\circ$ in the following way.

Theorem 4.6. *Let* $F\colon \mathsf{Alg}^{\mathrm{op}} \to \mathsf{Coalg}$ *be a subfunctor of the finite dual* (2.4). *If* $F(\mathbb{M}_d(k)) = \mathbb{M}_d(k)^*$ *for all integers* $n \geq 0$, *then* F *is equal to the finite dual.*

Proof. First let A be a finite-dimensional k-algebra, and set $d = \dim_k(A)$. Consider the embedding of A into its k-endomorphisms:

$$i\colon A \hookrightarrow \operatorname{End}_k(A) =: S.$$

Because $S \cong \mathbb{M}_d(k)$, our hypothesis yields $F(S) = S^*$. As F is a subfunctor of $(-)^\circ$, we obtain a commuting diagram

$$\begin{tikzcd} F(S) \arrow[r, "F(i)"] \arrow[d, equal] & F(A) \arrow[d, hook, "\alpha"] \\ S^* \arrow[r, two heads, "i^*"] & A^* \end{tikzcd}$$

The morphism α is an inclusion of a subcoalgebra by the subfunctor condition. Now a simple diagram chase incorporating surjectivity of i^* shows that α is also surjective, and therefore $F(A) = A^*$ (and α is the identity).

Since F is a subfunctor of the finite dual, we now deduce that $F = (-)^*$ on the full subcategory of finite-dimensional k-algebras. It is then straightforward to apply the directed colimit characterization (2.3) of the finite dual to deduce that in fact $F = (-)^\circ$ on the category Alg of all k-algebras. □

This is one way in which we might think of the finite dual as a "minimal" functorial extension of the coradical from commutative to noncommutative algebras. Let us return to the the description (4.3) of the coradical of A° once more. As discussed above, the coradical contains information about all finite-dimensional irreducible representations of A. While $\operatorname{corad} A^\circ$ fails to behave functorially in general, the theorem above suggests that we can view the finite dual as a minimal functorial substitute for the irreps of A. Or stated otherwise, A° provides us with a "functorial snapshot" of the representation theory of A.

Despite the general failure of functoriality for the coradical, there are conditions on an algebra homomorphism under which this coradical *does* behave functorially. These can be characterized in the following way.

Proposition 4.7. *For an algebra homomorphism $f\colon A \to B$, the following are equivalent:*

(a) $f^\circ(\operatorname{corad} B^\circ) \subseteq \operatorname{corad} A^\circ$ *(i.e., f° preserves the coradical);*
(b) *For every finite-dimensional semisimple left (resp., right) B-module M, the restriction of scalars ${}_AM$ (via f) is a semisimple A-module;*
(c) *For every maximal ideal $\mathfrak{m}$ of finite codimension in B, the ideal $f^{-1}(\mathfrak{m}) \subseteq A$ is semiprime.*

Proof. The simple subcoalgebras of B° are precisely those of the form $(B/\mathfrak{m})^*$ for a maximal ideal $\mathfrak{m}$ of finite codimension in B. The restriction of f° to one of these simple subcoalgebras coincides with the surjective map

$$\operatorname{Hom}_k(B/\mathfrak{m}, k) \to \operatorname{Hom}_k(A/f^{-1}(\mathfrak{m}), k)$$

arising from the algebra homomorphism $A/f^{-1}(\mathfrak{m}) \to B/\mathfrak{m}$ induced by f. Thus the image of this simple subcoalgebra of B° lies in the coradical of A° if and only if the finite-dimensional algebra $A/f^{-1}(\mathfrak{m})$ is semisimple, which occurs if and only if $f^{-1}(\mathfrak{m})$ is semiprime. This establishes (a) $\Longleftrightarrow$ (c).

To see that (b) $\Longrightarrow$ (c), let $\mathfrak{m}$ be a maximal ideal of finite codimension in B. Then $B/\mathfrak{m}$ is a semisimple left B-module, and under hypthesis (b) it is also a semisimple left A-module. But the natural ring embedding $A/f^{-1}(\mathfrak{m}) \hookrightarrow B/\mathfrak{m}$ is also an embedding of left A-modules. So $A/f^{-1}(\mathfrak{m})$ is semisimple and consequently $f^{-1}(\mathfrak{m})$ is semiprime.

Finally, assume (c). For (c) $\implies$ (b), it suffices to consider the case where ${}_BM$ is simple and prove that ${}_AM$ is semisimple. Setting $\mathfrak{m} = \operatorname{ann}_B(M)$, it follows that $B/\mathfrak{m}$ is a finite-dimensional simple algebra. Because $\operatorname{ann}_A(M) = f^{-1}(\mathfrak{m})$, the action of A on M factors through $A/f^{-1}(\mathfrak{m})$. This finite-dimensional algebra is semiprime by hypothesis, so it must be semisimple. Thus M is semisimple as a left A-module. $\square$

We remark that if $f\colon A \to B$ satisfies the equivalent conditions above, then it follows from [Rad12, Corollary 4.2.2] that if f° preserves the whole coradical filtration. Below are two situations in which the algebra B is "tame enough" for this to automatically hold.

Corollary 4.8. *An algebra homomorphism $f\colon A \to B$ satisfies $f^\circ(\operatorname{corad} B^\circ) \subseteq \operatorname{corad} A^\circ$ in each of the following cases:*

(1) *Every finite-dimensional simple quotient algebra of B is a division algebra. (This holds, for instance, if B is commutative.)*
(2) *B is a finite normalizing extension* [MR01, 10.1.3] *of the image $f(A)$.*

Proof. (1) This case is a very slight generalization of [Rad12, Exercise 4.1.2]. If $\mathfrak{m}$ is a maximal ideal of finite codimension B, then the image of $f(A)$ in $B/\mathfrak{m}$ is a finite-dimensional domain and therefore is a division algebra. So $f^{-1}(\mathfrak{m})$ is maximal in A, and the result follows from Proposition 4.7(c).

(2) This case also follows from Proposition 4.7(c), this time with the help of the "Cutting Down" theorem [MR01, Theorem 10.2.4]. $\square$

Note that condition (1) is satisfied when B is commutative, or more generally, when $B/J(B)$ is commutative where $J(B)$ is the Jacobson radical. Condition (2) is satisfied in case B is module-finite over a central subalgebra A and f is the inclusion of A into B. An important case of the latter occurs when B is a Cayley-Hamilton algebra and A is the image of its trace; see [DCP93, Theorem 4.5].

4.2. **Fully RFD algebras.** We now discuss the question of which algebras have "enough" finite-dimensional representations that A° can be considered as a reasonable substitute for a noncommutative maximal spectrum. If the algebra structure of A is to be faithfully represented by A°, we should at least ask that the naturally induced algebra homomorphism

$$A \to A^{\circ *} = \operatorname{Obs}(A^\circ)$$

is injective. Injectivity of this map is equivalent [Swe69, Lemma 6.1.0] to the property that the algebra is a subdirect product of finite-dimensional algebras. A k-algebra satisfying these conditions is said to be *residually finite-dimensional* (RFD). It is straightforward to verify that the RFD property is further equivalent to the requirement that intersection of all ideals in $\mathcal{F}(A)$ is zero, or that every nonzero element of A acts nontrivially on a finite-dimensional left (equivalently, right) module.

However, it seems prudent to ask for more than the property above. For instance, if I is an ideal of A and if we view A/I as the coordinate ring of a closed subscheme of the spectrum of A, then we should reasonably expect A° to carry sufficient information about A/I as well. This will not always be the case, as RFD algebras can have homomorphic images that are not RFD. For instance, the algebra $A = \prod_{n=1}^\infty \mathbb{M}_n(k)$ is RFD by construction, but for the ideal $T = \bigoplus_{n=1}^\infty \mathbb{M}_n(k)$ of A, the same argument as in the proof of [Kap51, Lemma 7.5] shows that A/T has no finite-dimensional homomorphic image. Similarly, the free algebra $k\langle x, y\rangle$ is known to be RFD (as the ideals I_n generated by words in $\{x, y\}$ of length n have finite codimension and satisfy $\bigcap I_n = 0$), but its homomorphic image $k\langle x, y\rangle/(xy - 1)$ is not Dedekind-finite and therefore cannot be a subdirect product of finite-dimensional algebras.

Definition 4.9. (Following [CS20] in the case of C*-algebras.) An algebra A is *strongly residually finite-dimensional* (strongly RFD) if A/I is residually finite-dimensional for every ideal I of A.

Every affine noetherian PI k-algebra A is strongly RFD. Indeed, a result of Anan'in [Ana92] shows that every affine right noetherian PI algebra is RFD. Because the property of being an affine noetherian PI algebra passes to homomorphic images, it follows that A/I is RFD for all ideals I of A.

From the discussion above, we see that A is strongly RFD if and only if every ideal I of A is equal to the intersection of all ideals of finite codimension containing I, if and only if every homomorphic image of A embeds in the algebra of observables on its finite dual. Thus the algebras for which A° can be considered a reasonable quantization of the maximal spectrum should form a subclass of the strongly RFD algebras. However, one could reasonably ask for a slightly stronger condition. Because A° contains information about finite-dimensional representations of A as in (4.3), we can ask that every finitely generated A-module be determined by all of its finite-dimensional homomorphic images. This reasoning leads to the following condition.

Definition 4.10. A k-algebra A is *left fully residually finite-dimensional* (RFD) if every finitely generated left A-module is a subdirect product of finite-dimensional left A-modules. *Right fully residually finite-dimensional* algebras are defined similarly, and we say that A is *fully residually finite-dimensional* if it is both left and right fully RFD.

In general, we have the following relationship between these properties of k-algebras:

$$\text{left fully RFD} \implies \text{strongly RFD} \implies \text{RFD}.$$

We have seen that the second implication is strict. We presume that the first implication is also strict in general, but we do not have a counterexample. Proposition 4.12 below will illustrate that any such counterexample cannot be a fully bounded noetherian algebra.

Below are some equivalent characterizations of the left fully RFD property. Condition (c), in particular, highlights the role of the finite dual for such algebras. A left A-module is *locally finite* if each of its finitely generated submodules is finite-dimensional. We let $A\text{-}\mathsf{Mod}_{lf}$ denote the full subcategory of $A\text{-}\mathsf{Mod}$ whose objects are the locally finite modules.

Theorem 4.11. *For an algebra A, the following are equivalent:*

(a) *A is left fully RFD;*
(b) *the injective hull of every simple left A-module is locally finite-dimensional;*
(c) *A° is a cogenerator in $A\text{-}\mathsf{Mod}$.*

If A is left noetherian, then these properties are further equivalent to:

(d) *every simple left A-module is finite-dimensional, and the subcategory $A\text{-}\mathsf{Mod}_{lf}$ is closed under injective hulls in $A\text{-}\mathsf{Mod}$;*
(e) *A° is an injective cogenerator in $A\text{-}\mathsf{Mod}$;*

Proof. Let $\{V_i\}$ be a complete set of simple left A-modules up to isomorphism. Because every left A-module is a subdirect product of the injective hulls of the simple left A-modules ([Lam99, Theorem 19.8]), every finitely generated left A-module is a subdirect product of finitely generated submodules of the $E(V_i)$. Furthermore, each $E(V_i)$ and its submodules are subdirectly irreducible (since they contain an essential simple submodule). The equivalence (a) $\iff$ (b) follows.

(b) $\implies$ (c): Let V be a simple left A-module, and let $\mathfrak{m} = \operatorname{ann}(V)$. Because the injective hull $E(V)$ is locally finite-dimensional, V must be finite-dimensional. Thus the semisimple algebra $\operatorname{End}_A(V) \cong A/\mathfrak{m}$ is also finite-dimensional and consequently is symmetric [Lam99, Example 16.59]. It follows that we have an embedding of finite-dimensional left A-modules

$$V \hookrightarrow A/\mathfrak{m} \cong (A/\mathfrak{m})^* \hookrightarrow A^\circ.$$

As explained in the proof of [HL20, Theorem 2.1], the left module A° is an injective object in the category locally finite left A-modules. Since (b) states that $E(V)$ is locally finite-dimensional, the embedding above must extend to an injective homomorphism $E(V) \hookrightarrow A^\circ$. It follws [Lam99, Theorem 19.8] that A° is a cogenerator in $A\text{-}\mathsf{Mod}$.

