Generalized Jonsson Cardinals

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We discuss some joint work with J. Holshauser on generalizing questions and results about Jonsson and similar notions to non-wellordered sets in the AD context.

We work throughout in ZF + AD.

We write $[\kappa]^{<\omega}_{<\delta} \to [\kappa]^{<\omega}_{\gamma}$ if for all $\lambda < \delta$ and $f \colon \kappa^{<\omega} \to \lambda$ there is an $H \subseteq \kappa$ of size κ such that $|f[H^{<\omega}]| \leq \gamma$.

Recall the following definitions.

- κ is Jonsson if whenever $f\colon \kappa^{<\omega} \to \kappa$, there is an $H \subseteq \kappa$ of size κ with $f[H^{<\omega}] \neq \kappa$.
- κ is Rowbottom if $[\kappa]^{<\omega}_{<\kappa} \to [\kappa]^{<\omega}_{\omega}$.
- κ is Ramsey if for every $f\colon \kappa^{<\omega} \to 2$, there is an $H \subseteq \kappa$ of size κ with $f \restriction H^n$ constant for each n.

In the wellordered case we have the following.

Theorem

(J,Ketchersid, Schlutzenberg, Woodin) Assume $\mathsf{AD} + V = L(\mathbb{R})$*. Let* $\kappa < \Theta$ *be an uncountable cardinal. then:*

- κ *is Jonsson.*
- *If* $\mathrm{cof}(\kappa) = \omega$ *then* κ *is Rowbottom.*
- $[\kappa]^{<\omega}_{<\kappa} \to [\kappa]^{<\omega}_{\mathrm{cof}(\kappa)}$ *and* $[\kappa]^{<\omega}_{<\mathrm{cof}(\kappa)} \to [\kappa]^{<\omega}_{\omega}$.
- *Let* $\lambda \leq \kappa$ *be an uncountable cardinal. Suppose* $f \colon \kappa^{<\omega} \to \lambda$*. Then there is an* $H \subseteq \kappa$ *of size* κ *such that* $|\lambda - f[H^{<\omega}]| = \lambda$*. In fact,* $\lambda - f[H^{<\omega}]$ *contains a club.*

We extend these questions/results to general sets.

For any set A, let $A^n = \{a \subseteq A : |a| = n\}$. Let $A^{<\omega} = \bigcup_n A^n$.

Definition

Let A, B be infinite sets.

- A is Jonsson if for any $f : A^{<\omega} \to A$ there is an $X \subseteq A$ with $|X| = |A|$ and $f[X^{<\omega}] \neq A$.
- A is strongly Jonsson if for any $f : A^{<\omega} \to A$ there is an $X \subseteq A$ with $|X| = |A|$ and $|A - f[X^{<\omega}]| = |A|$.
- (A, B) is a Jonsson pair if for any $f : A^{<\omega} \to B$ there is an $X \subseteq A$ with $|X| = |A|$ and such that $f[X^{<\omega}] \neq B$.
- (A, B) is a strong Jonsson pair if for any $f : A^{<\omega} \to B$ there is an $X \subseteq A$ with $|X| = |A|$ and such that $|B - f[X^{<\omega}]| = |B|$.
- (A, B) is Rowbottom if for any $f : A^{<\omega} \to B$ there is an $X \subseteq A$ with $|X| = |A|$ and $f[X^{<\omega}]$ is countable.
- (A, B) is Ramsey if for any $f : A^{<\omega} \to B$ there is an $X \subseteq A$ with $|X| = |A|$ and $f[X^n]$ is constant for each n.

Theorem

(AD $+ V = L(\mathbb{R})$) Let $\mathcal{C}$ be the closure of $\{\kappa, \mathbb{R}, \mathbb{R}/E_0 : \omega < \kappa < \Theta\}$ under $\cup$ and $\times$. Then for any $A, B \in \mathcal{C}$, (A, B) is a strong Jonsson pair.

