Question: Let $E$ be an equiv relation on $R$. Is there a $\Delta^1_1$ set $C$ so that $E \cap C$ is $\Delta^1_1$?

Silva: If $E$ is $\Delta^1_1$ equiv relation then either $E$ has countably many equiv classes or a perfect set of $E$-inequivalent elements.

Def: $I$ a $\sigma$-ideal on $R$

$$P^*_I = (\Delta^1_1 \setminus I, \subseteq)$$

Fact: There is a $P^*_I$-name for a real $x_{\check{g}}$ s.t. if $g \in P^*_I$ and $B$ is a $\Delta^1_1$-set coded in $V$ then

$$V[g]^I = x_{\check{g}} \in B \iff B \in g$$

Fact: $x_{\check{g}}$ is not in any ground model coded $I$-small set.

Def: $M \prec V_{\text{ad}}$

$x \in R$ is $P^*_I$-generic $/ M$ iff

$$\{ B \in \Delta^1_1^{V_{\text{ad}}} | x \check{\in} B \}$$

is a $P^*_I$-generic object $/ M$.

Fact: (Zapletal) TFAE:

1. $P^*_I$ is proper

2. all $\subseteq M \prec V_{\text{ad}}$

$B \in \Delta^1_1$ set

$$\{ x \check{\in} B \}$$

is $I^+ \Delta^1_1$.
Question. Let $E$ be a $\Sigma^1_1$ eqv rel. $B \weq_1 I^+ \mathcal{C}$.

Prop. Is there a $C \subseteq B$, $I^+ \weq_2 I$, s.t. $\mathcal{E} \cap C = \Delta^1_1$.

Fact (Kacurek-Sakamoto-Zapletal) N.

Fact (15-5-2) If $E$ is $\Sigma^1_1$ with all classes $\Pi^1_1$ then $E$ is $\Delta^1_1$-resolvable to the orbit eqv rel. of a Polish group action.

Hence yes.

Example $x \subseteq E \setminus y \iff w^x = w^y$.

$E$ is $\Sigma^1_1$, classes $\Delta^1_1$, uncountable many classes, is thin.

Let $I$ be the $\sigma$-ideal generated by $F_0$, classes.

Thus (Chmiel, Magidor, indep.) If there is a measurable cardinal then yes.

Fact (15+2) If $E$ is thin $\Sigma^1_3$ eqv rel. then there is a set $C \subseteq I^+ \weq_2 I$. $\mathcal{E} \cap C$ is a single class.

Fact (Let $\kappa$ be a measurable cardinal in $\mathbb{L}$. $g \subseteq \mathbb{C} \cap \mathcal{L}(\omega_1)$.

In $L[\mathbb{G}]$: if $I$ is s.t. $\mathcal{P}_I$ is proper, $E$ is $\Pi^1_3$ eqv rel. with countable many classes, then there is $C \subseteq I^+ \weq_2 I$. $\mathcal{E} \cap C$ is $\Delta^1_1$.

Question. What if $E$ is projective s.t. all equivalence classes $\Sigma^1_3$, $\Pi^1_3$. Can you find an $I^+ \weq_2 I$ s.t. $\mathcal{E} \cap C$ is $\Sigma^1_3$ or $\Pi^1_3$?
Fact (Indep. Diamon): In $L$, there is an epimorphic $E_L$ such that $E_L \models L$ for all $\mathcal{E}$ with all closed classes and the $\mathcal{E}$ has negative answer for all $\sigma$-ideals $E_L$.

$E_L y \models E_L y$ appear first at the same admissible level of $L$.

Question: What about all epimorphic relations $E$, with $E \models L$?

If needs can be $\omega$-ordered: No.

Open: Is a negative answer for $E \models L$ consistent?

REM All negative answers rely on Silver Dichotomy.

General framework for positive answer (joint with Magidor):

- Background on homogeneous tree

Let $G$ be a homogeneous tree on $\omega_1 \times \omega$ and $E_G = P[\mathcal{B}]$.

Let $I$ be a $\sigma$-ideal on $\mathcal{R}$ with $P^+$ proper.

$(2c + \omega)$ $\models \exists \mathcal{E}$ is a homogeneous tree.

