Question Let E be an equiv relation on $\mathbb{R}$. Is there a $\underset{\sim}{\Delta}^1_1$ set C so that $E \upharpoonright C$ is $\underset{\sim}{\Delta}^1_1$

Silver If E is $\underset{\sim}{\Pi}^1_1$ equiv relation then either E has countably many equiv classes or a perfect set of E-inequivalent elements.

Def I a σ-ideal on $\mathbb{R}$

$$\mathbb{P}_I = (\underset{\sim}{\Delta}^1_1 \setminus I, \subseteq)$$

Fact There is a $\mathbb{P}_I$-name for a real x_{gen} s.t. if $g \subseteq \mathbb{P}_I$ and B is a $\underset{\sim}{\Delta}^1_1$-set coded in V then

$$V[g] \models x^g_{gen} \in B \iff B \in g$$

Fact x^g_{gen} is not in any ground model coded I-small set.

Def $M \prec V_\Theta$.

$x \in \mathbb{R}$ is $\mathbb{P}_I$-generic / M iff
$\{B \in \underset{\sim}{\Delta}^1_1 \cap M \mid x \in B\}$ is a $\mathbb{P}_I$-generic filter / M.

Fact (Zapletal) TFAE

(1) $\mathbb{P}_I$ is proper

(2) all Ξ $M \prec V_\Xi$

B an I^+ $\underset{\sim}{\Delta}^1_1$ set

$\{x \in B \mid x \text{ is } \mathbb{P}_I\text{-generic}/M\}$ is I^+ $\underset{\sim}{\Delta}^1_1$

<u>Question</u> Let E be a $\underset{\sim}{\Sigma}^1_1$ eq. rel. B $\underset{\sim}{\Delta}^1_1$ I^+ set $\mathbb{P}_I$ proper. Is there a $C \subseteq B$, I^+ $\underset{\sim}{\Delta}^1_1$ s.t. $E \upharpoonright C$ is Δ^1_1.

<u>Fact</u> (Kanovei-Sabok-Zapletal) N.

<u>Fact</u> (K-S-Z) If E is $\underset{\sim}{\Sigma}^1_1$ with all classes ctble or E is $\underset{\sim}{\Delta}^1_1$-reducible to the orbit eq. rel. of a Polish group action then yes.

<u>Quest</u> Let E be $\underset{\sim}{\Sigma}^1_1$ (or $\underset{\sim}{\Pi}^1_1$) equiv rel. with all classes $\underset{\sim}{\Delta}^1_1$. B is I^+ $\underset{\sim}{\Delta}^1_1$. $\mathbb{P}_I$ proper. Is there $C \subseteq B$ I^+ Δ^1_1 s.t. $E \upharpoonright C$ is $\underset{\sim}{\Delta}^1_1$

<u>Example</u> $x E_{\omega_1} y \iff \omega_1^x = \omega_1^y$

E is $\underset{\sim}{\Sigma}^1_1$, classes $\underset{\sim}{\Delta}^1_1$, unctbl many classes, is thin.

Let I be the σ ideal generated by E_{ω_1} classes.

<u>Thm</u> (Chan, Magidor indep. Drucker) If there is a measurable cardinal then yes.

<u>Fact</u> (ZFC) If E is thin $\underset{\sim}{\Sigma}^1_1$ eq. rel. then there is a set C I^+ $\underset{\sim}{\Delta}^1_1$ s.t. $E \upharpoonright C$ has a single class.

<u>Fact</u> Let κ be a remarkable cardinal in L. $g \subseteq \mathrm{Coll}(\omega, <\kappa)$. In $L[g]$: If I is s.t. $\mathbb{P}_I$ is proper, E is $\underset{\sim}{\Pi}^1_1$ eq. rel. with countable equiv classes then there is C $\underset{\sim}{\Delta}^1_1$ I^+ s.t. $E \upharpoonright C$ is Δ^1_1.

<u>Question</u> What if E is projective s.t. all equivalence classes $\underset{\sim}{\Sigma}^1_1$ or $\underset{\sim}{\Pi}^1_1$. Can you find an I^+ Δ^1_1 C s.t. $E \upharpoonright C$ is $\underset{\sim}{\Sigma}^1_1$ or $\underset{\sim}{\Pi}^1_1$?

<u>Fact</u> (indep Drucker) In L there is thin $\underset{\sim}{\Delta}^1_2$ equiv rel E_L s.t. ~~for all~~ it has all ctbl classes and the Q has negative answer for all σ-ideals

$x \, E_L \, y \Leftrightarrow x, y$ appear first at the same admissible level of L

<u>Question</u>: What about all equiv relations with $\underset{\sim}{\Sigma}^1_1$ ($\underset{\sim}{\Pi}^1_1$) equiv classes?