Conjecture

Every set $A \in L_\Theta(\mathbb{R})$ is (strongly) Jonsson.

Question

Which sets are Rowbottom?

If we assume $V = L(\mathbb{R})$, then every $A \in L_\Theta(\mathbb{R})$ is the surjective image of $\mathbb{R}$, and thus can be identified with an equivalence relation on $\mathbb{R}$.

Thus, we are asking which equivalence relations on $\mathbb{R}$ are Jonsson?

Tentative result: If $\mathcal{C}'$ is the smallest collection containing $\{\kappa, \mathbb{R}, \mathbb{R}/E_0 \colon \kappa < \Theta\}$ and closed under $\times$ and increasing unions, then every $A \in \mathcal{C}'$ is strongly Jonsson.

Recall that any two non-smooth hyperfinite equivalence relations on $\mathbb{R}$ are isomorphic, thus we can replace $\mathbb{R}/E_0$ in the above with "hyperfinite."

Some easy cases

Recall a basic result of Mycielski.

Theorem

(Mycielski) Suppose $C_n \subseteq (2^\omega)^n$ are comeager. Then there is a perfect $P \subseteq 2^\omega$ such that $P^n \subseteq C_n$ for all n.

Theorem

- $\mathbb{R}$ *is strongly Jonsson.*
- *For all uncountable κ, $(\mathbb{R}, \kappa)$ is Rowbottom.*
- *For all uncountable κ, $(\kappa, \mathbb{R})$ is Rowbottom.*

To see $\mathbb{R}$ is strongly Jonsson, let $f\colon R^{<\omega} \to \mathbb{R}$. View f as $\{f_n\}$, with $f_n\colon \mathbb{R}^n \to \mathbb{R}$.

By taking comeager sets $C_n \subseteq (2^\omega)^n$ on which f_n is continuous, and using Mycielski's theorem, we may assume each f_n is continuous.

We build sequences σ_s, ρ_t for $s, t \in 2^{<\omega}$, extending in the usual way and with $|\sigma_s| = \ell(|s|)$ and $|\rho_t| = r(|t|)$.

We will have for all $k \leq 2^n$, $\vec{s} \in (2^n)^k$ and $t \in 2^n$:

$$f_k[N(\sigma_{s_1}), \dots, N(\sigma_{s_k})] \cap N(\rho_t) = \emptyset.$$

We let P, Q be the perfect sets defined by the σ_s and ρ_t, then $f[P^{<\omega}] \cap Q = \emptyset$.

Assume σ_s, ρ_t are defined for $s, t \in 2^n$.

For $s, t \in 2^{n+1}$, first let $\sigma_s^0 = \sigma_{s \restriction n}{}^\frown s(n)$, and $\rho_s = t^0 = \rho_{t \restriction n}{}^\frown t(n)$.

Let K be large enough so that

$$2^{K-r(n)} > \sum_{k \leq 2^{n+1}} \binom{2^{n+1}}{k}.$$

For each $k \leq 2^{n+1}$ and each $s_1, \ldots, s_k$ in $(2^{n+1})^k$, we successively extend the σ_s so that $N(\sigma_{s_1}) \times \cdots \times N(\sigma_{s_k})$ determines the first K values of f_k.

From the choice of K, there are ρ_t such that

$$f[N(\sigma_{s_1}) \times \cdots \times N(\sigma_{s_k})] \cap N(\rho_t) = \emptyset$$

for all $t \in 2^{n+1}$.

The fact that $(\mathbb{R}, \kappa)$ is Rowbottom follows from additivity of category and the fact that there are only countably many neighborhoods in $(2^\omega)^n$.

The fact that $(\kappa, \mathbb{R})$ is Rowbottom follows from the fact that any wellordered subset of $\mathbb{R}$ is countable.

We next show that $\mathbb{R}/E_0$ is Jonsson.

This requires establishing a generalization of Mycielski's theorem to E_0.

Mycielski for E_0

Recall the Glimm-Effros dichotomy for E_0.