(c) $\implies$ (b): Let V be a simple left A-module. If A° is a cogenerator in $A\text{-}\mathsf{Mod}$, then there is an injective homomorphism $E(V) \hookrightarrow A^\circ$ (see [Lam99, Theorem 19.8]). Because A° is locally finite-dimensional, the same must be true for $E(V)$.

(d) $\implies$ (e): As explained in the proof of [HL20, Theorem 2.1], if $A\text{-}\mathsf{Mod}_{lf}$ is closed under injective hulls then A° is an injective left A-module (even without the left noetherian hypothesis). It is also clear that (d) $\implies$ (b), so from (b) $\implies$ (c) established above we have that A° is also a cogenerator.

Obviously (e) $\implies$ (c). Now assume A is left noetherian; we will verify (b) $\implies$ (d). Suppose M is a locally finite left A-module. Let $\{M_i\}$ denote the set of all finite-dimensional submodules of M. Then $M = \varinjlim M_i$ is the directed union of the M_i. Because A is left noetherian, injective left A-modules are closed under direct limits [Lam99, Theorem 3.46] so that

$$E(M) = \varinjlim E(M_i).$$

Since each M_i has essential socle, its injective hull $E(M_i)$ is a finite direct sum of injective hulls of simple modules, which are locally finite-dimensional. Thus $E(M)$ is a directed union of locally finite-dimensional modules and must itself be locally finite-dimensional. □

This raises the obvious question of how to locate examples of fully RFD algebras. Fortunately, it turns out that there is a rich supply of these, in the sense that *every affine noetherian PI algebra is fully RFD.* This is deduced in the next result, which also shows that for fully bounded noetherian (FBN) rings [GW04, Chapter 9], the left and right fully RFD properties are equivalent to one another.

Proposition 4.12. *Let A be a fully bounded noetherian k-algebra. The following are equivalent:*

(a) *A is fully RFD;*

(b) *A is strongly RFD;*
(c) *every simple left (equivalently, right) A-module is finite-dimensional;*
(d) *every maximal ideal of A has finite codimension.*

In particular, every affine noetherian PI algebra is fully RFD.

Proof. The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) hold for any algebra A. We use the following results on FBN rings:

(i) The annihilator of every simple left or right module is maximal [GW04, Corollary 9.5].
(ii) The injective hull of a simple left or right module is locally of finite length [Jat74, Theorem 3.5, Corollary 3.6].

The equivalence (c) $\Longleftrightarrow$ (d) follows directly from (i). Thanks to (ii) and Theorem 4.11(c), A is left fully RFD if and only if every simple left A-module has finite dimension. Since the left and right module versions of (c) are equivalent, it follows that (a) $\Longleftrightarrow$ (c).

The final statement is a consequence of the following two facts: every PI algebra is fully bounded [MR01, Corollary 13.6.6], and condition (c) holds over any affine PI algebra [MR01, Theorem 13.10.3]. □

Remark 4.13. By a result of Amitsur and Small [AS96], every affine FBN algebra over an uncountable algebraically closed field k is PI, from which it follows that such a k-algebra is fully RFD. It seems natural to then ask if every affine FBN algebra over an arbitrary field k is fully RFD over k. However, because division algebras are FBN, this question is in fact a generalization of Kurosh's problem for division algebras, which asks whether every affine division algebra is finite-dimensional. Indeed, if A is an affine FBN algebra with maximal ideal M, then $A/M \cong \mathbb{M}_n(D)$ for a division algebra D that must be affine by the Artin-Tate Lemma. Assuming Kurosh's problem has a positive solution, D is finite-dimensional so that M has finite codimension and A is fully RFD by Proposition 4.12.

There are many more interesting questions that one can ask about the fully residually finite-dimensional property. Are the left and right fully RFD properties independent of one another? If A° is an *injective* cogenerator in A-Mod, must A be left noetherian? Are there any examples of affine but non-noetherian fully RFD algebras? How similar is the structure theory of fully RFD algebras to that of FBN rings? For now, we must content ourselves with the information learned above and turn to other matters regarding the finite dual.

4.3. **Morita equivalence and the quantized spectrum.**

A common philosophy within noncommutative geometry is that *Morita equivalent rings should represent the same noncommutative space.* The claim is natural and understandable when one works in a framework where a (commutative or noncommutative) space is represented solely by its category of sheaves of modules. However, this is problematic from the perspective laid out in Section 1. For instance, if we wish to view a noncommutative algebra as consisting of observables of a quantum system, then its commutative subalgebras are physically important as they are closely related to the information that can be accessed through all possible measurements [Heu17, 1.3]. In particular, the algebra $\mathbb{M}_n(\mathbb{C})$ of observables on an n-level system has many commutative subalgebras isomorphic to $\mathbb{C}^n$, representing different measurements with n possible outcomes that can be made on the system. However, all of these matrix algebras $\mathbb{M}_n(\mathbb{C})$ are Morita equivalent to one another and, in particular, to $\mathbb{C}$. Thus Morita equivalence is blind to this important aspect of algebraic quantum mechanics, while functorial spectral theory as in Subsection 1.1 aims to preserve exactly this kind of information.

This is not to say that Morita equivalence should be ignored in noncommutative spectral theory. Rather, from this perspective we expect that *Morita equivalent algebras will have Morita equivalent noncommutative spaces, but Morita equivalent noncommutative spaces need not be isomorphic.* This subsection illustrates this principle in action for dual coalgebras. Two coalgebras are said to be *Takeuchi equivalent* [Tak77b] if their categories of left comodules are k-linearly equivalent. We will show below that if A and B are algebras that are Morita equivalent (in an appropriately modified sense), then A° and B° are Takeuchi equivalent. So this quantized maximal spectrum reflects Morita equivalence, in accordance with the guideline above.

We will say that two k-algebras A and B are *k-linearly Morita equivalent* if there is a k-linear equivalence between the categories A-Mod and B-Mod. A number of standard facts [Lam99, Section 18] about Morita equivalence of rings are readily verified to carry over to k-linear Morita equivalence as long as suitable care is given to compatibility with the k-vector space structure. In particular, one can readily verify that within the class of k-algebras, the RFD, strongly RFD, and (left) fully RFD properties are each preserved by k-linear Morita equivalence.

Proposition 4.14. *Let A and B be k-algebras, and consider the following statements:*

(a) *A and B are k-linearly Morita equivalent algebras;*
(b) *A° and B° are Takeuchi equivalent k-coalgebras;*
(c) *There is a k-linear equivalence between the categories of pseudocompact left modules over $\operatorname{Obs}(A^\circ) \cong \widehat{A}$ and $\operatorname{Obs}(B^\circ) \cong \widehat{B}$.*

Then (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c)*.*

Proof. The equivalence (b) $\Longleftrightarrow$ (c) is a direct consequence of the k-linear duality between the categories of left A°-comodules and pseudocompact left modules over $(A^\circ)^* \cong \widehat{A}$ (as in Corollary 2.9(2)). Thus it suffices to show that (a) $\Longrightarrow$ (c).

Recall [Lam99, Proposition 18.33] that two rings R and S are Morita equivalent if and only if $S \cong e\mathbb{M}_n(R)e$ for some integer $n \geq 1$ and a full idempotent $e \in \mathbb{M}_n(R)$. Assuming (a) holds, we may thus write

$$B \cong e\mathbb{M}_n(A)e \tag{4.15}$$

for $n \geq 1$ and a full idempotent $e \in \mathbb{M}_n(A)$, where the above is a k-algebra isomorphism by k-linearity of the Morita equivalence. For each open ideal $I \in \mathcal{F}(B)$ in the cofinite topology on B, there is a corresponding $J \in \mathcal{F}(A)$ such that, under (4.15), both $I \cong e\mathbb{M}_n(J)e$ and $B/I \cong e\mathbb{M}_n(A/J)e$ (where we view $\mathbb{M}_n(A/J)$ as a bimodule over $\mathbb{M}_n(A)$ via its natural surjection onto $\mathbb{M}_n(A)/\mathbb{M}_n(J) \cong \mathbb{M}_n(A/J)$). Thus if we pass to the completion, we obtain an isomorphism of topological algebras

$$\begin{aligned} \widehat{B} &= \varprojlim_{I \in \mathcal{F}(B)} B/I \\ &\cong \varprojlim_{J \in \mathcal{F}(A)} e\mathbb{M}_n(A/J)e \\ &\cong e\mathbb{M}_n(\widehat{A})e, \end{aligned}$$

where we now view $\mathbb{M}_n(\widehat{A})$ as a bimodule over $\mathbb{M}_n(A)$ via the algebra homomorphism given by completion $\mathbb{M}_n(A) \to \widehat{\mathbb{M}_n(A)} \cong \mathbb{M}_n(\widehat{A})$. Letting $e_{ij} \in \mathbb{M}_n(\widehat{A})$ denote the matrix units, we define pseudocompact bimodules

$$
\begin{aligned}
{}_{\widehat{A}}U_{\widehat{B}} &= e_{11}\mathbb{M}_n(\widehat{A})e,\\
{}_{\widehat{B}}V_{\widehat{A}} &= e\mathbb{M}_n(\widehat{A})e_{11}.
\end{aligned}
$$

It is then straightforward to verify using the completed tensor product of pseudocompact bimodules (see [Bru66, Section 2] or [VdB01, Section 4]) we have $U\widehat{\otimes}_{\widehat{B}}V \cong \widehat{A}$ and $V\widehat{\otimes}_{\widehat{A}}U \cong \widehat{B}$, so that the k-linear functors

$$
\begin{aligned}
U \otimes_{\widehat{B}} - &: \widehat{B}\text{-}\mathsf{PC} \to \widehat{A}\text{-}\mathsf{PC},\\
V \otimes_{\widehat{A}} - &: \widehat{A}\text{-}\mathsf{PC} \to \widehat{B}\text{-}\mathsf{PC}
\end{aligned}
$$

yield a k-linear equivalence of categories. Thus (c) holds as desired. $\square$

5. Dual coalgebras of twisted tensor products

We hope that the preceding sections have convinced the reader that the finite dual of a fully RFD algebra is an invariant that is well worth studying. Our goal in this section is to provide a method by which the finite dual can in principle be computed for a large class of algebras: those arising from a twisted tensor product construction. This makes the problem of computing dual coalgebras approachable for algebras that do not necessarily arise in connection with a Hopf algebra. Even so, the method has some novel applications in the context of smash products of Hopf algebra, as shown in Subsection 5.3.

5.1. **Crossed product algebras and coalgebras.** A twisted tensor product of two k-algebras A and B, as introduced in [ČSV95], is (to speak informally) an algebra structure on the vector space $A \otimes B$ that is allowed to deform the usual tensor product algebra structure while retaining the respective algebra structures on A and B. Simliarly, crossed products were introduced for bialgebras and coalgebras in [BD99, CIMZ00]. We recall the definitions and basic notions associated with these constructions below. Our notation and terminology primarily follows the treatments in [CIMZ00, BCT13].

In the symmetric monoidal category $(\mathsf{Vect}_k, \otimes, k)$ of k-vector spaces, we will denote the "tensor swap" braiding for objects $V, W \in \mathsf{Vect}_k$ by

$$
\begin{aligned}
\sigma_{V,W} \colon V \otimes W &\to W \otimes V,\\
v \otimes w &\mapsto w \otimes v.
\end{aligned}
$$

Given a k-algebra A, we let (A, m_A, η_A) denote its structure as a monoid object in $(\mathsf{Vect}_k, \otimes, k)$. Thus its multiplication is considered as a linear map $m_A \colon A \otimes A \to A$, and its unit map $\eta_A \colon k \to A$ is given by $\eta_A(1_k) = 1_A$.

