On Steel's Conjecture

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We discuss some joint work with R. Atmai with also contributions by H. Woodin.

The theory ZF $+$ AD $+$ DC gives a complete picture of the scale property below the supremum of the Suslin cardinals.

This theory also gives a complete picture of the prewellordering property. For every Levy class $\boldsymbol{\Gamma}$ either pwo($\boldsymbol{\Gamma}$) or pwo($\check{\boldsymbol{\Gamma}}$).

There is an important closure question about pointclasses, Steel's conjecture which is open. We introduce a notion called the spectrum of a pointclass, show how it relates to the conjecture, and use it to prove some related results.

We henceforth assume ZF $+$ AD $+$ DC.

Basic Definitions

A pointclass is a $\Lambda \subseteq \mathcal{P}(\omega^\omega)$ closed under Wadge reduction, i.e., if $A \in \Lambda$ and $B \leq_w A$, then $B \in \Lambda$.

- We usually write $\mathbf{\Gamma}$ for nonselfdual pointclasses, $\boldsymbol{\delta}$ for selfdual classes, and Λ for either.

For Γ a pointclass, $\mathbf{\Delta}(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$.

We let $o(\Lambda) = \sup\{|A|_w : A \in \mathbf{\Delta}(\Lambda)\}$.

A Levy class is a nonselfdual pointclass $\mathbf{\Gamma}$ closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$ (or both).

Definition

A projective algebra is a selfdual pointclass $\boldsymbol{\delta}$ closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$, $\vee$, $\wedge$.

For any pointclass Λ, there is a largest projective algebra $\boldsymbol{\delta}$ contained in Λ.

Fact

If $\boldsymbol{\Delta}$ is a projective algebra, then

$$\begin{aligned} o(\boldsymbol{\Delta}) &= \sup\{|A|_w : A \in \boldsymbol{\Delta}\} \\ &= \sup\{|\preceq| : \preceq \textit{ is a } \boldsymbol{\Delta} \textit{ prewellordering}\} \end{aligned}$$

Let $\mathbf{\Gamma}$ be a Levy class. Let $\Lambda = \Lambda(\mathbf{\Gamma})$ be the largest projective algebra contained in $\mathbf{\Gamma}$. Let $\kappa = o(\Lambda)$.

Then $\mathbf{\Gamma}$ is in a projective hierarchy over Λ. The nature of this projective hierarchy splits into cases.

Case 1.) $\mathrm{cof}(\kappa) = \omega$.

Let $A_n \in \Lambda$ with $\sup |A_n| = \kappa$. Then $A = \oplus A_n$ is selfdual, defining $\boldsymbol{\delta}$, and has Wadge degree κ. Let

$$\mathbf{\Sigma}_0 = \bigcup_{\omega} \Lambda = \exists^{\omega^\omega} \mathbf{\Delta}.$$

Then $\mathrm{pwo}(\mathbf{\Sigma}_0)$, and $\mathbf{\Sigma}_0$ is closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$.

In the remaining cases assume $\text{cof}(\kappa) > \omega$. There is a nonselfdual pointclass (Steel pointclass) $\boldsymbol{\Gamma}_\kappa$ closed under $\forall^{\omega^\omega}$ with $o(\boldsymbol{\Gamma}_\kappa) = \kappa$.

We have $\text{pwo}(\boldsymbol{\Gamma}_\kappa)$.

$\boldsymbol{\Gamma}_\kappa$ is the collection of $\boldsymbol{\Sigma}^1_1$-bounded unions of $\boldsymbol{\Lambda}$ sets ($\boldsymbol{\Lambda} = \boldsymbol{\Delta}_\kappa$).

Case 2.) $\boldsymbol{\Gamma}_\kappa$ is not closed under $\vee$.

This includes the case κ is singular.

Then $\text{pwo}(\boldsymbol{\Gamma}_\kappa)$, and $\boldsymbol{\Gamma}_\kappa$ is not closed unions with $\boldsymbol{\Delta}_\kappa$ sets.

Case 3.) $\mathbf{\Gamma}_\kappa$ is closed under $\vee$, but not $\exists^{\omega^\omega}$.

