PROBLEM LIST
The Second Irvine Conference In Inner Model Theory
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Problem 1 (Steel): Formulate a background condition for the K^c-construction and prove a form of UBH for iteration trees on (levels) of K^c.

For instance, take the background conditions for K^c in CMIP (or require something even weaker, like ultrapowers by background extenders are closed under 2^ω sequences). Suppose M is transitive such that $M^\omega \subseteq M$ and there is a Σ_2 elementary embedding $\pi : M \to V$ and $\mathcal{T}$ is an iteration tree on $(K^c)^M$ such that $lh(\mathcal{T}) = \omega$ and $\mathcal{T}$ has no drops. Can there be distinct cofinal branches b, c of $\mathcal{T}$ such that there are realization maps $\sigma_b : \mathcal{M}^{\mathcal{T}}_b \to K^c$, $\sigma_c : \mathcal{M}^{\mathcal{T}}_c \to K^c$ with the property that

$$\pi = \sigma_b \circ i^{\mathcal{T}}_b = \sigma_c \circ i^{\mathcal{T}}_c?$$

Problem 2 (Zeman): Assume every function $f : \omega_2 \cap \mathrm{cof}(\omega_1) \to \omega_2$ is bounded by a canonical function on a club. Does there exist a sharp mouse $\mathcal{M}$ with a measurable κ such that $o^{\mathcal{M}}(\kappa) \geq 2$?

Remark 0.1.

- *The theory "for every function $f : \omega_1 \to \omega_1$ is bounded on a club by a canonical function" is equiconsistent with "in K, there is an inaccessible limits of measurable cardinals".*
- *The forcing direction is due to Larson-Woodin, Larson-Shelah; the inner model direction is due to Deiser-Donder.*

Problem 3 (Trang): Are the following theories equiconsistent?

(a) $\mathsf{ZF} + \mathsf{DC} + \omega_1$ is supercompact.

(b) $\mathsf{ZF} + \mathsf{DC} + \omega_1$ is strongly compact.

(c) ZFC + there is a proper class of Woodin limit of Woodin cardinals.

Remark 0.2.

1. *One can add AD^+ to the theories (a) and (b) above.*
2. *The theories*
   - *(i)* $\mathsf{ZF} + \mathsf{DC} + \omega_1$ *is $\mathbb{R}$-supercompact.*
   - *(ii)* $\mathsf{ZF} + \mathsf{DC} + \omega_1$ *is $\mathbb{R}$-strongly compact.*
   - *(iii)* ZFC + *there is a measurable cardinal.*

   are equiconsistent.
3. *The theory "$\mathsf{ZF} + \mathsf{AD} + \omega_1$ is $\mathbb{R}$-strongly compact" follows from $\mathsf{ZF} + \mathsf{AD}$. The theory "$\mathsf{ZF} + \mathsf{AD} + \omega_1$ is $\mathbb{R}$-supercompact" is equiconsistent with "ZFC + there are ω^2 many Woodin cardinals" (Woodin) and therefore is strictly stronger than $\mathsf{ZF} + \mathsf{AD}$.*

4. *The theory "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\mathcal{P}(\mathbb{R})$-strongly compact" is equiconsistent with $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ while the theory "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\mathcal{P}(\mathbb{R})$-supercompact" is strictly stronger (Trang-Wilson).*

Problem 4 (Adolf): Let κ be a singular cardinal and let $\mathcal{W}$ be a premouse with $ON \cap W = \kappa$. Suppose $\mathcal{M}_1, \mathcal{M}_2$ are sound, countably iterable premice such that $\rho_\omega(\mathcal{M}_1) = \rho_\omega(\mathcal{M}_2) = \kappa$. Is $\mathcal{M}_1 \trianglelefteq \mathcal{M}_2$ or $\mathcal{M}_2 \trianglelefteq \mathcal{M}_1$?

Remark 0.3. *The above question has positive answer in the case $\kappa > \omega$ is regular, cf. "Stacking mice" (Jensen, Schimmerlinng, Schindler, Steel) and the case κ is singular with $cof(\kappa) > \omega$ (Adolf).*

Problem 5 (Steel): Develop the theory of strategy mice with long extenders.

Problem 6 (Schlutzenberg): Are there models V, W of ZFC with V an inner model of W, W is not class generic over V such that no set in $W \backslash V$ is set generic over V?

Problem 7 (Wilson): Are the following theories equiconsistent?

(i) ZFC + there is a Woodin cardinal.

(ii) ZFC $+\omega_2$ is a weak Vopenka cardinal (i.e. there is no sequence $(B_\alpha : \alpha < \omega_2)$ of structures of size at most ω_1 in a common signature of size at most ω_1 such that for all $\alpha, \beta < \omega_2$, the number of homomorphisms from $B_\alpha \to B_\beta$ is 0 if $\alpha < \beta$ and 1 otherwise.