Theorem

Let $X \subseteq \mathbb{R}/E_0$. Then either

1. *X is countable, or*
2. *X is in bijection with $\mathbb{R}$, or*
3. *X is in bijection with $\mathbb{R}/E_0$.*

We say $A \subseteq \mathbb{R}$ has size E_0 if A is E_0-saturated and $A/(E_0 \restriction A)$ is in bijection with $\mathbb{R}/E_0$.

By Glimm-Effros, this is the same as saying that $E_0 \restriction A$ is not smooth.

For $X \subseteq \mathbb{R}$, we write $X^n_{E_0}$ for

$$\{\vec{x} \in X^n \colon x_1, \ldots, x_n \text{ are pairwise } E_0 \text{ inequivalent }\}.$$

Theorem

Suppose $C_m \subseteq \mathbb{R}^m$ are comeager for all m. Then there is an $A \subseteq \mathbb{R}$ of size E_0 so that for all m, $A^m_{E_0} \subseteq C_m$.

Proof. Let $C_m \subseteq \mathbb{R}^m = (2^\omega)^m$ be comeager, and wlog the C_m are E_0-saturated. We build $A \subseteq 2^\omega$ such that $E_0 \restriction A$ is not smooth and $A^m_{E_0} \subseteq C_m$ for all m.

We build a binary tree σ_s for $s \in 2^{<\omega}$. will have $|\sigma_s| = |\sigma_t|$ if $|s| = |t|$.

For $s \neq t \in 2^{<\omega}$ with $|s| = |t|$ we set

$$D(s,t) = \max\{n\colon s(n) \neq t(n).$$

For $s_1, \ldots, s_m \in 2^{<\omega}$, set

$$\lambda(s_1, \ldots, s_m) = \min\{D(u,v)\colon u \neq v \in \{s_1, \ldots, s_m\}\}.$$

The λ function records how "E_0-inequivalent" the pairs from $s_1, \ldots, s_m$ appear to be.

Definition

We say $s_1, \dots, s_m \in (2^{<\omega})^m$ is active if:

1. $|s_1| = |s_2| = \cdots = |s_m| \ (= n)$
2. $\lambda(s_1 \restriction n-1, \dots, s_m \restriction n-1) < \lambda(s_1, \dots, s_m)$
3. $\lambda(\vec{s}) \geq m$

We let $S_m \subseteq (2^{<\omega})^m$ denote the set of active m-tuples.

Let $S_m = \{\vec{s}_{m,n}\}_{n\in\omega}$ where $\vec{s}_{m,n} = (s_{m,n,1}, \dots, s_{m,n,m})$ enumerate S_m.

Let

$$v \in S_m(n,i) \Leftrightarrow s_{m,n,i} \sqsubseteq v$$
$$v \in S_m(n,-1) \Leftrightarrow v \notin \bigcup_{1\leq i\leq m} S_m(n,i)$$

Let $\sigma_\emptyset = \emptyset$. Assume σ_s has been defined for $|s| \leq n$.

First just split: $\sigma^0_{s^\frown i} = \sigma_{s \restriction n}{}^\frown i$.

For the $m \leq n$ such that $S_{m,n} \cap 2^{n+1} \neq \emptyset$:

- We define $\tau_{m,i}$ for $i \in \{1, 2, \ldots, m\} \cup \{-1\}$
- We define integers $i(m, s) \in \{1, 2, \ldots, m\} \cup \{-1\}$ for $s \in 2^{n+1}$.

We then let $\sigma_s = \sigma^0_s{}^\frown \tau_{1,i(1,s)}{}^\frown \ldots {}^\frown \tau_{m,i(m,s)} \cdots$

Suppose after stage m we have defined

$$\sigma_s^m = \sigma_s^0 \frown \tau_{1,i(1,s)} \frown \cdots \frown \tau_{m,i(m,s)}$$

We need to define $\tau_{m+1,i}$ and $i(m+1,s)$ (where $i \in \{1,\ldots,m+1\} \cup \{-1\}$).

