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I_0 (Woodin) There exists a λ and an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\operatorname{crit}(j) < \lambda$

Kunen (AC) There is no elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.

I_1 : $\exists\, j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$

Similarity: $L(\mathbb{R}) \sim L(V_{\lambda+1})$

$$\begin{array}{ccc} L(\mathbb{R}) & \sim & L(V_{\lambda+1}) \\ AD & & I_0 \end{array}$$

First considered by Woodin.

Woodin: I_0 implies that λ^+ is measurable in $L(V_{\lambda+1})$.

Theorem (Cramer) (I_0 at λ). In $L(V_{\lambda+1})$ every subset of $V_{\lambda+1}$ has the λ-splitting perfect set property.

Woodin's AD-conjecture

Question Do there always exist strong determinacy models corresponding to very large cardinals

AD-conjecture Roughly: "Yes, there is evidence for this"

The AD-conjecture for I_0 says:

I_0 at $\lambda \Rightarrow$ every subset of $V_{\lambda+1}$ has a $U(j)$-representation

$U(j)$-representation is analogous to weakly homogeneously Suslin representation.

Def $A \subseteq \mathbb{R}$ is κ-Suslin iff $A = p[T]$ for some T on $\omega \times \kappa$. A is κ-homogeneously Suslin iff such a T is augmented by κ-complete measures.

A is weakly homogeneously Suslin iff A is a projection of a homogeneously Suslin set.

Measures for $U(j)$ are found as follows.

Fix some n and consider

$$S = \langle j_\alpha \mid \alpha < \lambda \rangle$$

where for all $\alpha < \lambda$:

$$j_\alpha : L_n(V_{\lambda+1}) \to L_n(V_{\lambda+1})$$

Let

$$D^S = \{ a \in L_n(V_{\lambda+1}) \mid \forall \alpha < \lambda \ \ j_\alpha(a) = a \}$$

Partition $L_n(V_{\lambda+1})$ into $< \lambda$ many pieces on which the collection of sets of the form D^S generate ultrafilters.

<u>Theorem</u> (Cramer) AD conjecture for I_0 holds.

<u>Theorem (Woodin)</u> Assume the AD-conjecture for I_0 and let λ be a limit of supercompacts. ~~and~~ Also assume proper class of Woodin cardinals. Assume I_0 at λ.

Let $G \subseteq \mathrm{Coll}(\omega, \lambda)$ be V-generic and let Γ^∞_G be the set of all universally Baire sets in $L(V_{\lambda+1})[G]$. Then

(1) $L(\Gamma^\infty_G) \models LSA$

(2) $\Theta^{L(\Gamma^\infty_G)} = \Theta^{L(V_{\lambda+1})}$

<u>Question</u> What is the largest Suslin cardinal of $L(\Gamma^\infty_G)$?

<u>Consequence</u> Uniformization for $L(V_{\lambda+1})$ is unrelated to $U(j)$-representation.

<u>Question</u> (Woodin) Does the relation

$$R = \{(j,k) \mid j,k : V_\lambda \to V_\lambda \text{ elementary and they extend to elementary } j^+, k^+ : V_{\lambda+1} \to V_{\lambda+1} \text{ such that } k^+(k) = j\}$$

Have a uniformization in $L(V_{\lambda+1})$?

<u>Laver</u> If $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ then for every $a, b \in V_{\lambda+1}$ there is a $k : V_{\lambda+1} \to V_{\lambda+1}$ s.t.