Let A and B be algebras. Suppose that

$$\rho \colon B \otimes A \to A \otimes B$$

is a linear map. Define a multiplication $m_\rho \colon (A \otimes B) \otimes (A \otimes B) \to A \otimes B$ by

$$m_\rho = (m_A \otimes m_B) \circ (\mathrm{id}_A \otimes \rho \otimes \mathrm{id}_B). \tag{5.1}$$

If m_ρ is associative with identity $1_A \otimes 1_B$, then the resulting algebra

$$(A \#_\rho B, m_{A\#_\rho B}, \eta_{A\#_\rho B}) = (A \otimes B, m_\rho, \eta_A \otimes \eta_B) \tag{5.2}$$

is a *twisted tensor product* (or *crossed product*) of A and B, and ρ is called a *twisting map* for A and B. In this case, the “inclusion” maps

$$\begin{aligned} i_A &= \mathrm{id}_A \otimes 1_B \colon A \to A\#_R B \text{ and} \\ i_B &= 1_A \otimes \mathrm{id}_B \colon B \to A\#_R B \end{aligned} \tag{5.3}$$

are algebra homomorphisms. The twisted tensor product can be characterized in terms of a universal property [ČSV95, Proposition 2.7] via ρ and the inclusion maps above, but we will not recall that here since we do not make use of it.

Notice that the prototypical case where ρ is chosen to be the tensor swap $\sigma := \sigma_{B,A}$ results in the usual tensor product algebra $A\#_\sigma B = A \otimes B$. In this way, we may view twisted tensor products as deformations of the tensor algebra. On the other hand, if one takes an algebra A with an automorphism σ and a σ-derivation δ, then setting $B = k[t]$ and choosing $\rho\colon k[t] \otimes A \to A \otimes k[t]$ appropriately, we can recover the Ore extension $A[t;\sigma,\delta] \cong A\#_\rho k[t]$ as in [CIMZ00, Examples 2.11].

A fundamental problem addressed in [ČSV95, CIMZ00] is to isolate properties of a linear map $\rho\colon B \otimes A \to A \otimes B$ that make it a twisting map. The map ρ is defined to be *normal* if it satisfies the conditions

$$\begin{aligned} \rho \circ (\eta_B \otimes \mathrm{id}_A) &= \mathrm{id}_A \otimes \eta_B, \\ \rho \circ (\mathrm{id}_B \otimes \eta_A) &= \eta_A \otimes \mathrm{id}_B . \end{aligned} \tag{5.4}$$

(This is equivalent to saying that the restriction of ρ to the subspaces $A \otimes 1_B$ and $1_A \otimes B$ agrees with the tensor swap $\sigma = \sigma_{B,A}$.) Furthermore, ρ is defined to be *multiplicative* if it satisfies the conditions

$$\begin{aligned} \rho \circ (\mathrm{id}_B \otimes m_A) &= (m_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho) \circ (\rho \otimes \mathrm{id}_A), \\ \rho \circ (m_B \otimes \mathrm{id}_A) &= (\mathrm{id}_A \otimes m_B) \circ (\rho \otimes \mathrm{id}_B) \circ (\mathrm{id}_B \otimes \rho). \end{aligned} \tag{5.5}$$

These turn out to be necessary and sufficient conditions for ρ to define a twisted tensor product.

Proposition 5.6. *Let A and B be algebras. A linear map*

$$\rho\colon B \otimes A \to A \otimes B$$

is a twisting map if and only if it is normal and multiplicative.

Proof. See [ČSV95, Proposition/Definition 2.3, Remark 2.4] and [CIMZ00, Theorem 2.5]. □

Dually, let C and D be coalgebras with a linear map

$$\phi\colon C \otimes D \to D \otimes C.$$

Define a comultiplication $\Delta_\phi\colon (C \otimes D) \otimes (C \otimes D) \to C \otimes D$ by

$$\Delta_\phi = (\mathrm{id}_C \otimes \phi \otimes \mathrm{id}_D) \circ (\Delta_C \otimes \Delta_D).$$

If Δ_ϕ is coassociative with counit $\varepsilon_C \otimes \varepsilon_D$, then the resulting coalgebra

$$(C\#^\phi D, \Delta_{C\#^\phi D}, \varepsilon_{C\#^\phi D}) = (C \otimes D, \Delta_\phi, \varepsilon_C \otimes \varepsilon_D) \tag{5.7}$$

is called a *crossed product* of A and B, and ϕ is a *cotwisting map.*

As one would expect, there is a characterization of cotwisting maps dual to that of Proposition 5.6. A linear map $\phi\colon C \otimes D \to D \otimes C$ is defined to be *conormal* if it satisfies the equations

$$\begin{aligned}(\varepsilon_D \otimes \mathrm{id}_C) \circ \phi &= \mathrm{id}_C \otimes \varepsilon_D, \\ (\mathrm{id}_D \otimes \varepsilon_C) \circ \phi &= \varepsilon_C \otimes \mathrm{id}_D,\end{aligned} \tag{5.8}$$

and ϕ is defined to be *comultiplicative* if it satisfies the conditions

$$\begin{aligned}(\mathrm{id}_D \otimes \Delta_C) \circ \phi &= (\phi \otimes \mathrm{id}_C) \circ (\mathrm{id}_C \otimes \phi) \circ (\Delta_C \otimes \mathrm{id}_D), \\ (\Delta_D \otimes \mathrm{id}_C) \circ \phi &= (\mathrm{id}_D \otimes \phi) \circ (\phi \otimes \mathrm{id}_D) \circ (\mathrm{id}_C \otimes \Delta_D).\end{aligned} \tag{5.9}$$

These provide the following characterization of cotwisting maps.

Proposition 5.10 ([CIMZ00, Theorem 3.4]). *Let C and D be coalgebras. A linear map*

$$\phi\colon D \otimes C \to C \otimes D$$

is a cotwisting map if and only if it is conormal and comultipliative.

Finally, suppose that A and B each have the structure of both an algebra and a coalgebra. Let $\rho\colon B \otimes A \to A \otimes B$ be a twisting map of the underlying algebras and let $\phi\colon A \otimes B \to B \otimes A$ be a cotwisting map of the underlying coalgebras. If the algebra $A\#_\rho B$ and coalgebra $A\#^\phi B$ structures together make $A \otimes B$ into a bialgebra, then this is called the *crossed product bialgebra* $A\#^\phi_\rho B$ in [CIMZ00, Definition 4.1].

5.2. **Duality for crossed products.** The formal duality between crossed product algebras and coalgebras suggests the naive idea that the the dual of a crossed product algebra should be a crossed product algebra. Unfortunately, there are examples for which this fails; see Example 5.13 below for a specific instance. In order to establish sufficient conditions where this expectation is correct, we need appropriate assumptions in place on the twisting map. It turns out that the appropriate tool is the topological tensor product $- \otimes^! -$ discussed in Subsection 2.2 and its good behavior under the continuous dual functor $(-)^\circ\colon \mathsf{CF}_k \to \mathsf{Top}_k$. Recall that the cofinite topology on an algebra, defined before Theorem 2.12, is the linear topology whose open ideals are the ideals of finite codimension.

First we note that the topological tensor product has the following weak compatibility with twisted tensor products.

Lemma 5.11. *For algebras A and B and a twisting map $\rho\colon B \otimes A \to A \otimes B$, the linear isomorphism (given by the identity on underlying vector spaces)*

$$A \otimes^! B \to A\#_\rho B$$

is continuous, where A, B, and $A\#_\rho B$ are endowed with their cofinite topologies.

Proof. Denote $S = A\#_\rho B$. Fix $K \in \mathcal{F}(S)$ and denote the canonical projection by $\pi\colon S \twoheadrightarrow S/K =: F$, where the codomain is finite-dimensional. Composing with the inclusion $i_A\colon A \to S$ of (5.3) gives a homomorphism $A \to S \to F$ that factors through a finite-dimensional algebra $A \twoheadrightarrow A_1 \subseteq F$ with kernel $I \in \mathcal{F}(A)$. Similarly, the composition of i_B with π must factor through a finite-dimensional algebra $B \twoheadrightarrow B_1 \hookrightarrow F$ with kernel $J \in \mathcal{F}(B)$. Because there is a vector space isomorphism $S \cong i_A(A) \otimes i_B(B)$, it follows that multiplication in F yields a linear surjection $A_1 \otimes B_1 \twoheadrightarrow F$. It follows that

$$I \otimes B + A \otimes J \subseteq K,$$

which implies that K is open in $A \otimes^! B$ and proves that the linear isomorphism is continuous. $\square$

Given a twisting map $\rho\colon B \otimes A \to A \otimes B$, we will wish to apply the continuous dual functor to obtain a cotwisting map. But this will only be possible in the case where ρ is continuous. This can be equivalently characterized in terms of a stronger compatibility between the topological tensor and twisted tensor products as follows.

Proposition 5.12. *Let A and B be algebras with $\rho\colon B \otimes A \to A \otimes B$ a twisting map. Consider the following conditions:*

(i) *$\rho\colon B \otimes^! A \to A \otimes^! B$ is continuous.*
(ii) *The map $A \otimes^! B \to A \#_\rho B$ of Lemma 5.11 is a homeomorphism.*
(iii) *There exist neighborhood bases of zero $\{I_\alpha\} \subseteq \mathcal{F}(A)$ and $\{J_\beta\} \subseteq \mathcal{F}(B)$ for which $\rho(B \otimes I_\alpha) \subseteq I_\alpha \otimes B$ and $\rho(J_\beta \otimes A) \subseteq A \otimes J_\beta$.*

Then (i) $\iff$ (ii) $\impliedby$ (iii).

Proof. Note that in the arguments below, while we carefully distinguish between topologies in $A \otimes^! B$ and $A \#_\rho B$, some computations are made by identifying both with the underlying vector space $A \otimes B$. We trust that this will not cause confusion for the vigilant reader.

(ii) $\implies$ (i): Fix a basic open neighorhood of zero in $\mathcal{F}(A \otimes^! B)$ of the form $N = I \otimes B + A \otimes J$ for $I \in \mathcal{F}(A)$ and $J \in \mathcal{F}(B)$. To prove continuity of ρ, we must produce $K \in \mathcal{F}(B \otimes^! A)$ such that $\rho(K) \subseteq N$. By condition (ii), there exits an open ideal $U \in \mathcal{F}(A \#_\rho B)$ such that $U \subseteq N$ as subspaces of $A \otimes B$, and by Lemma 5.11 there exists a basic open subspace of $A \otimes^! B$ contained in U. Altogether, this means that there exist $I_0 \in \mathcal{F}(A)$ and $J_0 \in \mathcal{F}(B)$ such that

$$I_0 \otimes B + A \otimes J_0 \subseteq U \subseteq I \otimes B + A \otimes J.$$

Note that $K := B \otimes I_0 + J_0 \otimes A$ is an element of $\mathcal{F}(B \otimes^! A)$. We claim that $\rho(K) \subseteq N$ as required above. We compute by interpreting ρ in terms of multiplication in $A \#_\rho B$ and liberally applying the linear identification of $A \otimes^! B$ with $A \#_\rho B$:

$$\begin{aligned}\rho(K) &= \rho(B \otimes I_0) + \rho(J_0 \otimes A) \\ &= (1\#B)(I_0\#1) + (1\#J_0)(A\#1) \\ &\subseteq (1\#B)U + U(A\#1) \subseteq U,\end{aligned}$$

because U is an ideal of $A \#_\rho B$. It follows that $\rho(K) \subseteq U \subseteq N$ as desired.