We define $\tau_{m+1,i}$ for $i \in \{1,\ldots,m+1\}$ such that for all $\vec{s} = (s_1,\ldots,s_{m+1}) \in (2^{n+1})^{m+1}$ extending $(s_{m+1,n+1,1},\ldots,s_{m+1,n+1,m+1})$ we have:

$$N(\sigma_{s_1}^m \frown \tau_{m+1,1}) \times \cdots \times N(\sigma_{s_{m+1}}^m) \subseteq W_{m+1,n+1}$$

where $W_{m+1,n+1}$ is dense open in $(2^\omega)^{m+1}$, decreasing in n, and $\bigcap_n W_{m+1,n} \subseteq C_{m+1}$.

If $s \in 2^{n+1}$ extends $s_{m+1,n+1,i}$, we set $i(m+1,s) = i$.

If $s \in 2^{n+1}$ does not extend any of the $s_{m+1,n+1,i}$, let $i = i(m+,s) \in \{1,\dots,m+1\}$ be least such that $D(s, s_{m+1,n+1,i})$ is minimal among $i \in \{1,\dots,m+1\}$.

For such s set $\sigma_s^{m+1} = \sigma_s^{m}{}^\frown\tau_{m+1,i(m+1,s)}$.

This completes the definition of the σ_s^{m+1}, and so completes the definition of the σ_s for $s \in 2^{n+1}$.

Let $A = \bigcup_a \in 2^\omega \bigcap_{n\in\omega} N(\sigma_{a\restriction n})$ be the perfect set defined by the σ_s.

We first show that $E_0 \restriction A$ has size E_0, that is, $E_0 \restriction A$ embeds E_0. Let $\phi\colon 2^\omega \to A$ be the continuous map $\phi(a) = \bigcap_n N(\sigma_{a\restriction n})$. We show that ϕ is a reduction of E_0 to $E_0 \restriction A$.

If $a \neq b \in 2^\omega$ and a is not E_0 equivalent to b, it is clear that $\phi(a)$ is E_0-inequivalent to $\phi(b)$.

Suppose aE_0b and let $n_0 = \max\{n\colon a(n) \neq b(n)\}$.

Claim

There are only finitely many n such that there is an $m \leq n$ with $\vec{s}_{m,n}$ defined and such that there are $1 \leq i < j < m$ with $a \restriction n$ extending $s_{m,n,i}$ and $b \restriction n$ extending $s_{m,n,j}$.

Proof. We need only consider m with $m \geq n_0$ (from definition of active). Fix such am m. We may assume the values of i and j are fixed. But then for such n we have $\lambda(\vec{s}_{m,n}) \leq n_0$. As the value of λ increases for active tuples, there can be only finitely many such n.

□

We can repeat the argument to also get the conclusion for all $a' \neq b'$ in Y, the set of s which agree with a (and hence b) after n_0. Say the conclusion holds for all $n \geq n_1$ (for all a', b').

It suffices to show that for all $n \geq n_1$ that $i(m, a \restriction n) = i(m, b \restriction n)$ for all $m \leq n$. Fix $n \geq n_1$ and $m \leq n$.

Case 1. There are $1 \leq i \neq j \leq m$ with $a \restriction n$ extending $s_{m,n,i}$ and $b \restriction n$ extending $s_{m,n,j}$.

This case cannot occur from the claim and definition of n_1.

Case 2. There is an $1 \le i \le m$ with $a \restriction n \in S_m(n, -1)$ and $b \restriction n$ extending $s_{m,n,i}$ (or with a, b switched).

As $n > n_0$, $t_{a \restriction n} \in Y$. We must have $i(m, t_{a \restriction n}) = i$ as otherwise, since $n > n_1$, $D(a \restriction n, b \restriction n) > n_0$, a contradiction.

Case 3. $a \restriction n, b \restriction n \in S_m(n, -1)$.