Remark 0.4.
- *It is almost true that Con(i) implies Con(ii).*
- *(ii) implies $\neg\mathsf{CH}$.*

Problem 8 (Steel): Let AD_2 be the theory:

1. $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R} + \Theta$ is regular.
2. Letting $Hom = \mathcal{P}(\mathbb{R})$ and HS be the set of all homogeneity systems for subsets of $\mathbb{R}$, for $\mathcal{A} \subset Hom$, define $Code(\mathcal{A}) = \{\overline{\mu} \in HS : S_{\overline{\mu}} \in \mathcal{A}\}$. For every $\mathcal{A} \subseteq Hom$, $Code(\mathcal{A})$ is Suslin (i.e. there is a tree T on $\Theta \times \gamma$ for some γ such that $Code(\mathcal{A}) = p[T]$).
3. There is a normal fine measure on $\mathcal{P}_{\omega_1}(Hom)$.
4. Let $R \subseteq Hom \times \mathcal{P}(Hom)$ and $\forall A \exists \mathcal{B} R(A, \mathcal{B})$ then there is a function $f : Hom \times Hom \to \mathcal{P}(Hom)$ such that $\forall A, B R(A, f(A, B))$.
5. Let $R \subseteq \mathcal{P}(Hom) \times \mathcal{P}(Hom)$ and $\forall \mathcal{A} \subseteq Hom \exists \mathcal{B} \subseteq Hom\ R(\mathcal{A}, \mathcal{B})$ then there is an f such that whenever $\mathfrak{X} \subseteq \mathcal{P}(Hom)$ and $\mathfrak{X}$ is a surjective image of Hom then

   (i) $\forall \mathcal{A} \in \mathfrak{X} \exists \mathcal{B} \in f(\mathfrak{X})\ R(\mathcal{A}, \mathcal{B})$.

   (ii) $f(\mathfrak{X})$ is a surjective image of Hom.

(a) Is AD_2 consistent?

(b) Is (1) + (2) consistent?

(c) For $Y \subseteq Hom^\omega$, let G^1_Y be the game where players I, II take turns to play homogeneity systems $\overline{\mu}_i$ for $i < \omega$, II wins the run iff $\langle S_{\overline{\mu_i}} : i < \omega \rangle \in Y$. We say G^1-determinacy holds if for every $Y \subseteq Hom^\omega$, G^1_Y is determined. Does G^1-determinacy follow from AD_2?

(d) Does (1)-(4) imply Θ^+ is regular?

(e) Does Θ^+ is regular + (1)-(4) imply (5)?

(f) Define the class Hom^2_∞ to be the collection of homogeneously Suslin subsets of Hom in a natural way. What is the theory of $L(Hom^2_\infty)$? Does $L(Hom^2_\infty) \vDash \mathsf{AD}_2$?

(g) Is the UB powerset $\mathcal{A}_\infty$ a subset of Hom^2_∞?

Remark 0.5.
- *(1) implies every set of reals is homogeneously Suslin and every set of reals is coded by an element of Θ^ω.*
- *Given $X \subseteq Hom \times Hom$, let G^0_X be the game where I, II take turns to play measures $(\mu_{s_i} : i < \omega)$ (I's plays) and $(\nu_{s_i} : i < \omega)$ (II's plays), I wins iff $(S_{\bar{\mu}}, S_{\bar{\nu}}) \in X$. G^0-determinacy is the statement G^0_X is determined for every $X \subseteq Hom \times Hom$. G^0-determinacy follows from (1)-(3).*
- *(4) is equivalent to the statement in $V^{Coll(\omega,\mathcal{P}(\mathbb{R}))}$, the countable axiom of choice holds for relations on $\omega \times \mathbb{R}$.*
- *(1)-(5) implies Θ^+ is regular.*
- *Assume κ is supercompact and $g \subset Coll(\omega, 2^{2^\kappa})$ is V-generic, then Hom^2_∞ is sealed, and $L(Hom^2_\infty) \cap \mathcal{P}(\mathbb{R}) = Hom^2_\infty$; therefore, $L(Hom^2_\infty) \vDash \mathsf{AD}_\mathbb{R}$.*

Problem 9 (Trang): Suppose AD holds and $\mathbb{P}$ is a nontrivial set forcing.

(a) If $1 \Vdash_\mathbb{P} \check{\mathbb{R}} \subsetneq \dot{\mathbb{R}}$, must AD fail in $V^\mathbb{P}$?

(b) Can there be a nontrivial elementary embedding $j : V \to V[g]$ for any V-generic $g \subseteq \mathbb{P}$?

(c) Are there models M, N of AD such that M is an inner model of N, $\mathbb{R}^M$ is uncountable in N, and $\omega_1^M < \omega_1^N$?

Remark 0.6.
- *Chan-Jackson and Ikegami-Trang show that the answer to (a) is "yes" if $\mathbb{P}$ is a surjective image of $\mathbb{R}$.*
- *Schlutzenberg shows that there cannot be any elementary embedding $j : V[g] \to V$ if g adds a new set of ordinals.*
- *Using techniques of Velikovic-Woodin, if $\mathbb{P}$ adds a real, then $\omega_1^V < \omega_1^{V[g]}$.*

Problem 10 (Wilson): Let $\mathcal{C}$ be a nontrivial class of structures (e.g. graphs, trees, well-founded labeled trees). Let $VP(\mathcal{C})$ say for every proper class of structures in $\mathcal{C}$ there is a homomorphism from one to another. What are the possible consistency strengths of principles of the form $VP(\mathcal{C})$ for "natural" $\mathcal{C}$?

Remark 0.7. *The consistency strength of the following classes is known:*

(i) VP(graphs) : VP

(ii) VP(2-labeled trees): virtual VP/ ON is virtually Woodin for supercompactness

(iii) VP(wellfounded 2-labeled trees): virtual weak VP / ON is virtually Woodin

(iv) VP(well-founded trees) : ZFC