EXTERNAL ULTRAPowers OF HoD BY

We assume \( V = L(R) + \text{AD} \)

**Def.** \( M \subseteq L(R) \) is an inner model and \( \mu \in L(R) \)
a measure on some ordinal.

Define external ultrapower of \( M \), \( \text{Ext}(M, \mu) \)
using all functions \( f \in V \).

\[ J^M_{\mu} : M \rightarrow \text{Ext}(M, \mu) \]

If \( M = \text{AC} \) then \( J^M_{\mu} \) is fully elementary

Background: Woodin used internal ultrapowers
of HoD to show

\[ \text{HoD} \models \exists \gamma \text{ is strong up to } \Theta \text{ in HoD} \]

**Def.** (Jackson) \( \mathcal{W}_1 \) is the club measure on \( \omega_1 \)

Will be studying \( J_{\mathcal{W}_1} : \text{HoD} \rightarrow \text{Ext}(\text{HoD}, \mathcal{W}_1) \).

Outline

1. \((\mathcal{W}_1, \mathcal{W}_2)\) - extends from \( J_{\mathcal{W}_1} \)
2. codes for cardinals less than \( \omega_2 \)
3. as much \( J_{\mathcal{W}_1} \) as I can
4. Global results
Review of Jackson-Ketlencid paper

(An initial segment property + Aron (compactness quantification)

1. \( \text{HOD is minimal} \)

\[ M \vDash \text{min} \Leftrightarrow \forall N \text{ minimal iff a tenindle } \Theta \text{ s.t.} \]

\[ (1) \ L[M^{\Theta^*}] = \varnothing \]

\[ (2) \ \forall \gamma < \delta^\text{M} \ L[M^{\gamma}] = \varnothing \]

2. \( M, N \) are I.M. with minimal witnesses by \( \Theta \) and of \( M, N \) have comparison

\[ P \leftarrow M \quad \rightarrow N \leftarrow Q \]

\[ \text{no drop in either side} \]

\[ \text{Below we} \]

The HOD and Ext (HOD, \( \omega_1 \)) have a successful comparison (Steel)

Notation \( \mu = \omega_1 \cap \text{HOD} \)

\[ \text{Ult}^x(\text{HOD}, \mu) = \text{the } \alpha \text{-th } \text{ult} \]

\[ \iota_x: \text{HOD} \rightarrow \text{Ult}^x(\text{HOD}, \mu) \text{ the embedding} \]
Then if $\text{HOD} \models P$

$\text{Ext} \models \varphi$

Then in the first $w_2$ steps the $\text{Ext}$ does not move
while $\text{HOD}$ decides $\varphi$.

$w_2$ is the least measurable. Analogously $\text{HOD} \models \psi$.

Thus let $\mathcal{E}$ be $(w_1, w_2)$-extender derived from $\mathcal{J}_{w_1}$

and let $\{\mathcal{E}_x | x < z\}$ enumerate the generators. Let

$\mu_x = \text{the measure on } \mathcal{E}_x$ derived from

$\text{Ult}(\text{HOD}, \mathcal{E}) \rightarrow \text{Ult}(\text{HOD}, \mathcal{E})$

Thus

1. $\mathcal{E}_x = \mathcal{E}_x(w_1)$
2. $\mu_x = \mu_x(\mu)$

Lemma (3.12) Let $\mathcal{M}$ be an ultrafilter of $\text{HOD}$.

Let $\kappa \in \mathcal{M}$ be an $\mathcal{M}$-cardinal st. there are

no total extenders overlapping $\kappa$. Let $\mathcal{O}$ be a proper class. Then

$\mathcal{P}(\kappa)^\mathcal{M} \subseteq \text{Hull}(\mathcal{M}, (\kappa \cup \mathcal{O}))$

Proof: $\mathcal{M} \models \varphi$

$\text{Hull} \mathcal{M} \models \varphi$.

$\square$
Proof of Theorem

1. Fix $d < \omega_1$

   \[(i) \quad \beta \leq d \rightarrow \xi_\beta = \hat{\beta}(\omega_1)\]

   \[(ii) \quad \mu_\beta = \xi_\beta(\mu)\]

Verify for $d+1$:

Assume $\xi_{d+1} < \hat{d+1}(\omega_1)$

Now $\xi_{d+1} = \xi_{d+1}(d)$ ($f_0^d$)

where $f : \xi_d \rightarrow \xi_d$

$f$ is definable in $\text{L}^{\omega_1}_d (\text{HOD}, \eta)$ from ordinals in $\Sigma_1$ when $\eta < d$,

as fixed by $\eta$ and $d$. So $\xi_{d+1}(d)$ is definable in $\text{L}^{\omega_1}_d (\text{HOD}, \eta)$ from $\xi_\eta \cap \Sigma_1$ so $\xi_{d+1}$ is definable in $\text{L}^{\omega_1}_d (\text{HOD}, \eta)$ from $\xi_\eta \cup \{ \xi \}$.