First assume that $t_{a \restriction n} \in Y$, and so $t_{b \restriction n} \in Y$ as well. As $n > n_1$ we must have $i(m, t_{a \restriction n}) = i(m, t_{b \restriction n})$, as otherwise $D(a \restriction n, b \restriction n) > n_0$.

Next assume $t_{a \restriction n} \notin Y$, so $t_{b \restriction n} \notin Y$ as well. In this case $D(a \restriction n, s_{m,n,i}) > n_0$ for all $1 \leq i \leq m$, and likewise for $b \restriction n$.

It follows that $D(a \restriction n, s_{m,n,i}) = D(b \restriction n, s_{m,n,i})$ for all $1 \leq i \leq m$. It then follows that $i(m, a \restriction n) = i(m, s \restriction n)$.

Thus, ϕ is an embedding from E_0 to $E_0 \restriction A$.

Finally, we show that $A^m_{E_0} \subseteq C_m$ for all m.

Fix $x_1, \dots, x_m \in A^m_{E_0}$. Say $\phi(a_i) = x_i$.

Thus $a_1, \dots, a_m$ are pairwise E_0-inequivalent. Thus $\lambda(a_1 \restriction n, \dots, a_m \restriction n)$ is monotonically increasing and unbounded with n.

So, for infinitely many k we have $(a_1 \restriction k, \dots, a_m \restriction k) \in S_m$ (is an active m-tuple). So, for infinitely many n we have $(a_1 \restriction n, \dots, a_m \restriction n)$ extends $(s_{m,n,1}, \dots, s_{m,n,m})$, and so $\phi(\vec{a}) \in W_m, n$. Thus, $\vec{x} = \phi(\vec{a}) \in C_m$.

Proof that $\mathbb{R}/E_0$ is strongly Jonsson.

Let $f\colon [\mathbb{R}/E_0]^{<\omega} \to \mathbb{R}/E_0$ be given.

By countable uniformization, there are functions $f_n\colon \mathbb{R}^n \to \mathbb{R}$ which induce f ($\vec{x} E_0 \vec{y} \to f_n(\vec{x}) E_0 f_n(\vec{y})$).

Get comeager $C_m \subseteq \mathbb{R}^m$ such that $f_m \restriction C_m$ is continuous.

Build sequences σ_s, ρ_t. The σ_s are defined similarly to the E_0-Mycielski theorem.

Suppose σ_s, ρ_t have been defined for $|s|, |t| \leq n$.

Let $\sigma_s^0 = \sigma_{s\restriction n}{}^\frown s(n)$, $\rho_t^0 = \rho_{t\restriction n}{}^\frown t(n)$, for $|s| = |t| = n+1$.

We define σ_s^m, ρ_t^m for $m \leq n+1$. We will have $\rho_t^m = \rho_t^0{}^\frown \pi_1{}^\frown \cdots {}^\frown \pi_m$, where π_m doesn't depend on t.

For $m+1$, consider $\vec{s}_{m+1,n+1}$ as before. Let $\ell = |s_{m+1,n+1,i}|$. There are $p = (2^{n+1-\ell})^{m+1}$ many $m+1$-tuples of length $n+1$ extending $\vec{s}_{m+1,n+1}$. Let k be large enough so that $2^k > p$.

Then we may define the σ_s^{m+1} as before and such that for any $\vec{s}$ extending $\vec{s}_{m+1,n+1}$, the corresponding σ_s^{m+1} determine $f(\vec{s})$ on the k length block of digits after $|\rho_t| + |\pi_1| + \cdots + |\pi_m|$.

We can then choose π_{m+1} such that $f_m(N(\sigma_s^{m+1}) \times \cdots \times N(\sigma_s^{m+1})) \cap N(\rho_t{}^\frown \cdots {}^\frown \pi_{m+1}) = \emptyset$ for all $t \in 2^{n+1}$.

The perfect sets A, B defined by the σ_s and ρ_t witness that $\mathbb{R}/E_0$ is strongly Jonsson.