(i) $\implies$ (ii): To establish (ii), it suffices to show that the natural map $A \otimes^! B \to A \#_R B$ of Lemma 5.11 is open; because this map is linear, we need only test at neighborhoods of zero. So fix $I \in \mathcal{F}(A)$ and $J \in \mathcal{F}(B)$ which form a basic open neighborhood of zero

$$N = I \otimes B + A \otimes J \in \mathcal{F}(A \otimes^! B).$$

Let $\sigma = \sigma_{A,B}\colon A \otimes^! B \to B \otimes^! A$ denote the tensor swap, which is clearly continuous. Since ρ is assumed to be continuous, it follows that $\theta := \rho \circ \sigma\colon A \otimes^! B \to A \otimes^! B$

is also continuous. Then $\theta^{-1}(N)$ is open in $A \otimes^! B$, so there exist $I_0 \in \mathcal{F}(A)$ and $J_0 \in \mathcal{F}(B)$ such that

$$N_0 := I_0 \otimes B + A \otimes J_0 \subseteq \theta^{-1}(N).$$

Let U denote the ideal of $A \#_\rho B$ generated by the subspace N_0. Because U contains the subspace N_0 of finite codimension, we also have $U \in \mathcal{F}(A \#_\rho B)$. We will show below that $U \subseteq N$. This will imply that N is open in $A \#_\rho B$ so that (ii) will be established.

Since $N_0 = I_0 \otimes B + A \otimes J_0$, to prove that the ideal U generated by this subspace lies in the subspace N it suffices to show that the ideals separately generated by $I_0 \otimes B$ and $A \otimes J_0$ both lie in N. It is straightforward to see that $I_0 \otimes B$ is a right ideal within $A \#_\rho B$. To verify that it is also a left ideal, we make note of the following facts:

- $I_0 \otimes B \subseteq N_0 \subseteq \theta^{-1}(N)$ implies that $\rho(B \otimes I_0) = \theta(I_0 \otimes B) \subseteq N$,
- N is an (A, B)-subbimodule of $A \#_\rho B$.

Thus we may compute using the product in $A \#_\rho B$ as follows:

$$(A \#_\rho B) \cdot (I_0 \otimes B) = A \cdot \rho(B \otimes I_0) \cdot B \subseteq ANB = N.$$

So $I_0 \otimes B$ generates a subideal of N, and a symmetric argument shows that the same is true for $A \otimes J_0$. It follows that $U = (A \#_\rho B) N_0 (A \#_\rho B) \subseteq N$ so that (ii) is established.

(iii) $\implies$ (i): Fix a basic open neighborhood of zero $U = I \otimes B + A \otimes J$ in $A \otimes^! B$ with $I \in \mathcal{F}(A)$ and $J \in \mathcal{F}(B)$. Then there exist elements I_α and J_β of the neighborhood bases in (iii) such that $I_\alpha \subseteq I$ and $J_\beta \subseteq J$. Thus we may produce the basic open neighborhood $V = B \otimes I_\alpha + J_\beta \otimes A \in \mathcal{F}(B \otimes^! A)$ that satisfies

$$\begin{aligned}\rho(V) &= \rho(B \otimes I_\alpha) + \rho(J_\beta \otimes A)\\ &\subseteq I_\alpha \otimes B + A \otimes J_\beta\\ &\subseteq U.\end{aligned}$$

Because ρ is linear, this suffices to prove that ρ is continuous. □

The following example illustrates that twisting maps can easily fail to be continuous.

Example 5.13. For the algebras $A = k[x]$ and $B = k[y]$, it is possible to choose twisting maps $\rho \colon B \otimes A \to A \otimes B$ such that $A \#_\rho B$ has no proper ideals of finite codimension. For instance, if k has characteristic zero and we set

$$\rho(y^i \otimes x^j) = \partial_x^i x^j \otimes 1 + x^j \otimes y^i,$$

then $A \#_\rho B = A_1(k)$ is the first Weyl algebra, which infamously has no nonzero finite-dimensional representations. In such cases, the finite topology on $A \#_\rho B$ is the indiscrete topology, which radically differs from the topology on $A \otimes^! B$. It follows from Proposition 5.12 that ρ is not continuous. We also see that $(A \#_\rho B)^\circ = 0$ is not isomorphic to any crossed product of the form $A^\circ \#^\phi B^\circ$, since such a coalgebra is nonzero by construction.

We now arrive at the major result of this section. It shows that continuity of the twisting map with respect to the cofinite topologies is sufficient to allow the finite dual of a twisted tensor product to be a crossed product of the expected form.

Theorem 5.14. *Let A and B be k-algebras with a twisting map ρ. If $\rho\colon B \otimes^! A \to A \otimes^! B$ is continuous where A and B are endowed with their cofinite topologies, then the continuous dual*

$$\rho^\circ = \mathsf{Top}_k(\rho, k)\colon A^\circ \otimes B^\circ \to B^\circ \otimes A^\circ$$

is a cotwisting map and the continuous dual of the topological isomorphism $A \otimes^! B \to A \#_R B$ yields an isomorphism of coalgebras

$$(A \#_\rho B)^\circ \xrightarrow{\sim} A^\circ \#^{\rho^\circ} B^\circ.$$

Proof. Because ρ is continuous, its continuous dual $\rho^\circ = \mathsf{Top}_k, (\rho, k)$ is defined. Recall that the topologies on A and B are cofinite. Then by Theorem 2.6, we can view this as a map

$$\rho^\circ\colon A^\circ \otimes B^\circ \xrightarrow{\sim} (A \otimes^! B)^\circ \to (B \otimes^! A)^\circ \xrightarrow{\sim} B^\circ \otimes A^\circ$$

By Proposition 5.6, ρ is normal and multiplicative. Theorem 2.6 now implies that ρ° is conormal and comultiplicative, since these are formally dual properties. Thus it follows from Proposition 5.10 that ρ° is a cotwisting map for A° and B°.

Denote $S = A \#_\rho B$ and $C = A^\circ \#^{\rho^\circ} B^\circ$. By Theorem 2.12, the finite dual and its comultiplication respectively coincide with the continuous duals of A and its multiplication. By Proposition 5.12 the linear isomorphism

$$\Phi\colon A \otimes^! B \to S$$

is a homeomorphism. Thus the continuous dual functor yields a linear isomorphism

$$S^\circ \xrightarrow{\sim} (A \otimes^! B)^\circ \cong A^\circ \otimes B^\circ = C, \tag{5.15}$$

where we identify the vector space $A^\circ \otimes B^\circ$ with the crossed product coalgebra C. It only remains to show that this is a morphism of coalgebras.

To achieve this goal, we will treat the isomorphism (5.15) as the identity map. (This is reasonable because it is defined in terms of the linear homeomorphism Φ above, which the identity on the underlying vector space.) We must then show that the comultiplication of S° and C coincide. By definition of the twisted tensor product $S = A \#_\rho B$, we have

$$m_S = (m_A \otimes m_B) \circ (\mathrm{id}_A \otimes \rho \otimes \mathrm{id}_B).$$

Applying the continuous dual functor to the above formula and invoking Theorem 2.6 in the third equality below yields

$$\begin{aligned}
\Delta_{S^\circ} &= (m_S)^\circ \\
&= (\mathrm{id}_A \otimes \rho \otimes \mathrm{id}_B)^\circ \circ (m_A \otimes m_B)^\circ \\
&= (\mathrm{id}_A^\circ \otimes \rho^\circ \otimes \mathrm{id}_B^\circ) \circ (m_A^\circ \otimes m_B^\circ) \\
&= (\mathrm{id}_{A^\circ} \otimes \rho^\circ \otimes \mathrm{id}_{B^\circ}) \circ (\Delta_{A^\circ} \otimes \Delta_{B^\circ}) \\
&= \Delta_C
\end{aligned}$$

as desired. An easier computation similarly shows that the unit $\eta_S = \eta_A \otimes \eta_B$ dualizes to the counit $\varepsilon_C = \varepsilon_{A^\circ} \otimes \varepsilon_{B^\circ}$. $\square$

Example 5.16. For algebras A and B, the tensor product algebra has dual coalgebra given by

$$(A \otimes B)^\circ \cong A^\circ \otimes B^\circ.$$

While this is straightforward to prove from (2.3), it follows immediately from Theorem 5.14. This is the special case where $\rho = \sigma_{B,A}$ is the "tensor swap" map, which is evidently continuous.

5.3. **Applications of crossed product duality.** In order to apply Theorem 5.14 in any particular situation, we must know that the twisting map $\rho \colon B \otimes A \to A \otimes B$ is continuous. This raises the important question of how to recognize when a given twisting map is continuous. Our next goal will be to provide a sufficient condition for continuity of ρ in Theorem 5.18 below, amounting to the existence of large subalgebras of A and B that respectively centralize B and A.

We will say that an extension of rings $R_0 \subseteq R$ is *finite* if R is finitely generated as both a left R_0-module and a right R_0-module.

Lemma 5.17. *Suppose that $A_0 \subseteq A$ is a finite extension of k-algebras. Then an ideal $I \trianglelefteq A$ satisfies $I \in \mathcal{F}(A)$ if and only if there exists an ideal $I_0 \in \mathcal{F}(A_0)$ such that $I_0 \subseteq I$.*

Proof. If $I \in \mathcal{F}(A)$ then we may set $I_0 = A \cap A_0$. The embedding $A_0/I_0 = A_0/(A_0 \cap I) \hookrightarrow A/I$ ensures that $I_0 \in \mathcal{F}(A)$.

To establish the converse, note that if $I_0 \in \mathcal{F}(A)$ with $I_0 \subseteq I$, then the ideal $I' = AI_0A$ obtained by extending I_0 to A satisfies $I' \subseteq I$, and if I' has finite codimension then the same is true for I. Replacing I with I', we may thus reduce to the case where $I = AI_0A$ for some $I_0 \in \mathcal{F}(A_0)$. Because A is finite over A_0, we may fix $x_1, \dots x_m, y_1, \dots, y_n \in A$ with $A = \sum x_i A_0 = \sum A_0 y_j$. Then

$$\begin{aligned} A/I &= A/(AI_0A) \\ &\cong A \otimes_{A_0} (A_0/I_0) \otimes_{A_0} A \\ &= \sum_{i,j} x_i \otimes_{A_0} (A_0/I_0) \otimes_{A_0} y_j. \end{aligned}$$

Since A_0/I_0 is finite-dimensional, we see that the same will be true for A/I. So $I \in \mathcal{F}(A)$ as desired. ☐

Theorem 5.18. *Let $\rho \colon B \otimes A \to A \otimes B$ be a twisting map. Suppose that there exist subalgebras $A_0 \subseteq A$ and $B_0 \subseteq B$ satisfying the following properties:*

- *The extensions $A_0 \subseteq A$ and $B_0 \subseteq B$ are both finite.*
- *The restriction of ρ to each of the subspaces $B \otimes A_0$ and $B_0 \otimes A$ agrees with corresponding restriction of the tensor swap $\sigma_{B,A} \colon B \otimes A \to A \otimes B$.*

Then ρ is continuous, so that $(A \#_\rho B)^\circ \cong A^\circ \#^{\rho^\circ} B^\circ$.

