Proof of ~~Bukovsky~~ the 's lemma Verify the assumptions of Bukovsky's lemma. Assume f: be a counterexample.

$f(\xi)=\eta \Leftrightarrow M_{sw} \models \varphi(\eta,\xi)$

Let N be a point on the M_∞ system.

$M_{sw} = N[g]$ $\quad f(\xi)=\eta \Leftrightarrow N[g] \models \varphi(\xi,\eta)$

$g_N(\xi) = \{\eta \mid \exists p \in \mathbb{B}^N \ p \Vdash^N_{\mathbb{B}N} \varphi(\xi,\eta)$ and there is only one such η for $\xi\}$

$|g_N(\xi)| < \delta^N$ each ξ.

$g_{M_\infty} = \pi_{N,\infty}(g_N)$ $\quad g_{M_\infty}(\xi)$ has the corresponding definition in M_∞

$|g_{M_\infty}(\xi)| < \delta_\infty = \kappa^+$

Define g with domain Θ

$g(\xi) = \{\eta \mid \eta^* \in g_{M_\infty}(\xi^*)\}$ $\quad \xi \in V$

Let $\eta = f(\xi)$. Pick N in the system s.t. $\xi^* = \pi_{N,\infty}(\xi)$

$\eta^* = \pi_{N,\infty}(\eta)$ $\quad \eta^* = \pi_{N,\infty}(\eta) \in \pi_{N,\infty}(g_N)(\pi_{N,\infty}(\xi)) = g_{M_\infty}(\xi^*)$

$\Rightarrow \eta \in g(\xi)$. This verifies the assumptions of B. lemma.

Claim

(a) M_∞ is fully iterable inside M_{sw}
and also inside any of its generic extensions

(b) M_∞ is fully iterable inside $V = L[M_\infty, \rho \mapsto \rho^*]$

Claim $M_\infty = H^{M_{sw}}$

Let
$$H = HOD^{M_{sw}^{Coll(\omega, <\kappa)}} \quad \text{Fix } g \subseteq Coll(\omega, <\kappa)$$

Claim $L[M_\infty, \rho \mapsto \rho^*] \subseteq H$

Proof $\mathcal{C}$ = the collection of all N
s.t. N has a Woodin $\delta^N < \kappa^N$
a strong, and $N[h] = M_{sw}[g]$

generic for $Coll(\omega, <\kappa_N)$ — above

and such that N is "pseudo-iterable".

These conditions allow to form M_∞ ~~inside~~ for N, M_∞^N

Subclaim $N \in \mathcal{C} \Rightarrow M_\infty^N = M_\infty$.

This shows: M_∞ "$\in$" H.

To verify the subclaim: Show the two systems give rise to M^N_∞, M_∞ resp. have cofinally many points in common.

$$N = K(N|\delta^N) = K(N|\delta^N)^{N[g]} = K(N|\delta^N)^{M_{sw}[g]} =$$
$$= K(N|\delta^N)^{M_{sw}[g \upharpoonright \xi]} \quad \text{some } \xi < \kappa$$

So N "$\in$" $M_{sw}[g \upharpoonright \xi]$ some $\xi < \kappa$

M_{sw} "$\in$" $N[h|\xi']$ some $\xi' < \kappa$

$$H_{\kappa^+}^{\mathrm{Ult}(M_{sw}, E_1)[g]} = H_{\kappa^+}^{M_{sw}[g]} = H_{\kappa^+}^{N[h]} = H_{\kappa^+}^{\mathrm{Ult}(N, E_2)[h]}$$

last with cr.pt $= \kappa$ — " —

$$\left.\begin{array}{l}\mathrm{Ult}(M_{sw}, E_1)[g] = K(\bar{H})^{M_{sw}[g]} \\ \mathrm{Ult}(N, E_2)[h] = K(\bar{H})^{N[h]}\end{array}\right\} \text{ these are the same}$$

Denote by $K(\bar{H})$

$$\Rightarrow i_{E_1}(\kappa) = i_{E_2}(\kappa)$$

Let k be $\mathrm{coll}(\omega, [\kappa, i_{E_1}(\kappa)))$ - generic / $K(\bar{H})$.

$$M_{sw}[g] \xrightarrow{\tilde{i}_{E_1}} K(\bar{H})[k] = \mathrm{Ult}(M_{sw}, E_1)[g^\frown k]$$

$$N[h] \xrightarrow[\tilde{i}_{E_2}]{} K(\bar{H})[k] = \mathrm{Ult}(N, E_2)[h^\frown k]$$

$$i_{E_1}(N) = i_{E_1}\left(K(N|\delta^N)^{M_{sw}[g]}\right) = K(N|\delta^N)^{K(\bar{M})[k]} = \mathrm{Ult}(N, E_2)$$

$$\tilde{i}_E : M_{sw}[g] \longrightarrow \mathrm{Ult}(M_{sw}, E_1)[g^\frown k] = K(\bar{H})$$