Define $D(x, T) \models T$ is a tree on $\omega_1 \times \omega$ and

$E_G y \models T$ is self-founded.
\[ p(T) = f(x) \]

(B) \( \forall x \in T \) \( D(x,T) \) and \( \exists ! x \in T \) \( D(T,T) \)

(C) There are a hole in \( w \times w \times 1 \).

\[ P[U] = \{ (x,T) | D(x,T) \} \]

\[ \exists \exists p \subseteq P[U] = \{ (x,T) | D(x,T) \} \]

Thus if \( A, B, C \) hold, \( B \) \( y \in T \) \( T \) then

Fix \( V \subseteq B \) \( y \in T \) \( T \) \( X \in T \).

Claim if \( \exists x \in \exists_1 \) \( -n \) \( \) \( \exists M \) and \( x, T \in M \) for \( \) \( D(x,T) \) then \( V = D(T,T) \).

By (C) \( M(T) \subseteq D(x,T) \)

\[ M(T) = \{ (x,T) \} \in P[U] \]

Fix \( g_1 \) \( \exists -gen \) \( / M \) \( B(T) \)

\[ M(T) \subseteq \exists x \in T \) \( D(x,T) \).

So \( T \in M(T) \) \( \) \( M(T) \subseteq D(T,T) \)

\[ (T) \subseteq P \subseteq T \]

\[ \begin{array}{c|cc}
  & m_{0,0} & m_{1,0} \\
 \hline
 m_{0,0} & x_0 & x_1 \\
 \end{array} \]
**Rules:**

1. I plays $m_0$, II plays $m_1 < \gamma$
2. $(m_0 \ldots m_k, m_0 \ldots m_{k-1}) \in T$
3. $1 \in \{ m_1, m_2 \ldots m_{k-1}, d_0 \ldots d_{k-1} \} \in S$

The first who violates rules, loses. With the play run for $w$ steps, II wins.

**Claim:** In $M(g)$ II has a w.s.

**Proof:** If II has a w.s. $s^g \in T^w$. By (A) $M(g)$ is a homogeneous.

Fix a homogeneous system $S$.

We construct $(a_0, k_2w), (b_2, k_2w), (A_k, k_2w)$

$(a_0, b_0) = \tau^k(\varnothing), A_0 = \varnothing$

Suppose $a_0, \ldots a_{i+1}, b_0, \ldots b_{k-1}, A_0, A_{k-1}$ defined. Let

$h_k: \mathcal{S}^{\leq k} \rightarrow w \times w$

$((a_0, b_0), \ldots (a_{i+1}, b_{i+1})) \in \mathcal{S}^{\leq k}$

$\mu_{g_{\leq i}}, \varnothing$ concentrates on $\mathcal{S}^{\leq k} + 0 \circ \varphi \circ \mathcal{S}$

So there a unique $(a_i, b_i)$ s.t.

$h_k(a_i, b_i) \in \mu_{g_i, \varnothing}$

Let $L: w \rightarrow w \times w$. Let $k = (a_i, b_i)$
Claim that \( LE(T) \)

If not: \( L(aT_{k+1}) \subset L(\tau) \cup \{ \tau \} \).

Denote \( \mu_i = \mu_k \cap \omega_i \). If \( i \leq j \) let \( \mu_j \) extend \( \mu_i \).

\( \forall \mu_k \subseteq (A_1) \in M_\kappa \) so the intersection \( \mu_k \subseteq (A_1) \in M_\kappa \) in that set.

Play \( \bigotimes \tau \) when \( \Pi \) uses \( \tau \) and \( \Pi \) uses \( \beta \).

\( \mu_0, \ldots, \mu_{k-1} \). \( (\beta, \beta') \in A_1 = \mu_{k-1} \) \[ 3^{k-1,2} \]

\( \Pi \) does not lose at \( k \). But \( \beta \neq \beta \). \( \alpha_{k+1} \in L(\tau) \cup \{ \tau \} \)

So I do the first to lose. \( \beta \).

We showed: \( LE(T^+) = \{ q \} \in S \)

Let \( \alpha = (a_0, \ldots) \). This means \( q \in S \alpha_{\omega} \downarrow \). \( (q, \alpha) \notin \bigotimes \sigma \).