We have pwo($\mathbf{\Gamma}_\kappa$), and $\mathbf{\Gamma}_\kappa$ is closed under $\cup_\omega$, $\cap_\omega$.

Case 4.) $\mathbf{\Gamma}_\kappa$ is closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$.
We have pwo($\mathbf{\Gamma}_\kappa$), and the projective hierarchy is generated from $\mathbf{\Pi}_1 = \forall^{\omega^\omega}(\mathbf{\Gamma}_\kappa \vee \check{\mathbf{\Gamma}}_\kappa)$.

We have pwo($\mathbf{\Pi}_1$), pwo($\mathbf{\Sigma}_2$), $\dots$.

$\mathbf{\Pi}_1 = \mathbf{\Sigma}^1_1$-bounded unions of $\check{\mathbf{\Gamma}}_\kappa$ sets.

$\mathbf{\Sigma}_2 = \bigcup_\kappa \mathbf{\Delta}_\kappa$.

Steel conjecture: $\boldsymbol{\Gamma}_\kappa$ is closed under $\vee$ (case 3 or 4 holds) iff κ is regular.

- Steel showed the conjecture holds if κ is a limit of Suslin cardinals.
- First place where conjecture is unknown is above the least type IV hierarchy.

Spectrum of a pointclass

Definition

Let $\mathbf{\Lambda}$ be a pointclass. the spectrum of $\mathbf{\Lambda}$, spec($\mathbf{\Lambda}$), is the set of $\alpha \in \mathrm{On}$ such that there is a strictly increasing sequence $E = \bigcup_{\beta<\alpha} E_\beta$ with $E \in \mathbf{\Lambda}$ such that the union is $\mathbf{\Sigma}^1_1$ bounded.

Remark

There is no requirement on the complexity of the sets E_β, just on their union E. Note that $\alpha \in \mathrm{spec}(\mathbf{\Lambda})$ requires $\mathrm{cof}(\alpha) > \omega$.

The following is the basic fact about the spectrum.

Lemma

Let $\kappa = o(\boldsymbol{\Delta})$, where $\boldsymbol{\Delta}$ is a projective algebra with cof$(\kappa) > \omega$, and let $\boldsymbol{\Gamma}_\kappa$ be the corresponding Steel pointclass. If cof$(\kappa) \in$ spec$(\boldsymbol{\Lambda})$, then $\check{\boldsymbol{\Gamma}}_\kappa$ is not closed under intersection with $\boldsymbol{\Lambda}$ sets.

Proof. Let $E = \bigcup_{\beta < \mathrm{cof}(\kappa)} E_\beta$ be a $\boldsymbol{\Sigma}^1_1$-bounded union with $E \in \boldsymbol{\Lambda}$. Let A be $\boldsymbol{\Gamma}_\kappa$ complete, and write $A = \bigcup_{\alpha < \mathrm{cof}(\kappa)} A_\alpha$, an increasing union with each $A_\alpha \in \boldsymbol{\delta}_\kappa$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal $\check{\boldsymbol{\Gamma}}_\kappa$ set. Fix a map $\rho \colon \mathrm{cof}(\kappa) \to \kappa$ increasing and cofinal.

Consider the game where I plays x, II plays y, and II wins iff

$$(x \in E) \Rightarrow [\exists \gamma > |x| \, (U_y = A_\gamma)]$$

where $|x|$, for $x \in E$, denotes the least β such that $x \in E_\beta$. By $\boldsymbol{\Sigma}^1_1$-boundedness of the E_β union, II has a winning strategy τ for this game. We then have

$$z \in A \leftrightarrow \exists x \, [(x \in E) \wedge z \in U_{\tau(x)}].$$

Since $\check{\boldsymbol{\Gamma}}_\kappa$ is closed under $\exists^{\omega^\omega}$, and $A \notin \check{\boldsymbol{\Gamma}}_\kappa$, we must have that the expression inside the square brackets is not in $\check{\boldsymbol{\Gamma}}_\kappa$. This expression is the intersection of a $\check{\boldsymbol{\Gamma}}_\kappa$ set with E, a $\boldsymbol{\Lambda}$ set. □

Example

Let $\kappa = o(\boldsymbol{\Delta})$ where $\boldsymbol{\Delta}$ is a projective algebra and $\mathrm{cof}(\kappa) = \omega_2$. Let $\boldsymbol{\Gamma}_\kappa$ be the Steel pointclass. Then $\check{\boldsymbol{\Gamma}}_\kappa$ is not closed under intersections with $\boldsymbol{\Pi}^1_2$.