Proof. It is enough to verify that ρ satisfies the condition of Proposition 5.12(iii). By Lemma 5.17, the family $\{I_\alpha\} \subseteq \mathcal{F}(A)$ of ideals in A that are of the form $I_\alpha = AI_0A$ for some $I_0 \in \mathcal{F}(A_0)$ is a neighborhood basis of zero in A. We claim that every such ideal $I_\alpha = AI_0A$ satisfies $\rho(B \otimes I_\alpha) \subseteq I_\alpha \otimes B$. First note that the multiplicative property (5.5) of ρ yields

$$\begin{aligned}
\rho(B \otimes AI_0) &= \rho \circ (\mathrm{id}_B \otimes m_A)(B \otimes A \otimes I_0) \\
&= (m_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho) \circ (\rho \otimes \mathrm{id}_A)(B \otimes A \otimes I_0) \\
&= (m_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho)(\rho(B \otimes A) \otimes I_0) \\
&\subseteq (m_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho)(A \otimes B \otimes I_0) \\
&= (m_A \otimes \mathrm{id}_B)(A \otimes \rho(B \otimes I_0)) \\
&= (m_A \otimes \mathrm{id}_B)(A \otimes I_0 \otimes B) \\
&= AI_0 \otimes B.
\end{aligned}$$

It then follows from another application of (5.5) again that

$$\begin{aligned}
\rho(B \otimes I_\alpha) &= \rho(B \otimes (AI_0)A) \\
&= \rho \circ (\mathrm{id}_B \otimes m_A)(B \otimes AI_0 \otimes A) \\
&= (m_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho) \circ (\rho \otimes \mathrm{id}_A)(B \otimes AI_0 \otimes A) \\
&= (m_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho)(\rho(B \otimes AI_0) \otimes A) \\
&\subseteq (m_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho)(AI_0 \otimes B \otimes A) \\
&= (m_A \otimes \mathrm{id}_B)(AI_0 \otimes \rho(B \otimes A)) \\
&\subseteq (m_A \otimes \mathrm{id}_B)(AI_0 \otimes A \otimes B) \\
&= AI_0A \otimes B = I_\alpha \otimes B
\end{aligned}$$

as desired.

We may similarly define a neighborhood basis of zero $\{J_\beta\} \subseteq \mathcal{F}(B)$ consisting of those ideals of the form $J_\beta = BJ_0B$ for some $J_0 \in \mathcal{F}(B_0)$. A symmetric argument will verify that these ideals satisfy $\rho(J_\beta \otimes A) \subseteq A \otimes J_\beta$. We now conclude from Proposition 5.12 that ρ is continuous, and the isomorphism of dual coalgebras follows from Theorem 5.14. □

Ore extensions are one of the most common methods of constructing noncommutative algebras. The above criterion applies nicely in the case of an Ore extension by a finite order automorphism. Interpreted geometrically, it verifies the intuition that a skew polynomial ring is like a ring of functions on a "twisted product" of the affine line and the space corresponding to A.

Corollary 5.19. *Let A be an algebra with an automorphism θ of finite order d. Assume that A is finite over the fixed subring A^θ. Then the finite dual of the Ore extension $A[t;\theta]$ is isomorphic to a crossed product coalgebra*

$$A[t;\theta]^\circ \cong A^\circ \#^\phi \operatorname{Dist}(\mathbb{A}^1_k)$$

for a suitable cotwisting map ϕ.

Proof. Note that the Ore extension is a crossed product algebra $A[t;\theta] = A\#_\rho k[t]$ defined by

$$\begin{aligned}
\rho\colon k[t] \otimes A &\to A \otimes k[t], \\
t^i \otimes a &\mapsto \theta^i(a) \otimes t^i.
\end{aligned}$$

We will show that ρ satisfies the condition of Theorem 5.18. Let $A_0 = A^\theta \subseteq A$ denote the θ-invariant subalgebra of A. Furthermore, since A_0 centralizes $t \in B$ within the Ore extension $A[t;\theta]$, it follows that the restriction of ρ to $B \otimes A_0$

coincides with σ_{B,A_0}. Next let $B_0 = k[t^d] \subseteq k[t]$. Certainly B is generated as a (left or right) B_0-module by the finite set $\{1, t, \ldots, t^{d-1}\}$. Since t^d centralizes A in $A[t;\theta]$, it follows that the restriction of ρ to $B_0 \otimes A$ coincides with $\sigma_{B_0,A}$.

It now follows from Theorem 5.18 that ρ is continuous. So Theorem 5.14 yields the isomorphism

$$A[t;\theta]^\circ = (A\#_\rho B)^\circ \cong A^\circ \#^{\rho^\circ} B^\circ.$$

The conclusion now follows from the fact that $B^\circ = k[t]^\circ \cong \mathrm{Dist}(\mathbb{A}^1_k)$ by Proposition 3.12, where ϕ corresponds to ρ° under this isomorphism. □

We now turn our attention to smash product algebras. Suppose H a Hopf algebra and A is a left H-module algebra. The smash product is a particular case of a twisted tensor product

$$A\#H = A\#_\rho H$$

where the linear map $\rho\colon H \otimes A \to A \otimes H$ is given in Sweedler notation by

$$\rho(h \otimes a) = \sum h_{(1)}(a) \otimes h_{(2)}. \tag{5.20}$$

It is well known [Mon93, Chapter 4] that this endows $A\#H$ with the structure of an algebra, so that ρ is a twisting map. If we write the H-module product as $\lambda\colon H \otimes A \to A$, then the twisting map can be written as the composite

$$\rho\colon H \otimes A \xrightarrow{\Delta\otimes \mathrm{id}_A} H \otimes H \otimes A \xrightarrow{\mathrm{id}_H \otimes \sigma} H \otimes A \otimes H \xrightarrow{\lambda \otimes \mathrm{id}_H} A \otimes H.$$

Note that if $\lambda\colon H \otimes^! A \to A$ is continuous with respect to the finite topologies, then the composite map ρ above is continuous. Indeed, the comultiplication Δ is an algebra homomorphism and thus is continuous, and it follows then that the maps in the above will all be continuous with respect to the finite topologies and topological tensor product. Thus if the module action λ is continuous, we have an isomorphism of coalgebras

$$(A\#H)^\circ \cong A^\circ \#^{\rho^\circ} H^\circ.$$

A general description of the finite dual of a smash product (without restriction on λ) is given in [Rad12, Proposition 11.4.2], which requires a more complicated subcoalgebra $A^{\underline{\circ}} \subseteq A^\circ$ to achieve a similar decomposition. The isomorphism above shows that such technicalities can be avoided if λ is sufficiently well-behaved.

In practice, one might wish for more straightforward conditions on the H-action that can be verified instead of topological continuity. The next result provides more familiar algebraic properties of the action that are sufficient. Recall [Mon93, Definition 1.7.1] that the subalgebra of *invariants* of H is

$$A^H = \{a \in A \mid h \cdot a = \varepsilon(h)a \text{ for all } h \in H\}.$$

For instance, if $H = kG$ is a group algebra, so that G acts by automorphisms on A, then $A^H = A^G$ is the usual subalgebra of G-invariants.

Theorem 5.21. *Let H be a Hopf algebra and let A be a left H-module algebra. Suppose that the following hold:*

1. *A is finite over the subalgebra A^H of H-invariants;*
2. *the action of H on A factors through a finite-dimensional Hopf algebra.*

Then the twisting map ρ of (5.20) *is continuous and there is an isomorphism of coalgebras $(A\#H)^\circ \cong A^\circ \#^{\rho^\circ} H^\circ$.*

Proof. It follows by the construction of H-invariants that the restriction of the twisting map (5.20) to the subspace $H \otimes A^H$ agrees with σ_{H,A^H}, the "tensor swap" map: for $h \in H$ and $a \in A^H$,

$$\begin{aligned}\rho(h \otimes a) &= \sum h_{(1)}(a) \otimes h_{(2)} \\ &= \sum \epsilon(h_{(1)})a \otimes h_{(2)} \\ &= a \otimes (\textstyle\sum \epsilon(h_1)h_{(2)}) \\ &= a \otimes h.\end{aligned}$$

Now let K be a finite-dimensional Hopf algebra through which the action of H on A factors; more precisely, there is a Hopf algebra surjection $\pi\colon H \twoheadrightarrow K$ and a K-module algebra action $\overline{\lambda}\colon K \otimes A \to A$ so that the action λ of H factors as

$$\lambda\colon H \otimes A \xrightarrow{\pi \otimes \mathrm{id}_A} K \otimes A \xrightarrow{\overline{\lambda}} A.$$

The surjection π makes H into a left K-comodule via the coaction

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{\pi \otimes \mathrm{id}_H} K \otimes H,$$

which we denote by α. This data relates to the twisting map of $A\#H$ through the following commuting diagram, where the composite across the top row is equal to ρ and the vertical arrows are induced by π:

$$\begin{tikzcd} H \otimes A \arrow[r, "\Delta\otimes\mathrm{id}"] \arrow[dr, "\alpha\otimes\mathrm{id}_A"'] & H \otimes H \otimes A \arrow[r, "\mathrm{id}\otimes\sigma"] \arrow[d] & H \otimes A \otimes H \arrow[r, "\lambda\otimes\mathrm{id}"] \arrow[d] & A \otimes H \\ & K \otimes H \otimes A \arrow[r, "\mathrm{id}\otimes\sigma"] & K \otimes A \otimes H \arrow[ur, "\overline{\lambda}\otimes\mathrm{id}_H"'] & \end{tikzcd}$$

Denote the subalgebra of K-coinvariants [Mon93, Definition 1.7.1] in H by

$$H_0 = \{h \in H \mid \alpha(h) = 1_K \otimes h\}.$$

Then for any $h \in H_0$ and $a \in A$, an examination of the commuting diagram above reveals that

$$\rho(h \otimes a) = 1_K(a) \otimes h = a \otimes h,$$

so that the restriction of ρ to $H_0 \otimes A$ is equal to $\sigma_{H_0,A}$. Furthermore, K being finite-dimensional implies [Mon93, Theorem 8.2.4] that H is a K-Galois extension of H_0, from which we can conclude [KT81, Theorem 1.7 and Corollary 1.8] that H is finitely generated (and projective) as both a left and right module over H_0.

Thus we have produced subalgebras $A_0 \subseteq A$ and $H_0 \subseteq H$ satisfying the hypotheses of Theorem 5.18, from which the desired conclusions follow. □

Historically speaking, the finite dual has mainly been of interest in the the study of Hopf algebras and bialgebras, as mentioned in Subsection 1.2. Thus as a final application of these methods we provide sufficient conditions for the dual of a crossed product bialgebra algebra to be the crossed product of the duals in Corollary 5.23 below; we thank Hongdi Huang for an insightful question that inspired the result.

Remark 5.22. It is a well-known fact [Mon93, Theorem 9.1.3] that the finite dual of a bialgebra (resp., Hopf algebra) again has the structure of a bialgebra (resp., Hopf algebra). This can be interpreted in terms of topological duality as follows. Let H be a bialgebra with multiplication $m\colon H \otimes H \to H$ and comultiplication $\Delta\colon H \to H \otimes H$. We already know from Theorem 2.12 that the continuous dual of

m yields the comultiplication of H°. Because H is a bialgebra, the comultiplication is an algebra homomorphism and therefore is continuous as a map $\Delta\colon H \to H \otimes^! H$ where H is endowed with the cofinite topology. The strong monoidal functor $(-)^\circ\colon \mathsf{CF}_k^{\mathrm{op}} \to \mathsf{Top}_k$ sends comonoids to monoids, so the continuous dual of the comultiplication

$$\Delta^\circ\colon H^\circ \otimes H^\circ \cong (H \otimes^! H)^\circ \to H^\circ,$$

is a multiplication. Similarly, the unit and counit of H have continuous duals that provide (co)units for the dual (co)multiplication structures. In this way H° becomes both a coalgebra and an algebra. Because the bialgebra axioms are self-dual (see the diagrams in the proof of [Swe69, Proposition 3.1.1]), they pass by continuous duality to Δ° and m° in in order to show that H° becomes a bialgebra under these structures. (If H is a Hopf algebra, then its antipode considered as a map $H \to H^{\mathrm{op}}$ is an algebra homomorphism and thus is continuous. So $S\colon H \to H$ is also continuous, allowing us to define $S^\circ\colon H^\circ \to H^\circ$. Again by self-duality of the axioms, this will be an antipode for H°, making it into a Hopf algebra.)

