Say $V \models \text{ZFC}$. 
$M \subseteq V$ is a ground for $V$ if $\exists \text{Re } M \exists \text{ge } V \\& \text{ P-generic over } M \text{ s.t. } V = M[G]$.

The mantle of $V$ is the intersection of all grounds. A bedrock is a ground which does not contain a further ground.

We will study the collection of grounds for extendible models. Such models have proper grounds only if they have Woodin cardinals.

- $\mathcal{M}_\lambda$: the mantle is the result of iterating the least measurable out of the universe

  - with G. Fuchs: Models without strong cardinals.

Let $M_w$ be the least $L[E]$-model with a Woodin cardinal $\delta$ and a strong cardinal $\kappa > \delta$.

I want to describe its mantle.

Let $\eta$ be a cutpoint of $M_w$ with $\delta < \eta < \kappa$. 
Now the least measurable above $\delta$, $d \to \delta'$, and then start generically absorbing. Use generic reductions over $\mathbb{M}(\delta)$ to find $\mathbb{Q}$-structures, if they exist.

**Case 2** If a $\mathbb{Q}$-structure exists, continue.

**Case 1** If the $\mathbb{Q}$-structure does not exist, the last $P = \text{the result of generic reduction}$

$$P [\mathbb{M}_{\text{sw}} / \delta (\eta) ] = \mathbb{M}_{\text{sw}}$$

**Describe** $\mathbb{M}_{\text{sw}}$ of $\mathbb{M}_{\text{sw}}$.

In $\mathcal{V}$: Points in the system are all models $P = P^{\mathbb{M}_{\text{sw}}}(\mathbb{M}(\eta))$, $e$.

$$\delta (\eta) = \eta^+ \text{ for some cutpoint } \eta < \kappa$$

result of generic reduction.

Each $P$ is good.

$P'$ is above $P$ in this system in $\mathcal{V}$, see that $P'$ is an iterate of $P$, let $\pi_{P'}$ be the map.

$(\mathbb{M}_{\text{sw}}, \pi_{P'}) = \text{the abv him of the system}$
\( P \) is \( \omega \)-iterateable is a finite set of arithms \( \leq \) below \( \delta_0 \) for all \( \delta \) of limit length according to the "short tree fragment of \( \Sigma \) there is a branch \( T \) of \\

\[ \Gamma_{\text{max}}(s) \rightarrow \Gamma_{\text{max}}(s)(y,s) \rightarrow \Gamma_{\text{max}}(x)(y,s) \] \\

Internal system consists of the same points \( P \), \( \omega \)-iterateable

The maps \( \pi_\mathcal{S} \), the usual fragments of \( \tau_{\mathcal{P}, \mathcal{P}} \), which

The internal system gives the same \( \mathcal{M}_\omega \).

Facts

- \( n \) is the least measurable of \( \mathcal{M}_\omega \).
- \( \delta_\omega > n^+ < n^+ \mathcal{M}_\omega < n^+ \mathcal{M}_\omega < n^+ \mathcal{M}_\omega \) \( \mathcal{M}_\omega \) \( \mathcal{M}_\omega \) \( \mathcal{M}_\omega \) \( \mathcal{M}_\omega \) \( \mathcal{M}_\omega \)

\[ \mathcal{M}_{\text{sw}} = \mathcal{M}_\omega \] is fully iterateable in all generic definitions

\[ \mathcal{M}_{\text{sw}} \rightarrow \mathcal{M}_\omega \rightarrow \mathcal{M}_\omega \]

Let \( \mathcal{M}_\omega = \mathcal{M}_\omega \) as defined in \( \mathcal{M}_\omega \)

\[ \mathcal{M}_\omega = \pi_\mathcal{A}(\mathcal{M}_\omega) \]
Let $\rho \in \text{On}$. Let

$$
\rho^* = \min \{ \pi^s_{\rho_\alpha} (\rho) \mid \rho \text{ in the system } \pi^s_{\rho_\alpha} \text{ defined} \}
$$

$\rho^*$ in $\mathcal{M}_{\omega_1}$. Let

$$
\mathcal{V} = L[\mathcal{M}_{\omega_1}, \rho \mapsto \rho^*] = L[\mathcal{M}_{\omega_1}, \rho \mapsto \rho^* \cap S_{\omega_1}^0]
$$

$\rho^* = \pi^w_{\omega_1, \omega} (\rho)$, let $T_{\omega_1, \omega} (\rho) = \rho$ be suitable

$$
\rho^* = T_{\omega_1, \omega} (\rho) = T_{\omega_1, \omega} (\pi^w_{\omega_1, \omega} (\rho)) = T_{\omega_1, \omega} (\pi^w_{\omega_1, \omega} (\rho)) = T_{\omega_1, \omega} (\rho)
$$

**Theorem (Sargsyan, Schindler)**

$\mathcal{V} = L[\mathcal{M}_{\omega_1}, \rho \mapsto \rho^*]$ is the mantle of $\mathcal{M}_{\omega_1}$

hence the least ground and also the birthcard

$$
\mathcal{G} \cap \text{On} \cap \text{Ord} = \text{HOD}_{\mathcal{M}_{\omega_1}}
$$

$\mathcal{H}^0_{\omega_1} = \mathcal{M}_{\omega_1} + S_{\omega_1}^0$ is Woodin in $\mathcal{V}$

also $\mathcal{M}_{\omega_1}$ is fully iterable in $\mathcal{V}$

**Lemma** $\mathcal{V}$ is ground for $\mathcal{M}_{\omega_1}$

**Proof** (this uses Buronsky's Theorem)

**Claim** $\mathcal{V}$ uniform $\aleph^+ -$ count $\mathcal{M}_{\omega_1}$, i.e.

$\forall \alpha \mapsto \text{On}, \alpha \in \mathcal{M}_{\omega_1} = \exists g \in \mathcal{V} \forall \alpha \in \text{On}

\text{den} (g) = \emptyset \quad (f(3) + g(3), |f(3)| < \alpha + \text{all } \beta) \quad \emptyset$

Then $\mathcal{M}_{\omega_1}$ is an $\aleph^+$-cc. extension of $\mathcal{V}$. (End)}