(1) $k(k \restriction V_\lambda) = j \restriction V_\lambda$

(2) $k(a) = j(a)$

(3) $b \in \operatorname{rng}(k)$

Fix $j : L_3(V_{\lambda+1}) \to L_3(V_{\lambda+1})$

$R \upharpoonright \{ (j', k) \mid j' : L_2(V_{\lambda+1}) \to L_2(V_{\lambda+1})$ and

$j(j' \upharpoonright V_\lambda) = j \upharpoonright V_\lambda$ and $k \in \mathrm{rng}(j') \}$

<u>Note</u> $R \in L_2(V_{\lambda+1})$

Fix k s.t. $(j, k) \in R$.

set of all liftings of maps in field(R)

$\mathcal{U} = \{ (j', (j')^{-1}(k)) \mid j' \in A \}$

$(j, k) \in R = J(R)$, $J_0 \in A$

$(J_0^{-1}(j), J_0^{-1}(k)) \in J_0^{-1}(J(R))$

$(J_0 \upharpoonright V_\lambda, J_0^{-1}(k)) \in R$

<u>Fact</u> If $J_0(j_0) = J$ and $a \in V_{\lambda+1}$ is s.t. $a \in \mathrm{rng}(j_0)$ then $J_0(a) = J(a)$

Then $\mathcal{U}$ is an uniformization for $R \upharpoonright A$

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Other consequences of AD-conjecture for I_0

(1) Generic absoluteness result for I_0 (Woodin, Cramer)

(2) New proof of λ-splitting Perfect Set Property for I_0 (Woodin-Shi)

(3) Con(I_0 at λ + $\neg$SCH at λ) follows from Con($I_0^{\#}$)

Let M_ω be the ω-th iterate of $L(V_{\lambda+1})$ by j.
and let $j_{0,\omega} : L(V_{\lambda+1}) \to$ ~~$L(V_{\lambda+1})$~~ M_ω

Thm (Woodin) Let j be an I_0-embedding

$$L_\lambda(V_{\lambda+1} \cap M_\omega[\vec{\kappa}]) \prec L_\lambda(V_{\lambda+1})$$

where $\vec{\kappa}$ is the critical sequence of j. It is cofinal in λ and it is Prikry generic over M_ω.

Thm (Woodin) Suppose $\mathbb{P} \in j_{0,\omega}(V_\lambda)$ and $g \in V$ is $\mathbb{P}$-generic over M_ω and $cf(\lambda)^{M_\omega[g]} = \omega$.
Then

$$L_\lambda(V_{\lambda+1} \cap M_\omega[g]) \prec L_\lambda(V_{\lambda+1})$$

__Theorem (C.)__ Assume $I_0^\#$ at λ and let $\mathbb{P} \in M_\omega$ and $g \in L(V_{\lambda+1})$ is $\mathbb{P}$-generic over M_ω and $cf(\lambda)^{M_\omega[g]} = \omega$. Then there is an elementary embedding

$$\pi : L(V_{\lambda+1} \cap M_\omega[g]) \longrightarrow L(V_{\lambda+1})$$

and $\pi \restriction \lambda = id$. Also note $V_\lambda^{M_\omega} = V_\lambda$, as $\vec{\kappa}$ is cofinal in λ.

New theorem involves a representation called a __j-Suslin representation__; these representations are __not__ augmented by measures.

__Def__ $A \subseteq V_{\lambda+1}$ has a (j, κ)-Suslin representation T if the following hold: For some cofinal $\vec{\lambda}$ in λ:

(1) T is a tree on $V_\lambda \times L_\eta(V_{\lambda+1})$ s.t.

$\forall (s, a) \in T$:

$s = (s_0, \ldots, s_n)$, $s_i \subseteq V_{\lambda_i}$, $s_i = s_{i+1} \cap V_{\lambda_i}$..

(2) "$A = p[T]$"

(3) $\forall (s,a) \in T$ there is an n such that

$$j_n(s,a) = (s,a)$$

where j_n is the n-th iterate of j, a fixed I_0-embedding in advance.

(4) $\forall s \in V_x$ $\exists n$ such that

$$j_n(T_s) = T_s$$

where

$$T_s = \{ a \mid (s, a) \in T \}$$

<u>Remark</u> Since j is iterable, a could be a sequence of ordinals