Corollary 5.23. *Let A and B be bialgebras, and suppose that $H = A\#_\rho^\phi B$ is a crossed product bialgebra. If the twisting and cotwisting maps are continuous*

$$\rho\colon B \otimes^! A \to A \otimes^! B,$$
$$\phi\colon A \otimes^! B \to B \otimes^! A,$$

where both algebras are equipped with their cofinite topologies, then there is an isomorphism of bialgebras

$$H^\circ \cong A^\circ \#_{\phi^\circ}^{\rho^\circ} B^\circ.$$

Proof. As in Remark 5.22 above, H° is a bialgebra with multiplication Δ° and comultiplication m°. We know from Theorem 5.14 that as a coalgebra we have $H^\circ \cong A^\circ \#^{\rho^\circ} B^\circ$. Also, because the comultiplication is given by

$$\Delta_H = (\mathrm{id}_A \otimes \phi \otimes \mathrm{id}_B) \circ (\Delta_A \otimes \Delta_B),$$

we are similarly able to apply Theorem 2.6 to show that Δ_{H° coincides with the multiplication m_{ϕ° of $A^\circ \#_{\phi^\circ} B^\circ$. Indeed, with suitable harmless identifications as in the proof of Theorem 5.14, we have

$$\begin{aligned}
m_{H^\circ} &= (\Delta_H)^\circ \\
&= ((\mathrm{id}_A \otimes \phi \otimes \mathrm{id}_B) \circ (\Delta_A \otimes \Delta_B))^\circ \\
&= (\Delta_A \otimes \Delta_B)^\circ \circ (\mathrm{id}_A \otimes \phi \otimes \mathrm{id}_B)^\circ \\
&= (\Delta_A^\circ \otimes \Delta_B^\circ) \circ (\mathrm{id}_A^\circ \otimes \phi^\circ \otimes \mathrm{id}_B^\circ) \\
&= (m_{A^\circ} \otimes m_{B^\circ}) \circ (\mathrm{id}_{A^\circ} \otimes \phi^\circ \otimes \mathrm{id}_{B^\circ}) \\
&= m_{\phi^\circ}.
\end{aligned}$$

It now follows that $H^\circ \cong A^\circ \#_{\phi^\circ}^{\rho^\circ} B^\circ$ is a crossed product bialgebra. □

6. Prime affine algebras finite over their center

Our final goal will be to specialize the ideas developed above to the case of the quantum plane [Man88, 1.2] at a root of unity. In order to clarify the geometric picture, we will assume throughout this section that our field k is algebraically closed. Part of this analysis will use the technique of Section 5 to view its dual coalgebra as a crossed product of two copies of the affine line. But to provide a clearer picture, we wish to incorporate information about its representation theory.

For this reason, we provide one further method to help describe the finite dual of an algebra R if it satisfies the following hypothesis:

(**H**) R is a prime affine k-algebra that is module-finite over its center $Z(R)$.

Under this hypothesis, the center $Z(R)$ is also an affine algebra (by the Artin-Tate Lemma), so that $Z(R)$ is noetherian (by the Hilbert Basis Theorem). So R is an affine noetherian PI algebra and therefore is fully RFD by Proposition 4.12. If R satisfies (**H**), then it is known [MR01, 10.2] that the prime spectrum of R is closely related to $\operatorname{Spec} Z(R)$. Our goal will be to similarly relate the quantized spectrum R° to the maximal spectrum of its center. Specifically, we will describe how the Azumaya locus [BG97, Section 3] in $\operatorname{Max} Z(R)$ appears within R°.

6.1. **The Azumaya locus in the dual coalgebra.** We recall some of the theory of the Azumaya locus from [BG97, Section 3] and [BG02, Section III.1]. Suppose that R satisfies hypothesis (**H**). Let d denote the PI degree [MR01, 13.6] of R. Then d is equal to the maximal k-dimension of all simple R-modules. The *Azumaya locus* is the subset of the maximal spectrum $\operatorname{Max}(Z(R))$ that can be characterized in the following equivalent ways:

$$\begin{aligned}\mathcal{A}(R) &= \{\mathfrak{m} \in \operatorname{Max}(Z(R)) \mid \mathfrak{m} = Z(R) \cap \operatorname{ann}(V) \text{ for a simple } {}_RV,\ \dim_k(V) = d\}\\ &= \{\mathfrak{m} \in \operatorname{Max}(Z(R)) \mid R_\mathfrak{m} \text{ is an Azumaya algebra over } Z(R)_\mathfrak{m}\}\\ &= \{\mathfrak{m} \in \operatorname{Max}(Z(R)) \mid R\mathfrak{m} \text{ is a maximal ideal of } R\}\\ &= \{\mathfrak{m} \in \operatorname{Max}(Z(R)) \mid R/R\mathfrak{m} \cong \mathbb{M}_d(k)\}.\end{aligned}$$

This is a nonempty open (and therefore dense) subset of $\operatorname{Max} Z(R)$.

A prime ideal P of R is *regular* if the PI degree of R/P is also equal to d. Then a maximal ideal M of R is regular if and only if $M = R\mathfrak{m}$ for some $\mathfrak{m} \in \mathcal{A}(R)$. Lemma 6.2 below will facilitate a link between R° and the points in the Azumaya locus of R.

Lemma 6.1. *Let $(A, \mathfrak{m})$ be a complete local ring whose residue field is algebraically closed. Then every Azumaya algebra over A is isomorphic to a matrix ring over A.*

Proof. By completeness of A, the morphism of Brauer groups $\operatorname{Br}(A) \to \operatorname{Br}(A/\mathfrak{m})$ induced by the surjection $A \twoheadrightarrow A/\mathfrak{m}$ is injective [AG60, Corollary 6.2]. The Brauer group of the algebraically closed field $A/\mathfrak{m}$ is trivial, so $\operatorname{Br}(A)$ is also trivial. It follows [AG60, Proposition 5.3] that every Azumaya algebra over A is the endomorphism ring of a finitely generated projective module $P_A \neq 0$. Because A is local, P is in fact free, so that the conclusion follows. □

Lemma 6.2. *Let R be an algebra over an algebraically closed field k satisfying* (**H**)*, and let d be the PI degree of R. Let M be a regular maximal ideal of R and set $\mathfrak{m} = M \cap Z(R) \in \mathcal{A}(R)$. Then we have*

$$\varinjlim (R/M^i)^* \cong \mathbb{M}^d \otimes \operatorname{Dist}(\operatorname{Spec} Z(R), \mathfrak{m}).$$

Proof. Denote $C = Z(R)$. The algebra $R/M^i \cong (R/M^i)_\mathfrak{m} \cong R_\mathfrak{m}/(M_\mathfrak{m})^i$ is a homomorphic image of $R_\mathfrak{m}$, which is Azumaya over $C_\mathfrak{m}$. Note that the completion of $R_\mathfrak{m}$ in the cofinite topology coincides with its completion in the $\mathfrak{m}$-adic topology (since its unique maximal ideal is $M_{\mathfrak{m}} = R_{\mathfrak{m}}\mathfrak{m}$). Because $C_{\mathfrak{m}}$ is noetherian, this completion [Eis95, Theorem 7.2] is given by

$$\widehat{R_{\mathfrak{m}}} \cong \widehat{C_{\mathfrak{m}}} \otimes_{C_{\mathfrak{m}}} R_{\mathfrak{m}}.$$

Because the Azumaya property is stable under extension of scalars [AG60, Corollary 1.6], it follows that $\widehat{R_{\mathfrak{m}}}$ is Azumaya over the complete local ring $\widehat{C_{\mathfrak{m}}}$, whose residue field k is algebraically closed. Thus Lemma 6.1 implies that $\widehat{R_{\mathfrak{m}}}$ is a matrix algebra over $\widehat{C_{\mathfrak{m}}}$. Since $\widehat{R_{\mathfrak{m}}}/\widehat{R_{\mathfrak{m}}}\mathfrak{m} \cong R/R\mathfrak{m} \cong \mathbb{M}_d(k)$, we must in fact have

$$\widehat{R_{\mathfrak{m}}} \cong \mathbb{M}_d(\widehat{C_{\mathfrak{m}}}).$$

Thus the inversely directed system of algebras

$$R/M^i \cong R_m/M_{\mathfrak{m}}^i \cong \widehat{R_m}/(\widehat{R_{\mathfrak{m}}}\mathfrak{m})^i$$

is isomorphic to the system of algebras

$$\mathbb{M}_d(\widehat{C_{\mathfrak{m}}}/\widehat{C_{\mathfrak{m}}}\mathfrak{m}^i) \cong \mathbb{M}_d(C_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}^i) \cong \mathbb{M}_d(k) \otimes (C_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}^i).$$

The claim now follows because $\operatorname{Dist}(C, \mathfrak{m}) \cong C_{\mathfrak{m}}^\circ \cong \varinjlim (C_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}^i)^*$. □

We are now prepared to prove the key result describing the relationship between the Azumaya locus and R°. To facilitate its statement, we will define an open subscheme of $\operatorname{Spec} Z(R)$ whose closed points are exactly $\mathcal{A}(R)$. Denote the intersection of all non-regular maximal ideals of R by

$$N = \bigcap \{M \in \operatorname{Max}(R) \mid M \text{ is not regular}\}.$$

Then [BG02, Lemma III.1.2] a prime ideal of R is non-regular if and only if it contains N. Combining this observation with the characterizations of the Azumaya locus as well as the fact [BG02, Lemma III.1.5] that the contraction map from the prime spectrum of R to $Z(R)$ is closed, it follows that a maximal ideal $\mathfrak{m}$ of $Z(R)$ lies outside of the Azumaya locus if and only if it contains the ideal

$$I_R = N \cap Z(R).$$

Let $U_R = D(I_R)$ the open subscheme of $\operatorname{Spec} Z(R)$ that is the non-vanishing locus of I_R. We will call this the *Azumaya subscheme* of $\operatorname{Spec} Z(R)$. By the discussion above, its set of closed points is exactly equal to $\mathcal{A}(R)$.

Theorem 6.3. *Let R be an algebra over an algebraically closed field k satisfying* (**H**)*. Let d denote the PI degree of R, let $N \trianglelefteq R$ denote the intersection of all non-regular maximal ideals in R, and let U_R be the Azumaya subscheme of* $\operatorname{Spec} Z(R)$ *described above. Then there is an isomorphism of coalgebras*

$$R^\circ \cong \varinjlim_{i \geq 1} (R/N^i)^\circ \oplus (\mathbb{M}^d \otimes \operatorname{Dist}(U_R)).$$

Proof. Denote $C = Z(R)$, a commutative affine algebra over which R is a finitely generated module. Let $I \in \mathcal{F}(R)$. The image of the composite $C \to R \to R/I$ is a finite-dimensional commutative algebra and thus has the form

$$C_I := C/(C \cap I) \cong \prod_{i=1}^{s} C/\mathfrak{m}_i^{e_i}$$

for distinct maximal ideals $\mathfrak{m}_i$ of C. Up to reordering, suppose that $\mathfrak{m}_1, \dots, \mathfrak{m}_r \notin \mathcal{A}(R)$ and $\mathfrak{m}_{r+1}, \dots, \mathfrak{m}_s \in \mathcal{A}(R)$. Then we may fix idempotents $e_I \in C_I$ and $f_I = 1 - e_I \in C_I$ so that via the isomorphism above we have

$$e_I C_I \cong \prod_{i=1}^{r} C/\mathfrak{m}_i^{e_i} \qquad \text{and} \qquad f_I C_I \cong \prod_{i=r+1}^{s} C/\mathfrak{m}_i^{e_i}. \tag{6.4}$$

Since e_I and f_I are central in $\overline{R} := R/I$, we have $\overline{R} = e_I(R/I) \oplus f_I(R/I)$.