\( \delta: \omega \rightarrow \omega + \omega \)

\( \delta(\omega) = (q, \langle b \rangle \alpha b) \)

\( (\kappa, \kappa) \) is a w.f.t. \( A \in \mathcal{F}_{\kappa} \) all \( \kappa \).

Then is \( \Phi: \omega \rightarrow \omega \). \( \Phi(k) \in \mathcal{F}_{\kappa} \) all \( \kappa \).

Play \( \bigotimes \tau \). I uses \( \tau \). \( \Pi \) uses \( \Phi \)

No one loses at finite stage, so \( \Pi \) wins. \( \delta \). \( \Phi \).

Claim: For \( \tau \in \mathcal{M}(\tau) \) for \( \Pi \). Then \( \tau \) is a w.s. for \( \Pi \).

\( \Phi \) is finite. \( B/C \) the game is closed.
Claim 4: \( y \in \mathbb{R}^n \in \mathcal{E}_S \text{ and } g \in \mathcal{M}(\mathbb{R}^n) \) imply

By Claim 1: \( \mathcal{M}(g) \subseteq D(g, t) \Rightarrow \forall \mathcal{E} \subseteq D(g, t) \)

so \( (g)_{\mathcal{E}_S} = p(T) \Rightarrow y \in p(T) \) so \( (y, t) \in T \) since \( T \) wining in \( U \). But (in \( V \)) \( g \in T \), s.t. I am in \( y \) and \( T \) wins. Since \( (y, t) \in \mathcal{E}(T) \), I do not lose. I cannot lose as \( T \) is winning. Let \( \phi(k) = \phi_k = \text{the } k \text{'th response according to } k \).

Since \( T \in \mathcal{M}(g) \): \( \phi(k) \in \mathcal{M}(g) \). Since \( P_T \) is proper, \( M \) and \( \mathcal{M}(g) \) have same ordinals so \( \phi(k) \in M \).

\( g \in \mathcal{M}(\mathbb{R}^n) \). So \( \mathcal{E}_S \) \( g \Rightarrow \mathcal{E}(\mathbb{R}^n) \).

By proposition: let \( C \subseteq B \in \Delta^1 \), \( I^+ \) \( q, h \in C \)

\( q \in \mathcal{E}_C \Rightarrow (\mathbb{R}^n) \Rightarrow g \mathcal{E}_C \). \\

(A) \( I^- S \) is homogeneous

\( |P_{\pm 1} = 2 \phi_0 \)

(B) \( \forall x \in T \) \( D(x, T) \Rightarrow P \)

(C) \( \pi(T) = \cdots D(x, T) \)

Thus (with Magidor) \( P_T \) proper. Assume infinity may Woodin with a measurable above.

Then \( \mathcal{E}(\mathbb{R}) \) all classes \( \mathcal{E}^1 (\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n)) \) then

then is \( C \subseteq I^+ \) s.t. \( \mathcal{E}(\mathbb{R}) \subseteq (\mathbb{R}^n, \Delta^1) \)
\((\text{Martin})\) \(AD_{\mathbb{R}}\). All trees on \(w \times \theta\) in \(\Theta\) are weakly homogeneous.

\((\text{Martin})\) \(AD + DC\). \(\forall A \subseteq \mathcal{P}_\omega \mathbb{R} \exists \alpha \in \text{On} \ \forall B \subseteq \mathcal{P}_\omega \mathbb{R} \ \alpha \in \text{Sushâ}} \rightarrow \text{Sushâ}}\).

\((\text{Woodin})\) \(AD_{\mathbb{R}}\) all \(A \subseteq \mathcal{P}_\omega \mathbb{R}\) are Sushâ

In \(AD_{\mathbb{R}} + DC\) all sets are homomorphic.

\((\text{Neeman - Norwood})\) \(ZF + DC + AD_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))\)

\(A\) proper \(H \subseteq \mathcal{P}_\omega \mathbb{R}\) friend.

\(J : L(\mathcal{P}(\mathbb{R})) \rightarrow L(\mathcal{P}(\mathbb{R}))^{\mathcal{CH}}\)

does not move reals or ordinals.