If reals can be w. ordered: No

<u>Open</u>: I a negative answer for $\underset{\sim}{\Sigma}^1_1$ (or $\underset{\sim}{\Pi}^1_1$) with $\underset{\sim}{\Delta}^1_1$-classes consistent?

<u>REM</u> All ~~...~~ negative answers rely on Silver Dichotomy

<u>GENERAL FRAMEWORK FOR POSITIVE ANSWER</u> (Joint with Magidor)

- Background on homogeneous trees.

Let S be a homogeneous tree on $\omega \times \omega \times \gamma$ and $E_S = p[S]$.
Let I be a σ-ideal on $\mathbb{R}$ with $\mathbb{P}_I$ proper

(A) (ZF+DC) $\Vdash \check{S}$ is a homogeneous tree

Define $D(x, T) \Leftrightarrow T$ is a tree on $\omega \times \omega$ and

$\forall y \; x E_S y \Leftrightarrow T^y$ wellfounded

$p[T] = [x]_{E_S}$

(B) $\forall x \exists T\, D(x,T)$ and $\Vdash \forall x \exists T\, D(x,T)$

(C) There is a tree U on $\omega \times \omega \times \zeta$ s.t.

$p[U] = \{(x,T) \mid D(x,T)\}$

$\Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x,T) \mid D(x,T)\}$

<u>Thm</u> If A, B, C hold, B is I^+ $\underset{\sim}{\Delta}^1_1$ then $\exists C \subseteq B$ I^+ Δ^1_1 s.t. $E_S \restriction C$ is $\underset{\sim}{\Sigma}^1_1$.

<u>Pf</u> Fix U on $\omega \times \omega \times \zeta$ satisfying (C). Fix $M \prec V_\Theta$ containing everything needed.

<u>Claim</u> If g is $\mathbb{P}_I$-generic over M and $x, T \in M[g]$ and $M[g] \models D(x,T)$ then $V \models D(x,T)$.

<u>Pf</u> By (C) $M[g] \models D(x,T)$

$M[g] \models (x,T) \in p[\check{U}]$, $V \models (x,T) \in p[U]$, $V \models D(x,T)$

Fix g $\mathbb{P}_I$-gen$/M$. B(B)

$M[g] \models \forall x \exists T\, D(x,T)$. So $\exists\, T \in M[g]$ s.t. $M[g] \models D(g,T)$

$[g]_{E_S} = p[T]$.

$G^{g,T}$:

I	$m_0 n_0$		$m_1 n_1$		...
II		α_0		α_1	

Rules:

(1) I plays integers m_i, n_i; II plays $\alpha_i < \delta$

(2) $(m_0 \dots m_{k-1}, n_0 \dots n_{k-1}) \in T$

(3) $(g \restriction k, m_0 \dots m_{k-1}, \alpha_0 \dots \alpha_{k-1}) \in S$

The first who ~~violates~~ violates rules, loses. If the play runs for ω steps, II wins.

Claim In $M[g]$ II has a w.s.

Pf If I has a w.s., say τ^*. $B_\delta(A)$

$M[g] \models S$ is homogeneous.

Fix $\vec{\mu}$ a homogeneity system for S.

We construct $(a_k \mid k \in \omega)$ $(b_k, k \in \omega)$, $(A_k \mid k \in \omega)$

$(a_0, b_0) = \tau^*(\emptyset)$ $A_0 = \emptyset$

Suppose $a_0 \dots a_{k-1}, b_0 \dots b_{k-1}, A_0 \dots A_{k-1}$ defined. Let

$h_k : S^{g \restriction k, \vec{a}} \to \omega \times \omega$

$$(\beta_0 \dots \beta_{k-1}) \longmapsto \tau^*(a_0, b_0 \dots a_{k-1}, b_{k-1})$$

$\mu_{g \restriction k, \vec{a}}$ concentrates on $S^{g \restriction k, \vec{a}}$ + is σ-cplt

So there is unique (a_k, b_k) s.t.

$$\underbrace{h_k^{-1}(a_k, b_k)}_{A_k} \in \mu_{g \restriction k, \vec{a}}.$$

Let $L : \omega \to \omega \times \omega$ $L(k) = (a_k, b_k)$

Claim that $L \in [T]$

If not: least k s.t. $L \restriction (k+1) \notin T$.

