On Steel’s Conjecture

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We discuss some joint work with R. Atmai with also contributions by H. Woodin.

The theory \( \text{ZF} + \text{AD} + \text{DC} \) gives a complete picture of the scale property below the supremum of the Suslin cardinals.

This theory also gives a complete picture of the prewellordering property. For every Levy class \( \Gamma \) either \( \text{pwo}(\Gamma) \) or \( \text{pwo}(\check{\Gamma}) \).

There is an important closure question about pointclasses, Steel’s conjecture which is open. We introduce a notion called the spectrum of a pointclass, show how it relates to the conjecture, and use it to prove some related results.

We henceforth assume \( \text{ZF} + \text{AD} + \text{DC} \).
Basic Definitions

A pointclass is a $\Lambda \subseteq \mathcal{P}(\omega^\omega)$ closed under Wadge reduction, i.e., if $A \in \Lambda$ and $B \leq_w A$, then $B \in \Lambda$.

- We usually write $\Gamma$ for nonselfdual pointclasses, $\delta$ for selfdual classes, and $\Lambda$ for either.

For $\Gamma$ a pointclass, $\Delta(\Gamma) = \Gamma \cap \check{\Gamma}$.

We let $o(\Lambda) = \sup\{|A|_w : A \in \Delta(\Lambda)|$.

A Levy class is a nonselfdual pointclass $\Gamma$ closed under $\exists^\omega\omega$ or $\forall^\omega\omega$ (or both).
Definition
A **projective algebra** is a selfdual pointclass \( \delta \) closed under \( \exists^{\omega}, \forall^{\omega}, \lor, \land \).

For any pointclass \( \Lambda \), there is a largest projective algebra \( \delta \) contained in \( \Lambda \).

Fact
*If \( \Delta \) is a projective algebra, then*

\[
\sigma(\Delta) = \sup\{|A|_w : A \in \Delta\} = \sup\{|\leq| : \leq \text{ is a } \Delta \text{ prewellordering}\}
\]
Let $\Gamma$ be a Levy class. Let $\Lambda = \Lambda(\Gamma)$ be the largest projective algebra contained in $\Gamma$. Let $\kappa = o(\Lambda)$.

Then $\Gamma$ is in a **projective hierarchy** over $\Lambda$. The nature of this projective hierarchy splits into cases.

**Case 1.** $\cof(\kappa) = \omega$.

Let $A_n \in \Lambda$ with $\sup |A_n| = \kappa$. Then $A = \bigoplus A_n$ is selfdual, defining $\delta$, and has Wadge degree $\kappa$. Let

$$\Sigma_0 = \bigcup_\omega \Lambda = \exists^\omega \Delta.$$ 

Then $\text{pwo}(\Sigma_0)$, and $\Sigma_0$ is closed under $\exists^\omega$, $\land$, $\lor$. 

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In the remaining cases assume \( \text{cof}(\kappa) > \omega \). There is a nonselfdual pointclass (Steel pointclass) \( \Gamma_\kappa \) closed under \( \forall \omega^\omega \) with \( o(\Gamma_\kappa) = \kappa \). We have \( \text{pwo}(\Gamma_\kappa) \).

\( \Gamma_\kappa \) is the collection of \( \Sigma_1^1 \)-bounded unions of \( \Lambda \) sets (\( \Lambda = \Delta_\kappa \)).

**Case 2.**) \( \Gamma_\kappa \) is not closed under \( \lor \).

This includes the case \( \kappa \) is singular.

Then \( \text{pwo}(\Gamma_\kappa) \), and \( \Gamma_\kappa \) is not closed unions with \( \Delta_\kappa \) sets.
Case 3.) $\Gamma_k$ is closed under $\lor$, but not $\exists^\omega\omega$.
We have $pwo(\Gamma_k)$, and $\Gamma_k$ is closed under $\bigcup_\omega$, $\bigcap_\omega$.

Case 4.) $\Gamma_k$ is closed under $\exists^\omega\omega$, $\forall^\omega\omega$.
We have $pwo(\Gamma_k)$, and the projective hierarchy is generated from $\Pi_1 = \forall^\omega\omega (\Gamma_k \lor \check{\Gamma}_k)$.

We have $pwo(\Pi_1)$, $pwo(\Sigma_2)$, $\ldots$.

$\Pi_1 = \Sigma^1_1$-bounded unions of $\check{\Gamma}_k$ sets.

$\Sigma_2 = \bigcup_k \Delta_k$. 

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Steel conjecture: $\Gamma_\kappa$ is closed under $\lor$ (case 3 or 4 holds) iff $\kappa$ is regular.

- Steel showed the conjecture holds if $\kappa$ is a limit of Suslin cardinals.
- First place where conjecture is unknown is above the least type IV hierarchy.
Introduction

Spectrum of a pointclass

**Definition**
Let $\Lambda$ be a pointclass. the spectrum of $\Lambda$, $\text{spec}(\Lambda)$, is the set of $\alpha \in \text{On}$ such that there is a strictly increasing sequence $E = \bigcup_{\beta < \alpha} E_\beta$ with $E \in \Lambda$ such that the union is $\Sigma^1_1$ bounded.

