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Analyzing $i_{\omega_1} : \mathrm{HOD} \to \mathrm{Ext}(\mathrm{HOD}, \omega_1)$

Last time: (ω_1, ω_2)-extender

Thm Let $\kappa \in \mathrm{HOD}$ be the least with Mitchell order 1. Let $\gamma = j_{\omega_1}[\kappa]$ and let E be the (ω_1, γ)-extender from j_{ω_1} and $(\xi_\alpha)_\alpha$ enumerate its generators

(i) ξ_α is measurable in $\mathrm{Ult}(\mathrm{HOD}, E \restriction \xi_\alpha)$ and the measure derived from

$\mathrm{Ult}(\mathrm{HOD}, E \restriction \xi_\alpha) \to \mathrm{Ult}(\mathrm{HOD}, E)$

is the order 0 measure in $\mathrm{Ult}(\mathrm{HOD}, E \restriction \xi_\alpha)$

(ii) $\xi_{\alpha+1}$ is the least measurable $> \xi_\alpha$ in $\mathrm{Ult}(\mathrm{HOD}, E \restriction (\xi_\alpha + 1))$

(iii) For α limit

Case 1 there is a single measure that has been hit cofinally often in

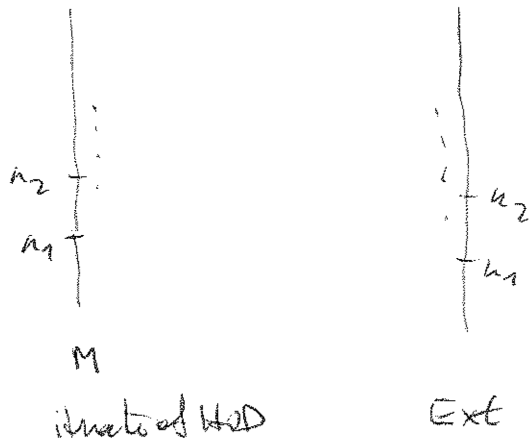
$\mathrm{Ult}(\mathrm{HOD}, E \restriction \xi_\beta) \longrightarrow \mathrm{Ult}(\mathrm{HOD}, E \restriction \xi_{\beta+1})$

Subcase a $\sup_{\beta<\alpha} \xi_\beta$ is measurable. ($\beta < \alpha$) in Ext. Then ξ_α is the least measurable $> \sup_{\beta<\alpha} \xi_\beta$

Subcase b $\sup_{\beta<\alpha} \xi_\beta$ not measurable in Ext

then $\xi_\alpha = \sup_{\beta<\alpha} \xi_\beta$

<u>Case 2</u> Not hitting a single measure cofinally often



then ξ_α is the least measurable in $Ult(HOD, E\restriction\xi_\alpha)$

$\geq \sup_{\beta<\alpha} \xi_\beta$

<u>Thm</u> (Steel's dichotomy) $\kappa \in HOD$ a HOD-regular cardinal. Then

(i) $cf^{L(\mathbb{R})}(\kappa) = \omega$

XOR

(ii) κ is measurable in HOD.

<u>Thm</u> Let κ be an Ext regular cardinal then

(i) $cf^{L(\mathbb{R})}(\kappa) \leq \omega_1$

XOR

(ii) κ is measurable in Ext

<u>Pf</u> κ measurable in Ext $\Rightarrow cf^{L(\mathbb{R})}(\kappa) > \omega_1$

Why: Say $\kappa = [f]_{\omega_1}$ $\forall^*\alpha$ $f(\alpha)$ is HOD regular and $cf^{L(\mathbb{R})}(f(\alpha)) \geq \omega_1$

Let $h(\alpha) = cf^{L(\mathbb{R})}(f(\alpha))$. Then

<u>Claim 1</u> $cf^{L(\mathbb{R})}([h]) > \omega_1$

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(3)

Generic codes for ctbl ordinals (Martin) Folklore

There is $G : \omega_1^\omega \to \omega^\omega$ s.t. for $\alpha < \omega_1$ then

$$\underbrace{\forall^* s \in \alpha^\omega}_{\text{comeager}} \; |G(\alpha^\frown s)|_{WO} = \alpha$$

Now assume C1 fails and let $\{\gamma_\alpha\}_{\alpha<\omega_1} \to [h]$

By uniformization + coding: get F s.t.