Denote $\mu_i = \mu_{g \restriction k, \vec{a}}$. If $i \le j$ let μ_j extend μ_i

$\pi_{k,i}^{-1}[A_i] \in \mu_k$ so the intersection $\in \mu_k$

$(\beta \cdots \beta_{k-1})$ in that set

Play $G^{g,T}$ when PI uses τ^* and II uses

$\beta_0 \cdots \beta_{k-1}$. $(\beta \cdots \beta_i) \in A_i = h_i^{-1}[S^{g \restriction i, \vec{a}}]$

II does not lose at k. But $a_0 b_0 \cdots a_k b_k$ is $L \restriction (k+1) \notin T$

So I is the first to lose. ⚡.

We showed: $L \in [T] = [g]_{E_S}$

Let $\bar{a} = (a_0, \ldots)$. This means $g E_S \bar{a}$, so $(g, \bar{a}) \in p[S]$.

$J: \omega \to \omega \times \omega$

$J(n) = (g(k), a_k)$

$(\mu_{J \restriction i})_i$ a w.f.T $A_i \in \mu_{J \restriction i}$ all $i \in \omega$

there is $\Phi: \omega \to \gamma$ s.t. $\Phi \restriction k \in A_k$ all k

Play $G^{g,T}$. I uses τ^* II uses Φ

No one loses at finite stage, so II wins. ⚡. ☒ Claim

Claim Fix $\tau \in M[g]$ w.s. for II. then τ is a w.s. for II in V.

Pf ~~[illegible]~~ B/C the game is closed.

<u>Claim 4</u> If $y \in \mathbb{R}^V$ ~~and~~ then $g E_s y \iff (S \cap M)^{(g,y)}$ illf.

<u>Pf</u> By Claim 1: $M[g] \models D(g,T) \implies V^{\mathbb{P}} \models D(g,T)$

So $[g]_{E_S} = p[T] \implies y \in p[T]$ so $(y,f) \in [T]$ some f.

T winning in U. Play (in V) $G^{g,T}$ s.t. I uses y and f II uses τ. Since $(y,f) \in S[T]$, I does not lose II cannot lose as τ is winning. Let $\phi(h) = \alpha_h$ = the II's h-th response according to k.

Since $T \in M[g]$: $\phi \upharpoonright k \in M[g]$. Since $\mathbb{P}_I$ is proper: M and $M[g]$ have same ordinals so $\phi \upharpoonright k \in M$.

$g E_s y \iff (S \cap M)^{(g,y)}$ illfd.

By properness: Let $C \subseteq B$ be $\underset{\sim}{\Delta}^1_1$ I^+ $\Big\backslash \frac{\text{generic for } M}{g,h \in C}$

$g E_s h \iff (S \cap M)^{(g,h)}$ illfd.

$E_s \upharpoonright C$ is $\underset{\sim}{\Sigma}^1_1$

(A) $\Vdash_{\mathbb{P}_I}$ S is homogeneous

$|\mathbb{P}_I| = 2^{\aleph_0}$

(B) $\forall x \exists T\, D(x,T)$ $\quad\Vdash_{\mathbb{P}} \ldots$

(C) $p[U] = \{\ldots D(x,T)\}$

<u>Thm</u> (with Magidor) $\mathbb{P}_I$ proper. Assume infinitely many Woodins with a measurable above.

$E \in L(\mathbb{R})$ all classes $\underset{\sim}{\Sigma}_1$ ($\underset{\sim}{\Pi}^1_1$, in $\underset{\sim}{\Delta}^1_1$) then there is C I^+ $\underset{\sim}{\Delta}^1_1$ s.t. $E \upharpoonright C$ is $\underset{\sim}{\Sigma}^1_1$ ($\underset{\sim}{\Pi}^1_1$, $\underset{\sim}{\Delta}^1_1$).

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(Martin) $AD_{\mathbb{R}}$. All trees on $\omega \times \gamma$, $\gamma < \Theta$ are weakly homogeneous

(Martin) AD + DC. $\forall A \subseteq \mathbb{R}$ ~~is Suslin~~ A is homogeneous iff A is Suslin co-Suslin

(+Woodin) $AD_{\mathbb{R}}$ all $A \subseteq \mathbb{R}$ are Suslin

In $AD_{\mathbb{R}}$ + DC all sets are homo Suslin

(Neeman - Norwood) $ZF + DC + AD_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))$

$\mathbb{P}$ proper $H \subseteq \mathbb{P}$ generic

$$j : L(\mathbb{R}^V) \longrightarrow L(\mathbb{R})^{V(H)}$$

does not move reals or ordinals