**Remark**
There is no requirement on the complexity of the sets $E_\beta$, just on their union $E$. Note that $\alpha \in \text{spec}(\Lambda)$ requires $\text{cof}(\alpha) > \omega$. 
The following is the basic fact about the spectrum.

**Lemma**

Let $\kappa = o(\Delta)$, where $\Delta$ is a projective algebra with $\text{cof}(\kappa) > \omega$, and let $\Gamma_\kappa$ be the corresponding Steel pointclass. If $\text{cof}(\kappa) \in \text{spec}(\Lambda)$, then $\check{\Gamma}_\kappa$ is not closed under intersection with $\Lambda$ sets.

**Proof.** Let $E = \bigcup_{\beta < \text{cof}(\kappa)} E_\beta$ be a $\Sigma^1_1$-bounded union with $E \in \Lambda$. Let $A$ be $\Gamma_\kappa$ complete, and write $A = \bigcup_{\alpha < \text{cof}(\kappa)} A_\alpha$, an increasing union with each $A_\alpha \in \delta_\kappa$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal $\check{\Gamma}_\kappa$ set. Fix a map $\rho : \text{cof}(\kappa) \to \kappa$ increasing and cofinal.
Consider the game where I plays $x$, II plays $y$, and II wins iff

$$(x \in E) \Rightarrow [\exists \gamma > |x| (U_y = A_{\gamma})]$$

where $|x|$, for $x \in E$, denotes the least $\beta$ such that $x \in E_\beta$. By $\Sigma^1_1$-boundedness of the $E_\beta$ union, II has a winning strategy $\tau$ for this game. We then have

$$z \in A \iff \exists x [(x \in E) \land z \in U_{\tau(x)}].$$

Since $\check{\Gamma}_\kappa$ is closed under $\exists^\omega^\omega$, and $A \notin \check{\Gamma}_\kappa$, we must have that the expression inside the square brackets is not in $\check{\Gamma}_\kappa$. This expression is the intersection of a $\check{\Gamma}_\kappa$ set with $E$, a $\Lambda$ set. □
Example
Let $\kappa = o(\Delta)$ where $\Delta$ is a projective algebra and $\text{cof}(\kappa) = \omega_2$. Let $\Gamma_\kappa$ be the Steel pointclass. Then $\check{\Gamma}_\kappa$ is not closed under intersections with $\Pi^1_2$.

Proof.
Let $\Lambda = \Pi^1_2$. Then $\omega_2 \in \text{spec}(\Pi^1_2)$. For example, we can let $E$ be the set of $x$ such that $T_x$ is wellfounded, where $T \subseteq \omega \times \omega_1$ is the Kunen tree. Then $E = \bigcup_{\beta < \omega_2} E_\beta$, where

$$E_\beta = \{x \in E : [\gamma \mapsto |T_x| \upharpoonright \gamma]_{W^1_1} = \beta\}.$$

This is a $\Sigma^1_1$-bounded union, and $E \in \Pi^1_2$ (here $W^1_1$ is the normal measure on $\omega_1$). □
Remark

Every $\Pi^1_2$ set is an $\omega_1$ intersection of $\delta^1_1$ sets, and the class $\check{\Gamma}^\kappa$ of the example is closed under intersections with $\delta^1_1$ sets (in fact with $\Sigma^1_2$ sets) by Steel’s theorem. This shows that having $B = \bigcap_{\beta < \lambda} B^\beta$ with $\lambda < \text{cof}(\kappa)$, and $\check{\Gamma}^\kappa$ closed under intersections with a pointclass containing all the $B^\beta$ is not sufficient to guarantee that $\check{\Gamma}^\kappa$ is closed under intersections with $B$.

In contrast, the corresponding statement for unions is true by an easy argument.
First theorem

We let $\tilde{C} \subseteq \Theta$ be the canonical c.u.b. set where we define Steel pointclasses.

**Definition**

$\kappa \in \tilde{C}$ iff $\kappa = o(\delta)$ for some projective algebra $\Delta$.

$\tilde{C}$ is c.u.b in both $\delta_1^2$ and $\Theta$.

**Theorem**

Let $\mu$ be a normal measure on $\delta_1^2$. Let $\kappa = j_\mu(\delta_1^2)$. If $\kappa \in \tilde{C}'$, then $\Gamma_\kappa$ is a counterexample to Steel’s conjecture. In fact $\tilde{\Gamma}_\kappa$ is not closed under intersection with $\Pi_1^2$. 
Proof. Fix a $\Delta^2_1$ pwo $(P, \leq)$ of length $\delta^2_1$. We view this as $P = \bigcup_{\alpha} P_{\alpha}$ an increasing, discontinuous union of $\Delta^2_1$ sets.

Let $h : \delta^2_1 \to \delta^2_1$ be given by $h(\alpha) = |P_{\alpha}|_W$.