We have arbitrarily large $\eta < \kappa$ s.t. η is a cutpoint of both M_{sw} and N s.t. setting $H = H_{\eta^+}^{M_{sw}[g\restriction\eta]}$

$$M_{sw} = \mathcal{P}^{M_{sw}[g\restriction\eta]}(M_{sw} | \eta^+ M_{sw})$$

↑ generic reduction

Note: $Ult(M_{sw}, E_1) = \mathcal{P}^{\underbrace{ult(M_{sw}, E_1)[g]}_{}}(M_{sw} | \kappa^+)$; $ult(M_{sw},E_1)[g] = g^{\wedge}k\restriction\kappa$

$$N = \mathcal{P}^{N[h\restriction\eta]}(N | \eta^+ N) \qquad K(H)$$

Note: $Ult(N, E_2) = \mathcal{P}^{\overbrace{Ult(N, E_2)[h]}}(N | \kappa^+)$

$= i_{E_2}(N)$

Fix s M suitable in M_{sw}.

M' suitable in N

want to find M^* in both systems.

N "$\in$" $M_{sw}[g\restriction\xi] \subseteq N[h\restriction\xi'']$ some $\xi'' < \kappa$

N, M_{sw} are both grounds of $M_{sw}[g\restriction\xi]$.

Pick $P \subseteq N \cap M_{sw}$ which is a ground of $M_{sw}[g\restriction\xi]$

$P^{Col} \ni M$ } candidates for M inside P^{Coll}

$P^{Coll} \ni M'$ } candidates for M'

$$P^{Coll} \quad \vdots \; M \Big\} \xrightarrow{\sigma} M_{sw}$$

$$P^{Coll} \quad \vdots \; M' \Big\} \xrightarrow{\sigma'} N$$

Each board is a joint comparison which makes at the same time M_{sw} generic (σ) and N generic (σ').

We proved that $L[M_\omega, \rho \mapsto \rho^*] \subseteq H = \mathrm{HOD}^{M_{sw}^{Coll(\omega,<\kappa)}}$

Claim $L[M_\omega, \rho \mapsto \rho^*] = H$

Pf Let $X \subseteq On$

$$\xi \in X \iff \Vdash^{M_{sw}}_{Coll(\omega,<\kappa)} \varphi(\xi, \alpha_0 \ldots \alpha_k)$$

Let $N \in$ System $N[g] = M_{sw}$. Hence

$$\xi \in X \iff \Vdash^{Coll(\omega,<\kappa)}_{N} \varphi(\xi, \alpha_0 \ldots \alpha_k)$$

Pick N s.t. $\pi_{N,N'}(\alpha_0 \ldots \alpha_k) = \alpha_0 \ldots \alpha_k$

then $\pi_{N,N'}(X) = X$.

$X \in \bigcap$ all points in the system

Given ξ^* s.t. $\pi_{N,\infty}(\xi) = \xi^*$

Want to compute X in $L[M_\infty, p \mapsto p^*]$. Fix ξ

$\xi \in X \iff \xi^* = \pi_{N,\infty}(\xi) \in \pi_{N,\infty}(X) = \pi_{N,\infty}(X) = \tilde{X} \in M_\infty$

$X = (p \mapsto p^*)^{-1}[\tilde{X}] \in L[M_\infty, p \mapsto p^*]$.

Claim $H_{\delta_\infty} \cap H = H_{\delta_\infty} \cap M_\infty$

Claim $M_\infty = K^{M_{sw}}$

$K^{M_{sw}} = K^{\mathcal{V}[g]} = K^{\mathcal{V}}$

$j : K \longrightarrow M_\infty$ in $\mathcal{V}$ and Assume $j \neq id$. Set $\lambda = crit(j)$

- $j(\lambda) < \delta_\infty$ Then $j \restriction \lambda^{+K}$ is cofinal in $j(\lambda)^{+M_\infty}$

  So $cf(j(\lambda)^{+M_\infty}) < j(\lambda)$ ↯

- $j(\lambda) = \delta_\infty$ Then λ not Woodin in the target, but then K is missing the Q-structure ☒

- ~~In fact: if $j(\lambda) > \delta$ then~~ λ is a limit

- $j(\lambda) > \delta_\infty$. So $\lambda > \delta_\infty$. So $j : K \to M_\infty$ is an iteration map. ↯

<u>Claim</u> $L[M_\infty, \rho \mapsto \rho^*]$ is the mantle
(Hence the smallest ground and a bedrock.)

<u>Pf</u> Let $N[g] = M_{sw}$

$$\mho = \underset{\substack{\| \\ K^{M_{sw}} = K^N}}{L[M_\infty, \rho \mapsto \rho^*]} \subset N \qquad \boxtimes$$

<u>Claim</u> δ_∞ is a Woodin cardinal of $L[M_\infty, \rho \mapsto \rho^*]$
(Proof pending) $\boxtimes$ $\boxtimes$

<u>More generally</u>

<u>Theorem</u>: For a cone of x:

$M_s(x)$ = the least model over x with a strong cardinal

$M_s(x)$ has a 2-small core model K s.t.
the mantle of $M_s(x)$ is equal to
$L[K, \rho \mapsto \rho^*]$.
$L[K, \Sigma]$
$\quad\wedge$ I.S. for $K^{M_s(x)}$