$I_0$ (Woodin) There exists a $\alpha$ and an elementary embedding $j: L(\mathcal{V}_{\alpha+1}) \rightarrow L(\mathcal{V}_{\alpha+1})$ with $\text{crit}(j) = \alpha$

Kunen (AC) There is no elementary embedding $j: \mathcal{V}_{\alpha+2} \rightarrow \mathcal{V}_{\alpha+2}$

$I_1$: $\exists j: \mathcal{V}_{\alpha+1} \rightarrow \mathcal{V}_{\alpha+1}$

Similarity $L(\mathbb{R}) \sim L(\mathcal{V}_{\alpha+1})$

AD $\& I_0$

First considered by Woodin.

Woodin: $I_0$ implies that $\alpha^+$ is measurable in $L(\mathcal{V}_{\alpha+1})$.

Theorem (Cramer) ($I_0$ at $\chi$) in $L(\mathcal{V}_{\alpha+1})$ every subset of $\mathcal{V}_{\alpha+1}$ has the $\chi$-splitting perfect set property.
Woodin's AD-conjecture

Question: Do there always exist strongly determinacy models corresponding to very large cardinals?

AD-conjecture: Roughly: "Yes, there is evidence for this."

The AD-conjecture for $I_0$ says:

$I_0 \Rightarrow \forall x \exists y \forall M (M \models \text{AD} \rightarrow \forall A \subset y, A \in V_{x+1} \exists \mathcal{U}(y) \text{-representation})$.

$\mathcal{U}(y)$-representation is analogous to weakly homogeneously Suslin representation.

Def: $A \subseteq \mathbb{R}$ is $n$-Suslin iff $A = \mathcal{P}(T)$ for some $T$ on $\omega \times \omega$. $A$ is $n$-homogeneously Suslin iff such a $T$ is augmented by $n$-complete measures.

$A$ is weakly homogeneously Suslin iff $A$ is a projection of a homogeneously Suslin set.
Measures for \( U_j \) are found as follows.

Fix some \( n \) and consider

\[
S = \{ \phi \mid \phi \in A \}
\]

where for all \( a \in A \):

\[
J_a : L_n(U_{\lambda+1}) \to L_n(U_{\lambda+1})
\]

Let

\[
S = \{ a \in L_n(V_{\lambda+1}) \mid \forall x \in x J_a(x) = a \}
\]

Partition \( L_n(U_{\lambda+1}) \) into \( \prec \)-maximal pieces on which the collection of sets of the form \( S \) generate ultrafilters.

**Theorem (Cramer)** AD conjecture for \( I_0 \) holds.

**Theorem (Woodin)** Assume the AD-conjecture for \( I_0 \) and let \( \lambda \) be a limit of supercompacts, and also assume proper class of Woodin cardinals.

Assume \( I_0 \) at \( \lambda \).

Let \( G \in \text{coll}(\mu, \lambda) \) be \( V \)-generic and let \( \Gamma^G \) be the set of all universally Baire sets in \( L(U_{\lambda+1})[G] \). Then

1. \( L(\Gamma^G) = L S A \)
2. \( \Theta^L(\Gamma^G) = \Theta^L(U_{\lambda+1}) \)
Question: What is the largest Suslin cardinal of $L(\mathcal{G}^\omega)$?

Consequence: Uniformization for $L(V_{\kappa+1})$ is unrelated to $U(y)$-representation.

Question (Woodin): Does the relation

$$R = \{(j,k) | j,k: V_\lambda \rightarrow V_\kappa \text{ elementary and they extend to elementary } j^*: V_{\kappa+1} \rightarrow V_{\lambda+1},$$

such that $j^*(k) = j\}$$

have a uniformization in $L(V_{\lambda+1})$?

Lemma: If $j: L(V_{\lambda+1}) \rightarrow L(V_{\kappa+1})$ then for every $a,b \in V_{\lambda+1}$ there is a $k: V_{\kappa+1} \rightarrow V_{\lambda+1}$ s.t.