- First, since $\delta^2_1$ has the strong partition property, there is a c.u.b. $D \subseteq \bar{C}$ such that $j_\mu(D) \subseteq \bar{C}$.
- Fix $f : \delta^2_1 \to \delta^2_1$ increasing, discontinuous, and $f(\alpha)$ a type-4 limit of $\bar{C}$ with pointclass $\Gamma_{f(\alpha)}$.
- Choose $f$ so that $|\Gamma_{f(\alpha)}|_W > h(\alpha)$. 

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Let $\kappa = [f]_{\mu}$. $\kappa$ is a limit of $\tilde{C}$ so $\Gamma_{\kappa}$ is defined. $\kappa$ is regular. This follows from the finite exponent block partition property.

We may assume that for each $\alpha$ there is a sequence $A_{\beta}^{\alpha}$, $\beta < f(\alpha)$, of $\Delta_{f(\alpha)}$ sets which union to a $\Gamma_{f(\alpha)}$ set $A^{\alpha}$.

To show the conjecture fails at $\kappa$, it suffices to show that $\kappa \in \text{spec}(\Pi^2_1)$.
Let $E$ be the set of codes $x$ for functions $f_x : \delta_1^2 \to \delta_1^2$ such that $f(\alpha) \in (\sup_{\beta<\alpha} f(\beta), f(\alpha)]$.

Let

$$E_\beta = \{ x \in E : [f_x]_\mu = \beta \}.$$ 

Say $y$ is an $\alpha$-code if $y \in A^\alpha$, and let $|y|^\alpha = \text{least } \gamma < f(\alpha)$ with $y \in A_\gamma^\alpha$.

Say $x$ is $\alpha$-good if $U(x, \leq \alpha, < \alpha) \neq \emptyset$ and

- $U(x, \leq \alpha, < \alpha) \neq \emptyset$ \quad [U universal $\Sigma_1^1(\leq, <)$.
- $y, z \in U(x, \leq \alpha, < \alpha) \to |y|^\alpha = |z|^\alpha$. 

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Then $x \in E$ iff $x$ is $\alpha$-good for all $\alpha < \delta_1^2$. Clearly $E \in \Pi_1^2$.

**Claim.** The sequence $E_\beta, \beta < \kappa$, is $\Sigma_1^1$-bounded.

**Proof.** Let $S \subseteq E$ be $\Sigma_1^1$. Fix $\alpha < \delta_1^2$.

Then the set $B \subseteq A^\alpha$ defined by

$$y \in B \leftrightarrow \exists x \ [(x \in S) \land (y \in U(x, \leq \alpha, < \alpha))]$$

is in $\Delta_{f(\alpha)}$. Since the $(A^\alpha_\gamma)_{\gamma < f(\alpha)}$ are $\Delta_{f(\alpha)}$-bounded (this is because $\Gamma_{f(\alpha)}$ was Type-4), $\{f_x(\alpha) : x \in S\}$ is bounded below $f(\alpha)$.

$\Box$
Question

Is there a normal measure $\mu$ on $\delta_1^2$ such that $j_\mu(\delta_1^2)$ is a limit point of $\bar{C}$?

We only need a normal measure $\mu$ on $\delta_1^2$ such that $j_\mu(\bar{C}) \cap [g]_\mu = \bar{C} \cap [g]_\mu$, where $g$ enumerates the type-4 $\kappa$ which are limits of $\bar{C}$ below $\delta_1^2$, and $g(\alpha) > h(\alpha)$. 
The leads to the next result. It says that the function $\alpha \mapsto h(\alpha) = |P_\alpha|_W$ must be badly discontinuous.

**Theorem**
*(with H. Woodin)* Let $\delta_1^2 < \kappa < \Theta$ with $\kappa \in C'$ and $\kappa$ of type 4. Let $\mu$ be a normal measure on $\delta_1^2$ (following Woodin) such that $[f]_\mu = \kappa$ and $\forall^* \alpha f(\alpha) \in C'$ is of type 4. Then $\forall^*_\mu \alpha f(\alpha) < h(\alpha)$.

**Corollary**
For any $P = \bigcup_\alpha P_\alpha$ increasing, discontinuous, there are $\alpha < \delta_1^2$ such that $|P_\alpha|_W >$ the next type 4 pointclass after $\alpha$ (or type 4 limit of type 4’s etc.).
Fix a normal measure $\mu$ on $\delta_1^2$. then $\kappa = J_\mu(\delta_1^2)$ is regular (as $\delta_1^2$ has the strong partition property).

If $\kappa \notin C'$, then let $[g]_{\mu}$ be the largest point in $C$ below $\kappa$. So, the $g' > g$ taking values in type 4 $\alpha < \delta_1^2$ do not represent type 4 ordinals.

So either Steel’s conjecture fails, or there are many $g: \delta_1^2 \to \delta_1^2$ taking type 4 values but with $[g]_{\mu}$ not of type 4.