Proof.

Let $\boldsymbol{\Lambda} = \boldsymbol{\Pi}^1_2$. Then $\omega_2 \in \mathrm{spec}(\boldsymbol{\Pi}^1_2)$. For example, we can let E be the set of x such that T_x is wellfounded, where $T \subseteq \omega \times \omega_1$ is the Kunen tree. Then $E = \bigcup_{\beta<\omega_2} E_\beta$, where

$$E_\beta = \{x \in E \colon [\gamma \mapsto |T_x| \restriction \gamma]_{W^1_1} = \beta\}.$$

This is a $\boldsymbol{\Sigma}^1_1$-bounded union, and $E \in \boldsymbol{\Pi}^1_2$ (here W^1_1 is the normal measure on ω_1). □

Remark

Every $\mathbf{\Pi}^1_2$ set is an ω_1 intersection of δ^1_1 sets, and the class $\check{\mathbf{\Gamma}}_\kappa$ of the example is closed under intersections with δ^1_1 sets (in fact with $\mathbf{\Sigma}^1_2$ sets) by Steel's theorem. This shows that having $B = \bigcap_{\beta<\lambda} B_\beta$ with $\lambda < \mathrm{cof}(\kappa)$, and $\check{\mathbf{\Gamma}}_\kappa$ closed under intersections with a pointclass containing all the B_β is not sufficient to guarantee that $\check{\mathbf{\Gamma}}_\kappa$ is closed under intersections with B.

In contrast, the corresponding statement for unions is true by an easy argument.

First theorem

We let $\bar{C} \subseteq \Theta$ be the canonical c.u.b. set where we define Steel pointclasses.

Definition

$\kappa \in \bar{C}$ iff $\kappa = o(\boldsymbol{\delta})$ for some projective algebra $\boldsymbol{\Delta}$.

$\bar{C}$ is c.u.b in both δ^2_1 and Θ.

Theorem

Let μ be a normal measure on δ^2_1. Let $\kappa = j_\mu(\delta^2_1)$. If $\kappa \in \bar{C}'$, then $\boldsymbol{\Gamma}_\kappa$ is a counterexample to Steel's conjecture. In fact $\check{\boldsymbol{\Gamma}}_\kappa$ is not closed under intersection with $\boldsymbol{\Pi}^2_1$.

Proof. Fix a $\mathbf{\Delta}^2_1$ pwo $(P, \leq)$ of length $\boldsymbol{\delta}^2_1$. We view this as $P = \bigcup_\alpha P_\alpha$ an increasing, discontinuous union of $\mathbf{\Delta}^2_1$ sets.

Let $h \colon \boldsymbol{\delta}^2_1 \to \boldsymbol{\delta}^2_1$ be given by $h(\alpha) = |P_\alpha|_W$.

- First, since $\boldsymbol{\delta}^2_1$ has the strong partition property, there is a c.u.b. $D \subseteq \bar{C}$ such that $j_\mu(D) \subseteq \bar{C}$.
- Fix $f \colon \boldsymbol{\delta}^2_1 \to \boldsymbol{\delta}^2_1$ increasing, discontinuous, and $f(\alpha)$ a type-4 limit of $\bar{C}$ with pointclass $\mathbf{\Gamma}_{f(\alpha)}$.
- Choose f so that $|\mathbf{\Gamma}_{f(\alpha)}|_W > h(\alpha)$.

Let $\kappa = [f]_\mu$. κ is a limit of $\bar{C}$ so $\mathbf{\Gamma}_\kappa$ is defined. κ is regular. This follows from the finite exponent block partition property.

We may assume that for each α there is a sequence A^α_β, $\beta < f(\alpha)$, of $\mathbf{\Delta}_{f(\alpha)}$ sets which union to a $\mathbf{\Gamma}_{f(\alpha)}$ set A^α.