1. $k(b \cap V_\kappa) = j(b \cap V_\lambda)$
2. $k(a) = j(a)$
3. $b \in \text{rng}(k)$
Fix $j : L_3(V_{x+1}) \to L_3(V_{x+1})$

R\{j'' \mid j' : L_2(V_{x+1}) \to L_2(V_{x+1}) \text{ and } j''(V_{x+1}) = j'(V_{x+1}) \text{ and } \ker(y(j''))\}

Note $R \in L_2(V_{x+1})$

Fix $k$ s.t. $(j, k) \in R$.

$U = \{ (j', (j')^{-1} k) \mid j' \in A \}$

$(j, k) \in R = J(R)$ 1 1 0 0 0

$(j_0^{-1}(k), j_0^{-1}(k)) \in j_0^{-1}(J(R))$

$(j_0^{-1}(V_x), j_0^{-1}(k)) \in R$

Fact: If $j_0(j_0) = j$ and $a \in V_{x+1}$ s.t. $a \in \ker(j_0)$ then $j_0(a) = j(a)$

Then $U$ is an abbreviation for $R \in A$
Other consequences of AD-conjecture for \( I_0 \)

1. Genuine absoluteness result for \( I_0 \) (Woodin, Chameleon)

2. New proof of \( \lambda \)-splitting perfect set property for \( I_0 \) (Woodin–Shi)

3. \( \text{Con}(I_0 \text{ at } \lambda + \text{TSCH at } \lambda) \) followed from \( \text{Con}(I_0^\#) \)

Let \( M_\omega \) be the \( \omega \)-th iterate of \( L(V_{\omega+1}) \) by \( j_\omega \), and let \( j_{\omega+1} : L(V_{\omega+1}) \rightarrow M_\omega \).

Then (Woodin) let \( j \) be an \( I_0 \)-embedding

\[
L_x(V_{\lambda+1} \cap M_\omega^{j_{\omega+1}}) < L_x(V_{\lambda+1})
\]

where \( \vec{w} \) is the critical sequence of \( j \). It is cofinal in \( \lambda \) and it is Prikry generic over \( M_\omega \).

Then (Woodin) suppose \( R \subseteq j_{\omega+1}(V_\lambda) \) and \( g \in V \) is \( R \)-generic over \( M_\omega \) and \( cf(\lambda)M_\omega[G] = \omega \).

Then

\[
L_x(V_{\lambda+1} \cap M_\omega[G] \cap j_{\omega+1}(V_\lambda)) < L_x(V_{\lambda+1})
\]
Theorem (C.) Assume \( I_0^+ \) at \( \lambda \) and let \( P \in \mathcal{M}_\omega \) and \( q \in L(V_{\lambda+1}) \) be \( P \)-generic over \( \mathcal{M}_\omega \) and of \( \lambda \) \( \mathcal{M}_\omega \)-club. Then there is an elementary embedding

\[
\mathbb{E} : L(V_{\lambda+1} \cap \mathcal{M}_\omega) \rightarrow L(V_{\lambda+1})
\]

and \( \mathbb{E}^\lambda = \text{id} \). Also note \( V_{\lambda}^{\mathcal{M}_\omega} = V_{\lambda} \). \( \mathcal{M}_\omega \) is cofinal in \( \lambda \).

New theorem involves a representation called a \( j \)-Suslin representation; these representations are not augmented by measures.

\textbf{Def} \( A \subseteq V_{\lambda+1} \) has a \( (j, k) \)-Suslin representation \( T \) if the following hold: For some club \( \delta \) in \( \lambda \):

1. \( T \) is a tree on \( V_\lambda \times L_n(V_{\lambda+1}) \)

\[
H(s, a) \in T
\]

\[
s = (s_0, \ldots, s_\nu), \quad s_\nu \subseteq V_\lambda, \quad s_\nu = s_{\nu+1} \cap V_\lambda,
\]

2. “\( A = p[T] \)”

3. \( H(s, a) \in T \) there is an \( n \) such that

\[
J_n(s, a) = (s, a)
\]

where \( J_n \) is the \( n \)-th iterate of \( J \),
a fixed $I_0$-embedding in advance.

(4) Use $U_x$ in such that

$$J_m(T_x) = T_x$$

where

$$T_x = \{ a : (s, a) \in T \}$$

Remark: Since $J$ is iterable, $a$ could be a sequence of ordinals.