By construction, the maximal ideals of $f_I\overline{R}$ correspond to regular maximal ideals of R, while those of $e_I\overline{R}$ correspond to non-regular maximal ideals of R. Furthermore, if $I \subseteq I'$ are ideals of finite codimension in R, then the image of $e_I \in R/I$ under $R/I \twoheadrightarrow R/I'$ is $e_{I'}$, and similarly f_I maps to $f_{I'}$. Thus the decomposition $(R/I)^* \cong (e_I R/I)^* \oplus (f_I R/I)^*$ is compatible with the directed limit taken over the $I \in \mathcal{F}(R)$, from which it follows that

$$R^\circ \cong \varinjlim_{I \in \mathcal{F}(R)} (e_I R/I)^* \oplus \varinjlim_{I \in \mathcal{F}(R)} (f_I R/I)^*.$$

Let $\mathcal{F}_n \subseteq \mathcal{F}(R)$ denote those ideals of finite codimension that are contained only in non-regular maximal ideals of R, and let $\mathcal{F}_r \subseteq \mathcal{F}(R)$ be those ideals that are contained only in regular maximal ideals. Then the decomposition above amounts to

$$R^\circ \cong \varinjlim_{I \in \mathcal{F}_n} (R/I)^* \oplus \varinjlim_{I \in \mathcal{F}_r} (R/I)^*. \tag{6.5}$$

First we will prove that $\varinjlim_{I \in \mathcal{F}_n} (R/I)^* \cong \varinjlim_i (R/N^i)^\circ$. To do so, it is enough to prove that

$$\mathcal{F}_n = \{I \in \mathcal{F}(R) \mid N^i \subseteq I \text{ for some } i \geq 1\}.$$

First suppose that $I \in \mathcal{F}_n$, and let $M_1, \dots, M_n \subseteq R$ be the maximal ideals above I, so that $N \subseteq M_1 \cap \cdots \cap M_n$. Then $(M_1 \cap \cdots M_n)/I$ is the Jacobson radical of R/I which is nilpotent, say of order i. This means that $N^i \subseteq (M_1 \cap \cdots \cap M_n)^i \subseteq I$. Conversely, suppose that $I \in \mathcal{F}(R)$ with some $N^i \subseteq I$. Then every maximal ideal containing I also contains N and therefore is not regular. So $I \in \mathcal{F}_n$, establishing the desired equality.

Now we describe $\varinjlim_{I \in \mathcal{F}_r} (R/I)^*$. By nilpotence of the Jacobson radical of each R/I, this is the same as computing the direct limit of the $(R/I)^*$ where I is a product of regular maximal ideals $M = R\mathfrak{m}$ for some $\mathfrak{m} \in \mathcal{A}(R)$. Given such maximal ideals $M_i = R\mathfrak{m}_i$, note that their products commute in the monoid of ideals of R (as $\mathfrak{m}_i \subseteq C$ are central). Then if I is a product of powers of distinct such M_i, we have

$$R/I = R/(M_1^{e_1} \cdots M_r^{e_r}) \cong \bigoplus R/M_i^{e_i},$$

so that

$$(R/I)^* \cong \bigoplus (R/M_i^{e_i})^*.$$

Taking the direct limit over all I is the same as taking the limit over all possible products of maximal ideals, which combines with Lemma 6.2 to result in

$$\begin{aligned}\varinjlim_{I\in\mathcal{F}_r}(R/I)^* &\cong \bigoplus_{\mathfrak{m}\in\mathcal{A}(R)} \varinjlim_i (R/R\mathfrak{m}^i)^* \\ &\cong \bigoplus_{\mathfrak{m}\in\mathcal{A}(R)} \mathbb{M}^d\otimes \mathrm{Dist}(\operatorname{Spec} C,\mathfrak{m}) \\ &\cong \mathbb{M}^d\otimes \bigoplus_{\mathfrak{m}\in\mathcal{A}(R)} \mathrm{Dist}(\operatorname{Spec} C,\mathfrak{m}).\end{aligned}$$

Finally, taking into account that $\mathcal{A}(R)$ consists precisely of the closed points of the open subscheme $U = U_R$ of $\operatorname{Spec} C$ and that the distributions based at the point $\mathfrak{m}$ are independent of the open subscheme in which we compute them, we have

$$\bigoplus_{\mathfrak{m}\in\mathcal{A}(R)} \mathrm{Dist}(\operatorname{Spec} C,\mathfrak{m}) = \bigoplus_{\mathfrak{m}\in U_0} \mathrm{Dist}(U,\mathfrak{m}) \cong \mathrm{Dist}(U)$$

thanks to Proposition 3.16. So $\varinjlim_{I\in\mathcal{F}_r}(R/I)^* \cong \mathbb{M}^d\otimes\mathrm{Dist}(U_R)$, and the conclusion now follows from (6.5). □

How should we interpret isomorphism of Theorem 6.3? The canonical surjection $R \twoheadrightarrow R/N$ yields an embedding $(R/N)^\circ \hookrightarrow R^\circ$, which we can view as the inclusion of a closed subspace [Smi02, Section 4]. The direct limit $\varinjlim (R/N^i)^\circ$ thus represents the underlying discrete part of a formal neighborhood of this closed subspace within R°. The complementary summand is the underlying discrete part of a complementary open subspace. Thus it can be imagined as a "direct product" [Man88, 1.1] of a qu*d*it (where d is determined by the representation theory of R) with the Azumaya locus (an open subspace of the maximal spectrum of the center). In light of the decomposition of $\operatorname{corad} R$ from (4.3), this summand contains the qu*d*its within the spectrum of R, while $(R/N)^\circ$ contains the qu-*n*its for all values $n < d$.

This specializes nicely to the case of an Azumaya algebra. By the Artin-Procesi Theorem [BG02, III.1.4], this is the case where $R/N = 0$, so that summand corresponding to the formal neighborhood of that closed subspace is trivial. In this case the Azumaya locus is the entire maximal spectrum $\mathcal{A}(R) = \operatorname{Max} Z(R)$, so that the Azumaya subscheme is $U_R = \operatorname{Spec} Z(R)$. So for Azumaya algebras the center governs the entire structure of the dual coalgebra in the following way.

Corollary 6.6. *Suppose that R is an algebra over an algebraically closed field k satisfying* (**H**)*, and let d be its PI degree. If R is Azumaya over its center $Z(R)$, then*

$$A^\circ \cong \mathbb{M}^d \otimes \mathrm{Dist}(\operatorname{Spec} Z(R)).$$

6.2. **A case study of the quantum plane.** In this final subsection we will analyze the dual coalgebra of the quantum plane at a root of unity. From the perspective set out in Sections 3 and 4 above, this will give us a glimpse of the quantized set of closed points in the quantum plane. We will apply the techniques developed in Sections 5 and 6.1 to achieve this goal.

We continue to assume that our field k is algebraically closed. In the discussion below, we will alternatively view the affine plane over k as the classical algebraic variety k^2 and as the scheme $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, depending on the best context for a particular observation. We trust that this fluctuating perspective can be reasonably navigated by the careful reader.

Let $q \in k^*$. Recall that the (algebra of functions on the) *quantum plane* is the affine domain

$$\mathcal{O}_q(k^2) = k_q[x, y] = k\langle x, y \mid yx = qxy\rangle.$$

This is also the crossed product algebra $A\#_\rho B$ where $A = k[x]$, $B = k[y]$, and

$$\rho = \rho_q \colon B \otimes A \to A \otimes B$$
$$\rho_q(y^i \otimes x^j) = q^{ij} x^j \otimes y^i.$$

If q is not a root of unity, then it is known [BG02, Example II.1.2] that the only maximal ideals of finite codimension in $\mathcal{O}_q(k^2)$ are of the form $(x, y-\lambda)$ or $(x-\lambda, y)$ for $\lambda \in k$. Using a strategy similar to that of Example 5.13, one can verify in this case that $k_q[x, y] = A\#_\rho B$ is not homeomorphic to $A \otimes^! B$, so that ρ_q is not continuous.

Suppose from now on that q is a primitive nth root of unity, so that n does not divide the characteristic of k. A number of facts stated without justification below can be found in [DCP93, Section 7] and [BG02, Examples III.1.2(3), III.1.4]. The center of the quantum plane is given by

$$Z(\mathcal{O}_q(k^2)) = k[x^n, y^n].$$

Thus $\mathcal{O}_q(k^2)$ is module-finite over its center and satisfies (**H**). Furthermore, the PI degree of $\mathcal{O}_q(k^2)$ is equal to n.

Letting σ denote the k-algebra automorphism of $k[x]$ given by $\sigma(x) = qx$, then $\mathcal{O}_q(k^2) \cong k[x][y; \sigma]$ satisfies the hypotheses of Corollary 5.19, and it follows (as in the proof of that result) that the twisting map ρ above is continuous. In this case $\rho^\circ \colon A^\circ \otimes B^\circ \to B^\circ \otimes A^\circ$ is a cotwisting map for the dual coalgebras of A and B. By Theorem 5.14 the dual coalgebra of the quantum plane is the crossed product coalgebra

$$\mathcal{O}_q(k^2)^\circ = (A\#_\rho B)^\circ \cong A^\circ \#^{\rho_q^\circ} B^\circ.$$

The commutative algebra $A \cong B$ is the coordinate ring of the affine line. By Proposition 3.12 we have $A^\circ \cong B^\circ \cong \mathrm{Dist}(\mathbb{A}^1_k)$, a coalgebra of distributions that was described geometrically in Example 3.19. Under this isomorphism, the cotwisting map ρ_q° corresponds to a cotwisting map $\phi_q \colon \mathrm{Dist}(\mathbb{A}^1_K) \otimes \mathrm{Dist}(\mathbb{A}^1_k) \to \mathrm{Dist}(\mathbb{A}^1_K) \otimes \mathrm{Dist}(\mathbb{A}^1_k)$. Thus we in fact have

$$\mathcal{O}_q(k^2)^\circ \cong \mathrm{Dist}(\mathbb{A}^1_k) \#^{\phi_q} \mathrm{Dist}(\mathbb{A}^1_k). \tag{6.7}$$

Note that the underlying vector space is independent of q, while the cotwisting map ϕ_q varies with the root of unity q.

Thus as the algebras $\mathcal{O}_q(k^2)$ are deformed throughout the family, their spectral coalgebras have identical underlying vector spaces and counits, but their comultiplications vary with the choice of parameter q. We interpret this by saying that the linear span of their quantum states remains unchanged, but that the quantum diagonal structure varies with q and causes the underlying quantum set to deform. This picture fits quite intuitively within the framework of quantum groups and q-deformations [BG02, Chapter I.1]. Of course, when $q = 1$ then $\rho^\circ = \sigma^\circ_{B,A}$ is simply a tensor swap, and the resulting coalgebra $\mathrm{Dist}(\mathbb{A}^1_k) \otimes \mathrm{Dist}(\mathbb{A}^1_k) \cong \mathrm{Dist}(\mathbb{A}^2_k)$ consists of distributions on the plane, recovering the affine plane as the limit of the quantum planes as $q \to 1$.