$x \in WO$ $\quad F(x)$ is a code for a function representing $\gamma_{|x|}$.

For $\alpha < \omega_1$, $s' \in \alpha^{<\omega}$ and $\beta < \omega_1$ $\quad$ here $s \supseteq s'$.

$$H(\alpha, s', \beta) = \begin{cases} \delta & \text{if } \forall^*_{B_{s'}} s \; F(G(\alpha^\frown s))(\beta) = \delta \\ 0 & \text{ow} \end{cases}$$

Then define

$$h_\alpha(\beta) = \sup_{s' \in \alpha^{<\omega}} H(\alpha, s', \beta)$$

<u>Claim</u> $[h_\alpha] = \gamma_\alpha$. all $\alpha < \omega_1$

If not: let $[f_\alpha] = \gamma_\alpha$. Then $\exists A \in \omega_1^\omega$

$\forall \beta \in A \quad h_\alpha(\beta) < f_\alpha(\beta)$ but

$\forall^* s \in \alpha^\omega$ $\cancel{\forall \beta \in A}$ $\exists \beta_0 \in A \quad F(G(\alpha^\frown s))(\beta_0) = f_\alpha(\beta_0)$

Under AD can switch quant-fiers for $s \in N_{s'}$ some s'.

Then get a contradiction

$[h_\alpha] = \gamma_\alpha$. Let

$$f(\beta) = \sup_{\alpha<\beta} h_\alpha(\beta)$$

then $[f] > [h_\alpha]$ $\forall\alpha$ and $[f] < [\bar{h}]$. ↯

So we have:

$\forall^*_{W^1_1}\alpha$ $f(\alpha)$ has $\mathrm{cof}^{L(\mathbb{R})} \geq \omega_1$

Set $h(\alpha) = \mathrm{cf}^{L(\mathbb{R})}(f(\alpha))$. Have $\mathrm{cf}^{L(\mathbb{R})}([h]) > \omega_1$.

<u>Claim 2</u> $\mathrm{cf}^{L(\mathbb{R})}([f]) = [h]$

Jackson, using SPP on ω_1.

<u>Fact</u> $\vec{\eta} < \gamma$ and (η, f) where

$$\forall^*_{E_\eta}\bar{x}\ \ \mathrm{cf}^{L(\mathbb{R})}(f(\bar{x})) = \omega_1$$

then

$$\mathrm{cf}^{L(\mathbb{R})}([\vec{\eta}, f]_{E \restriction \sup(\vec{\eta})}) \in (\omega, \omega_1)$$

Use these two facts to prove the theorem.

<u>Q</u> κ least Mitchell order 1 in HOD. We can show this process continues until the image of κ is measurable in Ext. Does the image of κ have Mitchell order 1 in Ext?

Probably yes.

Lemma (J-K) M an iterate of HOD. κ with no total extenders overlapping it in M. Then

$$(2^\kappa)^M \subseteq \mathrm{Hull}^M(\kappa \cup \Gamma)$$

for some proper class of ordinals.

Notation $\xi \in On$, P some property

$$\forall^*_{\omega_1^\xi} \alpha \quad P(\xi(\alpha))$$

means: $\forall f$: if $[f] = \xi$ then $\forall^*_{\omega_1^\xi} \alpha \; P(f(\alpha))$

Let E be the full extender from j_{ω_1}. Given $f \in \mathrm{HOD}$, $\bar{\eta}$ some generators: what is the $g \in L(\mathbb{R})$ s.t. $[\bar{\eta}, f]_E = [g]_{\omega_1^\xi}$?