To show the conjecture fails at κ, it suffices to show that $\kappa \in \operatorname{spec}(\mathbf{\Pi}^2_1)$.

Let E be the set of codes x for functions $f_x \colon \delta^2_1 \to \delta^2_1$ such that $f(\alpha) \in (\sup_{\beta<\alpha} f(\beta), f(\alpha)]$.

Let

$$E_\beta = \{x \in E \colon [f_x]_\mu = \beta\}.$$

Say y is an α-code if $y \in A^\alpha$, and let $|y|^\alpha =$ least $\gamma < f(\alpha)$ with $y \in A^\alpha_\gamma$.

Say x is α-good if $U(x, \leq_\alpha, <_\alpha) \neq \emptyset$ and

- $U(x, \leq_\alpha, <_\alpha) \neq \emptyset$ [U universal $\boldsymbol{\Sigma}^1_1(\leq, <)$.]
- $y, z \in U(x, \leq_\alpha, <_\alpha) \to |y|^\alpha = |z|^\alpha$.

Then $x \in E$ iff x is α-good for all $\alpha < \boldsymbol{\delta}^2_1$. Clearly $E \in \boldsymbol{\Pi}^2_1$.

Claim. The sequence E_β, $\beta < \kappa$, is $\boldsymbol{\Sigma}^1_1$-bounded.

Proof. Let $S \subseteq E$ be $\boldsymbol{\Sigma}^1_1$. Fix $\alpha < \boldsymbol{\delta}^2_1$.

Then the set $B \subseteq A^\alpha$ defined by

$$y \in B \leftrightarrow \exists x \, [(x \in S) \wedge (y \in U(x, \leq_\alpha, <_\alpha))]$$

is in $\boldsymbol{\Delta}_{f(\alpha)}$. Since the $(A^\alpha_\gamma)_{\gamma < f(\alpha)}$ are $\boldsymbol{\Delta}_{f(\alpha)}$-bounded (this is because $\boldsymbol{\Gamma}_{f(\alpha)}$ was Type-4), $\{f_x(\alpha) : x \in S\}$ is bounded below $f(\alpha)$.
□

Question

Is there a normal measure μ on δ^2_1 such that $j_\mu(\delta^2_1)$ is a limit point of $\bar{C}$?

We only need a normal measure μ on δ^2_1 such that $j_\mu(\bar{C}) \cap [g]_\mu = \bar{C} \cap [g]_\mu$, where g enumerates the type-4 κ which are limits of $\bar{C}$ below δ^2_1, and $g(\alpha) > h(\alpha)$.

Second theorem

The leads to the next result. It says that the function $\alpha \mapsto h(\alpha) = |P_\alpha|_W$ must be badly discontinuous.

Theorem

(with H. Woodin) Let $\delta^2_1 < \kappa < \Theta$ with $\kappa \in C'$ and κ of type 4. Let μ be a normal measure on δ^2_1 (following Woodin) such that $[f]_\mu = \kappa$ and $\forall^* \alpha\ f(\alpha) \in C'$ is of type 4. Then $\forall^*_\mu \alpha\ f(\alpha) < h(\alpha)$.

Corollary

For any $P = \bigcup_\alpha P_\alpha$ increasing, discontinuous, there are $\alpha < \delta^2_1$ such that $|P_\alpha|_W >$ the next type 4 pointclass after α (or type 4 limit of type 4's etc.).

Fix a normal measure μon $\boldsymbol{\delta}^2_1$. then $\kappa = \jmath_\mu(\boldsymbol{\delta}^2_1)$ is regular (as $\boldsymbol{\delta}^2_1$ has the strong partition property).

If $\kappa \notin C'$, then let $[g]_\mu$ be the largest point in C below κ. So, the $g' > g$ taking values in type 4 $\alpha < \boldsymbol{\delta}^2_1$ do not represent type 4 ordinals.

So either Steel's conjecture fails, or there are many $g \colon \boldsymbol{\delta}^2_1 \to \boldsymbol{\delta}^2_1$ taking type 4 values but with $[g]_\mu$ not of type 4.