Next we will describe the cotwisting map ρ_q° as a deformation away from the classical case $q = 1$ where $\rho_1 = \sigma_{B,A} \colon B \otimes A \to A \otimes B$. This is facilitated by invoking the $\mathbb{Z}/n\mathbb{Z}$-grading on the polynomial algebra induced from the $\mathbb{N}$-grading via $\mathbb{N} \hookrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ as follows:

$$k[t] = k[t^n] \oplus k[t^n]t \oplus \cdots \oplus k[t^n]t^{n-1}.$$

The twisting map acts in a predictable way relative to the $\mathbb{Z}/n\mathbb{Z}$-gradings on A and B: given homogeneous elements $f_i \in k[x^n]x^i$ and $g_j \in k[y^n]y^j$, it is straightforward to compute that

$$\rho(g_j \otimes f_i) = q^{ij} f_i \otimes g_j.$$

In the computation below we make use of the q-number associated to an integer $m \geq 0$, given by $[m]_q = 1 + q + \cdots + q^{m-1}$. So $[m]_1 = m$, and if $q \neq 1$ then $[m]_q = (1 - q^m)/(1 - q)$. Then for $f \in A$ and $g \in B$ with $\mathbb{Z}/n\mathbb{Z}$-graded decompositions $f = \sum f_i \in k[x]$ and $g = \sum g_j \in k[y]$, we have

$$\begin{aligned} \rho(g \otimes f) &= \sum_{i,j=0}^{n-1} \rho(g_j \otimes f_i) \\ &= \sum_{i,j=0}^{n-1} q^{ij} f_i \otimes g_j \\ &= f \otimes g - \sum_{i,j=1}^{n-1} (1 - q^{ij}) f_i \otimes g_j \\ &= f \otimes g - (1 - q) \sum_{i,j=1}^{n-1} [ij]_q f_i \otimes g_j. \end{aligned}$$

Thus for $\sigma = \sigma_{B,A}$ we have

$$\rho_q = \sigma - (1 - q)\tau_q,$$

where we define the *q-twist* $\tau_q \colon k[y] \otimes k[x] \to k[x] \otimes k[y]$ by the formula

$$\tau_q(g \otimes f) = \sum_{i,j=1}^{n-1} [ij]_q f_i \otimes g_j.$$

(Note that $\tau_1 = 0$ since the sum is empty.) Because ρ_q and σ are both continuous, the same must be true for τ_q. Then passing to the continuous dual, we have

$$\rho_q^\circ = \sigma^\circ - (1 - q)\tau_q^\circ.$$

This explicitly represents the (co)twisting maps as deformations away from the classical case as q varies.

We will now incorporate the representation theory of the quantum plane and the form of its Azumaya locus to provide further insight into the structure of its dual coalgebra. While the center $Z(\mathcal{O}_q(k^2)) = k[x^n, y^n]$ is abstractly isomorphic to the coordinate ring of the affine plane $\mathbb{A}^2_k$, we will not view it in this way because we wish to identify the subalgebras $k[x]$ and $k[y]$ with the coordinate rings of the axes that form the twisted product (6.7). We will rather view this as the coordinate ring of a quotient of the affine plane by a group action.

Let $G_q = \langle q \rangle^2 \subseteq (k^*)^2$ denote the subgroup of the 2-torus whose coordinates are powers of q. This is a finite group with $G_q \cong (\mathbb{Z}/n\mathbb{Z})^2$. The usual action of the 2-torus on k^2 by coordinate scaling restricts to an action of G_q on the plane:

$$(q^i, q^j) \cdot (\alpha, \beta) = (q^i \alpha, q^j \beta). \tag{6.8}$$

This corresponds to an action of G_q on the coordinate ring $k[x, y] = \mathcal{O}(k^2)$ given by $(q^i, q^j) \cdot f(x, y) = f(q^i x, q^j y)$, which has fixed subalgebra

$$k[x, y]^{G_q} = k[x^n, y^n] \cong Z(\mathcal{O}_q(k^2)).$$

For this reason we view the center as the coordinate ring of the quotient scheme

$$\operatorname{Spec} k[x^n, y^n] = \mathbb{A}^2_k / G_q.$$

To explicitly describe the coradical of $\mathcal{O}_q(k^2)^\circ$ as in (4.3) amounts to enumerating the finite-dimensional irreducible representations of the quantum plane. These have been parametrized, for instance, in [DCP93, Section 7], but we have been unable to find an explicit list of these irreps with the exception of [RW18, Section 8], which studies the case where $k = \mathbb{C}$ and $q = -1$. Thus we pause to record the following.

Lemma 6.9. *Let k be an algebraically closed field, and let $q \in k^\times$ be a primitive nth root of unity. The irreducible representations of $V(\alpha, \beta)$ of $R = \mathcal{O}_q(k^2)$ are parametrized by $(\alpha, \beta) \in k^2$ and are given by a homomorphism $R \to \operatorname{End}_k(V(\alpha, \beta))$ as follows:*

1. *For $\alpha\beta = 0$, we obtain pairwise inequivalent 1-dimensional representations, with $R \to k$ given by $x \mapsto \alpha$ and $y \mapsto \beta$.*
2. *For $\alpha\beta \neq 0$, there are irreducible representations of the form $V(\alpha, \beta) = k^n$ with $x \mapsto \alpha D$ and $y \mapsto \beta P$, where*

$$D = \begin{pmatrix} 1 & & & \\ & q & & \\ & & \ddots & \\ & & & q^{n-1} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Two of these representations satisfy $V(\alpha, \beta) \cong V(\alpha', \beta')$ if and only if (α, β) and (α', β') lie in the same orbit of the G_q-action (6.8) the plane.

Proof. Let M be a maximal ideal of R, and denote $\mathfrak{m} = M \cap k[x^n, y^n]$. Then $\mathfrak{m} = (x^n - c, y^n - d)$ for some $(c, d) \in k^2$. Because k is algebraically closed, we may fix $(\alpha, \beta) \in k^2$ such that $(c, d) = (\alpha^n, \beta^n)$, so that

$$\mathfrak{m} = (x^n - \alpha^n, y^n - \beta^n).$$

Let $S = R/R\mathfrak{m}$, which is a finite-dimensional algebra with k-basis $\{x^i y^j \mid 0 \leq i, j \leq n - 1\}$.

(1) Suppose $\alpha\beta = 0$. Without loss of generality, we may assume that $\beta = 0$ (the case $\alpha = 0$ being symmetric). Then $\mathfrak{m} = (x^n - \alpha^n, y^n)$. Note that the image of y lies in the Jacobson radical of S. Thus S has the same irreducible representations as $S/(y) \cong k[x]/(x^n - \alpha^n)$, whose irreps are all 1-dimensional and given by evaluating x at the roots of $x^n - \alpha^n = x^n - c$, one of which is α. Thus we obtain the representation $S \to k$ given by $x \mapsto \alpha$ and $y \mapsto \beta$ whenever $\alpha\beta = 0$.

(2) Now suppose that $\alpha\beta \neq 0$. In this case $\mathfrak{m} = (x^n - \alpha^n, y^n - \beta^n)$ lies in the Azumaya locus $\mathcal{A}(R)$, so that it lies below the single maximal ideal $M = R\mathfrak{m}$. One may readily verify that for D and P as in the statement above, the matrices

$X = \alpha D$ and $Y = \beta P$ satisfy $YX = qXY$ as well as $X^n = \alpha^n I$ and $Y^n = \beta^n I$. Thus for $V(\alpha,\beta) = k^n$ we have an algebra homomorphism $\phi\colon R \to \operatorname{End}_k(V(\alpha,\beta))$ given by $x \mapsto X$ and $y \mapsto Y$ whose kernel equals M. To see that this representation is irreducible, we only need to verify that ϕ is surjective. This follows easily from the fact that the regular maximal ideal M has codimension n^2. (It can also be verified explicitly by noting that $\phi(\alpha^{-1}x) = D$ and $\phi(\beta^{-1}y) = P$ generate the algebra $\operatorname{End}_k(V)$, an argument that is facilitated by noticing that $\frac{1}{n}(I + D + D^2 + \cdots + D^{n-1}) = E_{11}$ is a standard matrix unit.)

Every algebra automorphism of $S = R/M \cong \operatorname{End}_k(k^n)$ is inner and thus is induced by an intertwiner. So we have $V(\alpha,\beta) \cong V(\alpha',\beta')$ if and only if the kernels of the representations are the same regular maximal ideals, if and only if the central maximal ideals $(x^n - \alpha^n, y^n - \beta^n) = (x^n - (\alpha')^n, y^n - (\beta')^n)$ are equal. This occurs exactly when $\alpha^n = (\alpha')^n$ and $\beta^n = (\beta')^n$, which is equivalent to saying that the points (α,β) and (α',β') are in the same G_q-orbit. □

Thus we can describe the structure of the coradical quite precisely in the following way. Let $X = V(xy) \subseteq k^2$ be the union of the coordinate axes, and let $Y_q \subseteq k^2/G_q$ be the open subset of the quotient space that is the image of the G_q-invariant open complement $k^2 \setminus X \subseteq k^2$. Then Lemma 6.9 yields:

$$\begin{aligned} \operatorname{corad} \mathcal{O}_q(k^2) &\cong \bigoplus_X k \oplus \bigoplus_{Y_q} \mathbb{M}^n \\ &\cong kX \oplus (\mathbb{M}^n \otimes kY_q). \end{aligned} \tag{6.10}$$

This hints at a decomposition of the full finite dual, which can be rigorously demonstrated using the method of Subsection 6.1. Let N be the intersection of the non-regular maximal ideals in $\mathcal{O}_q(k^2)$. From Lemma 6.9(1) we have

$$N = (x) \cap (y) = (xy),$$

whose intersection with the center is $(xy) \cap k[x^n, y^n] = (x^n y^n)$. Let $V = V(xy)$ be the closed subscheme of $\mathbb{A}^2_k$ that is the union of the coordinate axes. If we view $\operatorname{Spec} Z(R) = \mathbb{A}^2_k/G_q$ as discussed above, then the Azumaya subscheme $U_q = D(xy)/G_q$ is the open subscheme that is the image of the G_q-invariant complement of V. Theorem 6.3 now gives us

$$\mathcal{O}_q(k^2)^\circ \cong \varinjlim_{i \geq 1} (\mathcal{O}_q(k^2)/(xy)^i)^\circ \oplus (\mathbb{M}^n \otimes \operatorname{Dist} U_q). \tag{6.11}$$

Since the varieties X and Y_q precisely correspond to the closed points of the schemes V and U_q, this decomposition restricts to (6.10) when restricting to the coradical.

Note that the algebra $\mathcal{O}_q(k^2)/N \cong k[x,y]/(xy)$ is independent of q, commutative, and isomorphic to the coordinate ring of $V = V(xy)$. In particular,

$$(\mathcal{O}_q(k^2)/N)^\circ \cong \operatorname{Dist} V.$$

Thus the first summand in (6.11) is a formal neighborhood of the classical union of the coordinate axes. But even though this portion of the summand is classical (i.e., cocommutative), the formal neighorhood has a truly quantum (non-cocommutative) nature. For instance, its first-order neighborhood $(\mathcal{O}_q(k^2)/N^2)^\circ$ contains the dual of the 4-dimensional algebra $\mathcal{O}_q(k^2)/(x^2, y^2)$ which is not commutative if $q \neq 1$.

Figure 2 illustrates the real part of the complex quantum plane $\mathcal{O}_q(\mathbb{C})^\circ$ in the case where $q = -1$, so that $n = 2$. The axes are classical sets of points, but with quantum formal neighborhoods. Meanwhile, the points off of the axes have orbits

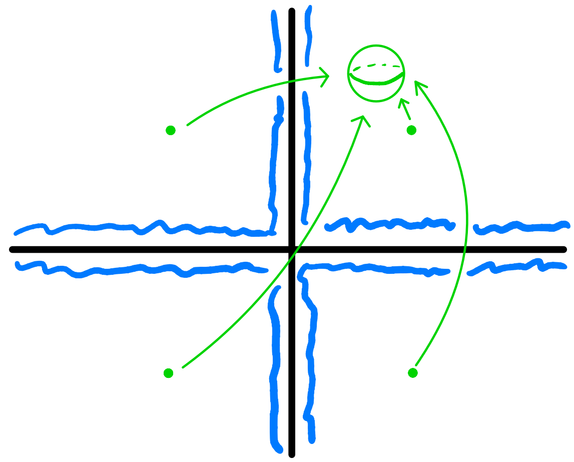

FIGURE 2. Depiction of $\mathcal{O}_q^\circ(\mathbb{C}^2)$ at $q=-1$

under the action of the group $G_{-1} = \{\pm 1\}^2$ of order 4. The points in each orbit become identified in the quotient by the group action and then "replaced" by the qubit $\mathbb{M}^2$.

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