Point $\xi_{\alpha+1} = \tau(\bar{\beta})$ when $\bar{\beta}$ is fixed

by all maps. So

$\xi_{\alpha+1} = \tau^\alpha(\bar{\beta}) = \tau^\alpha(\bar{\beta}) = \xi_\beta(\mu_{\xi_\beta(\omega_1)}) > \xi_\beta(\omega_1)$

$\xi_\beta(\omega_1) \Rightarrow \xi_{\alpha+1}$
Assume \( A \in \mu_{\omega+1} \cap i_{\omega+1}(\mu) \).

Standard \( A = \tau \mu^{\omega+1}(\bar{\beta}) \) where \( \bar{\beta} \) is fixed by \( k \).

Then

\[ i_\omega(\omega_1) \in k(A) = \tau \mathcal{E}xt(\bar{\beta}) \]
\[ i_\omega(\omega_1) \not\in i_\gamma(A) = \tau \mathcal{P}(\bar{\beta}) \]

\[ \text{exit } (i_\gamma) \rightarrow i_{\omega+1}(\omega_1) \]
\[ i_{\omega+1}(\omega_1) \in i_\gamma(k(A)) = \tau \mathcal{Q}(\bar{\beta}) = \tau \mathcal{P}(\bar{\beta}) \]

Codes for ordinals \( \omega_1 \)

Fact (Martin): \( j_{\omega_1}^\omega(\omega_1) = \tau \omega_{\omega+1} \).

(essentially \( \delta_3 = \omega_{\omega+1} \))

So \( f: \omega_1 \rightarrow \omega_1 \); view \( f \) as coding \( \mathcal{C}(f) = \omega_1 \)

Question (1-1): Which ordinals are coded by \( f \)?

Det: \( f: \omega_1 \rightarrow \omega_1 \) "codes a gap" \( \omega_1 \)

\( \sup \{ f(\xi) | \xi \in \text{HOD} \} \subset \omega_1 \) \( \subset \mathcal{C}(f) \subset \omega_1 \) \( \subset \mathcal{C}(f) \).
Thus \( f : \omega \to \text{HOD} \) then \( f \) begins a gap.

Let \( \text{Ext} \) be extended that

Analyse \( k \). Let \( F \) be extend that comes from iterating \( \text{Ex} \) \( w_2 \) times.

Fact 1 \( \text{Ext} ( \text{Ext} (HOD, \mu^2), F) = \text{Ext} w_{2+1} (HOD, \mu) \)

**Proof** \( \text{Ext} M \triangleright \text{Ext} (HOD, \mu^2) \)

\( \text{WTG} : \text{Ex} (M) \triangleright \text{Ext} (HOD, \mu) \)

\( \text{Ext} (HOD, \mu) \not\models M \) is an L.S. of \( \text{Ex} \) \( \mu \) by \( \text{Ex} (\mu) \)

Now apply \( F \).
Fact. For the $(\langle w_1, w_2 \rangle, \mu)$-extender defined from $\kappa$.

Proof.

Now run the argument as above using definability from a class of ordinals that are fixed. □

$\text{HOD} \cong \text{Ult}(\text{HOD}, \mu) \\
\text{Ult}(\text{HOD}, \mu) \cong \text{Ult}(\text{HOD}, \mu^+)$

$f$ being a gap iff $\kappa$ is discontinuous at $(\mu)^{\mu^+}$.

$\text{Ult}(\text{HOD}, \mu) = \text{Ult}(\text{HOD}, \mu^+)$ and $\omega_{\omega+1}(w_2) > \omega_{\omega+1}(w_1) > \omega_{\omega+1}(w_1)$.

So $\kappa$ is discontinuous at $(\mu)^{\mu^+}$. □
Let \( \kappa \) be the least measurable of Mitchell order 1 in HOD and let \( \gamma = \sup J_{\kappa_1}[\kappa] \). We can analyze the \((\kappa, \gamma)\)-extend from \(J_{\kappa_1}\).

\[ \begin{array}{cc}
\kappa & \text{the 2nd measurable} \\
F & \text{measures on same} \\
\kappa_2 & \\
\kappa_1 & \\
\kappa & \text{HOD} \\
\kappa & \text{for } \kappa_2 \\
\kappa_1 \text{ (HOD)} & \text{Ext} \\
\kappa_2 \text{ (HOD)} & \text{Ext} \\
\kappa_1 & \\
\end{array} \]

\( \kappa_2 \) is continuous at \( \kappa_2 \) while \( J_{\kappa_1} \) is not.