Theorem Define π_ξ for $\bar{\eta} < \xi$ and $f \in \mathrm{HOD}$ by

$$\forall^*_{\omega_1^\xi} \alpha \quad \pi_\xi([\bar{\eta}, f]_{E \upharpoonright \xi})(\alpha) = f(\bar{\eta}(\alpha))$$

Then $\pi_\xi = j_\xi : \mathrm{Ult}(\mathrm{HOD}, E \upharpoonright \xi) \to \mathrm{Ult}(\mathrm{HOD}, E)$

$\eta \in On$, (η, f)

Theorem Let $\alpha \in On$ s.t. $\mathrm{cf}^{L(\mathbb{R})}(\alpha) \neq \omega_1$. Let c_α be the constant function with value α. Then (ω_1, c_α) is never a generator; this means:

$\xi \neq [\bar{\omega}_1, c_\alpha]_{E \upharpoonright \xi}$ when $\xi = \mathrm{cr}(j_\xi)$,

$$j_\xi : \mathrm{Ult}(\mathrm{HOD}, E \upharpoonright \xi) \to \mathrm{Ult}(\mathrm{HOD}, E)$$

Pf If there were such a ξ, say $\xi = [\bar{\omega}_1, c_\alpha]_{E|\xi}$ and $\xi = \mathrm{cr}(j_\xi)$. Then $\pi_\xi([\bar{\omega}_1, c_\alpha])_{E|\xi} > \xi$.

In $L(\mathbb{R})$ there is g s.t.

$\forall^*_{w_1^1} \beta \;\; g(\beta) < \alpha$ and f.a. $\bar{\eta} < \xi$ and $f \in \mathrm{HOD}$ s.t. $[\bar{\eta}, f]_{E|\xi} < [\bar{\omega}_1, c_\alpha]$

$\forall^*_{w_1^1} \alpha' \;\; f(\bar{\eta}(\alpha')) < g(\alpha')$

Impossible, as $\mathrm{cf}^{L(\mathbb{R})}(\alpha) \neq \omega_1$. So $\mathrm{rng}(g)$ bounded in α.

If γ is the upper bound then $\pi_\xi([\bar{\omega}_1, c_\gamma]_{E|\xi}) > [g]$.

Theorem $\bar{\eta} \in \mathrm{Ord}$ and $g \in \mathrm{HOD}$ s.t.

$\forall^*_{E_\xi} \alpha \;\; \mathrm{cf}^{L(\mathbb{R})}(g(\alpha)) \neq \omega_1$ then $(\bar{\eta}, g)$ is never a generator, i.e. $\xi \neq [\bar{\eta}, g]_{E|\xi}$ where $\xi = \mathrm{cr}(j_\xi)$.

$$\begin{tikzcd} \mathrm{HOD} \arrow[r, "i_E"] \arrow[d, "j_{w_1^2}"'] & \mathrm{Ext}(\mathrm{HOD}, w_1^1) \arrow[dl, "k"] \\ \mathrm{Ext}(\mathrm{HOD}, w_1^2) & \end{tikzcd}$$

$\mathrm{cr}(k) = \omega_2$

$w_1^2 = w_1^1 \times w_1^1$

Then the extender derived from k is $i_E(E)$.

Theorem $\xi < \Theta$. $j_\xi : \mathrm{Ult}(\mathrm{Ext}(\mathrm{HOD}, w_1^1), i_E(E)|\xi) \to \mathrm{Ext}(\mathrm{HOD}, w_1^2)$

$[\eta, f]$ $\eta \in \xi^{<\omega}$ $[\eta, f]_{i(E)|\xi} \in M$. Say $[\eta, f] = [\beta, \rho]_E \in \mathrm{Ext}(\mathrm{HOD}, w_1^1)$.

$\forall^*_{E_\beta} \alpha \quad g(\alpha) = \xi(\eta_\alpha, f_\alpha)$ where

$[\bar{\beta}, \alpha \mapsto \eta_\alpha]_E = \eta$ and $[\bar{\beta}, \alpha \mapsto f_\alpha]_E = f$

$\forall^*_{w_1} \alpha \ \forall^*_{w_2} \beta \quad \pi_\xi([\eta, f]_{i_E(E) \restriction \xi})(\alpha, \beta) = f_{\beta(\alpha)}(\eta_{\beta(\alpha)}(\beta))$

then $\pi_\xi = j_\xi$.