Say V ~~HZgay~~ $\models$ ZFC.

$M \subseteq V$ is a ground for V if $\exists \mathbb{P} \in M \ \exists g \in V$ g $\mathbb{P}$-generic / M s.t. $V = M[g]$

The mantle of V is the intersection of all grounds

A bedrock is a ground which does not contain a further ground.

We will study the collection of grounds for extender models. Such models have proper grounds only if they have Woodin cardinals.

- M_1: the mantle of the result of iterating the least measure out of the universe

- with G. Fuchs: Models without strong cardinals.

Let M_{sw} be the least $L[E]$-model with a Woodin cardinal δ and a strong cardinal $\kappa > \delta$.

~~Rg~~ Want to describe its Mantle.

Let η be a cutpoint of M_{sw} s.t. $\delta < \eta < \kappa$.

Move the least measurable above δ, $\delta \mapsto \delta'$ and then start generically absorbing. Use generic reductions over $M(\sigma)$ to find Q-structures, if they exist.

Case 2 If a Q-structure exist, continue

Case 1 If the Q-structure does not exists, then letting:
$P =$ the result of gen. reduction,
$P[M_{sw} \restriction \delta(\sigma)] = M_{sw}$

Describe M_∞ of M_{sw}.

In V: Points in the system are all models
$P = P^{M_{sw}}(M(\sigma))$ s.t.
$\delta(\sigma) = \eta^+$ for some cutpoint $\eta < \kappa$
result of generic reduction.

Each P is good.

P' is above P in this system in V sees that P' is an iterate of P, let $\pi_{PP'}$ be the map.

$(M_\infty, \pi_{P,\infty}) =$ the dir lim of the system

P is s-iterable (s a finite set of ordinals) iff for all $\mathcal{T}$ of limit length below δ^P according to the short tree fragment of Σ there is a branch b s.t.

$$Th^{P|\max(s)}(\delta^P \cup \{s^-\}) \longmapsto Th^{P^{M_{sw}}(\mathcal{M}(\mathcal{T}))|\max(s)}(\delta^{P'} \cup \{s^-\})$$

Internal system consists of the same points P, s-iterable, the maps $\pi^s_{P,P'}$, the usual fragments of $\pi_{P,P'}$ which. The internal system gives the same M_∞.

Facts

- κ is the least measurable of M_∞.
- $\delta_\infty = \kappa^+ < \kappa_\infty < \kappa_\infty^{+M_\infty} < \kappa_\infty^{++M_\infty} = \kappa^{++M_{sw}}$

$\uparrow$ Woodin of M_∞ $\quad\nwarrow$ $\nexists$ strong of M_∞

$M_{sw} \models M_\infty$ is fully iterable in all generic extensions

$$M_{sw} \xrightarrow{\pi_\infty} M_\infty$$

Let $M_\infty^\infty = M_\infty$ as defined in M_∞

$= \pi_\infty(M_\infty)$

Let $\rho \in On$. Let

$$\rho^* = \min \{ \pi^S_{P,\infty}(\rho) \mid P \text{ in the system} \wedge \pi^S_{P,\infty} \text{ defined} \}$$

$\rho \mapsto \rho^*$ in M_{sw}. Let

$$\mathcal{V} = L[M_\infty, \rho \mapsto \rho^*] = L[M_\infty, \rho \mapsto \rho^* \restriction \delta_\infty]$$

$\rho^* = \pi^A_{\mathcal{V},\infty}(\rho)$ Let $\pi_{N,\infty}(\bar\rho) = \rho$ N suitable

$$\rho^* = \pi_{N,\infty}(\rho) = \pi_{N,\infty}(\pi_{N,\infty}(\bar\rho)) = \pi^A_{\mathcal{V},\infty}(\pi_{N,\infty}(\bar\rho)) = \pi^A_{\mathcal{V},\infty}(\rho)$$

<u>Theorem</u> (Sargsyan, Schindler)

$\mathcal{V} = L[M_\infty, \rho \mapsto \rho^*]$ is the mantle of M_{sw},
hence the least ground and also the bedrock,
$= HOD^{M_{sw}^{Coll(\omega, <\kappa)}}$

$H^{\mathcal{V}}_{\delta_\infty} = H^{M_\infty}_{\delta_\infty}$ δ_∞ is Woodin in $\mathcal{V}$

Also M_∞ is fully iterable in $\mathcal{V}$.

<u>Lemma</u> $\mathcal{V}$ is a ground for M_{sw}

<u>Pf</u> This uses Bukovsky's Thm.

<u>Claim</u> $\mathcal{V}$ uniformly κ^+-covers M_{sw}, i.e.

$f : \theta \to On$, $f \in M_{sw} \Rightarrow \exists g \in \mathcal{V}$ s.t.

$dom(g) = \theta$ $f(\xi) \in g(\xi)$, $|g(\xi)| < \kappa^+$ all $\xi < \theta$

Then (Bukovsky) M_{sw} is a κ^+-c.c. extension of $\mathcal{V}$.