2016 - 07 - 20 9:30 FARMER SCHLUTZENBERG (1)

$\text{HOD}^{L[E]}$ above $\omega_3^{L[E]}$ below a Woodin limit of Woodins
($\omega_2^{L[E]}$ most likely)

$M = L[E]$ $e = E^M \upharpoonright \omega_1^M$

HOD^M means ordinal definability in the language of set theory, i.e. with no reference to E.
for $M \models PS$

OLD RESULTS

- Models satisfying $V = \text{HOD}$: $L[\mu]$, $\underbrace{M_1 \models V = K}_{\text{Steel}}$
- $L(M_1^{\#}) \models V \neq \text{HOD}$, as ~~but~~ $\mathbb{R} \cap OD = M_1 \cap \mathbb{R}$
or ω_2 Woodin in HOD

$L(M_1^{\#}) = \text{HOD}[G]$ G Vopěnka generic
Note if $L[M_1^{\#}] = H[g]$ then $\text{card}(\mathbb{P}) \geq \omega_2$ ~~Rey~~.
where g is $\mathbb{P}$-generic

Question (Steel) If $M \models ZFC$ (M tame?), does
$M \models V = \text{HOD}_X$ some $X \subseteq \omega_1^M$ or $X \in \mathbb{R}$

<u>Theorem 1</u> Let $M \models PS$ or $M \models ZF^- + \omega_1$ exists. Then $\mathbb{E}^M$ is definable from e^M.

If $M \models$ "e^M (ω_1+1)-iterable" then $M \models V = HOD$

<u>Fact</u> (Schindler/Steel) If M is tame then $M \models e^M$ is ω_1-iterable above some $\alpha < \omega_1$

Corollary: $M_{adr} \models V = HOD$.

<u>Example</u> $L[\vec{M}_1^\#] = R$ is s.t. $R = L[R|\omega_1^R]$ and $R|\omega_1^R$ is closed under $M_1^\#$.

$R \not\models e^R$ is (ω_1+1)-iterable above some $\alpha < \omega_1$.

<u>Theorem 2</u> Let M be tame and $\models$ "ω_1 exists". Then e^M is M-definable from some $x \in \mathbb{R}$.

<u>Corollary & proof</u> $\mathbb{R}^M$ has a well-order which is

$$\left(\Sigma_2^{H_{\omega_2}}(\{x\})\right)^M$$

where x is a real.

<u>Def</u> Tame premouse: no extender overlapping a local Woodin. An Extender F is non-tame if for some δ: $\kappa_F \leq \delta < lh(F)$ and

$M|lh(F) \models \delta$ Woodin.

<u>Question</u> Does there exist a version of Thm 2 for non-tame?

<u>How does HOD^M relate to M?</u>

Since $M = HOD^M_{\{M\}}$ we have $M = HOD^M[g]$ where g is $(HOD^M, \mathbb{P})$-generic where $\mathbb{P}$ = Vopěnka algebra. Here get $\mathrm{card}(\mathbb{P})^{HOD^M} \leq \omega_3^{HOD^M}$

$\mathbb{P} \simeq$ OD codes for subsets of $\mathcal{P}(\omega_1)$, which is of size ~~M~~ ω_2. So in M, $\mathrm{card}(\mathbb{P}) \leq \omega_3 = \delta$

$\Rightarrow \mathrm{card}(\mathbb{P})^{HOD^M} < \omega_4^M$. $\Rightarrow HOD^M \models \mathbb{P}$ is δ-c.c.

$\Rightarrow \delta^{+H} = \omega_4^M \Rightarrow \mathbb{P} \subseteq \omega_3$.

For tame case get $\mathbb{P} \subseteq \omega_2^M$. This is optimal as HOD^M might satisfy "ω_2^M is Woodin".

<u>Theorem 3</u> Assume M is below Woodin limit of Woodins and $M \models PS$ and e^M is not M-definable. Let $H = HOD^M$ and $t = (Th_{\Sigma_3}^{H_{\omega_2}}(\omega_2))^M$ and $\delta = \omega_2^M$.

<u>Then</u>: there is a p.c. premouse $W \subseteq M$ s.t.

- $W \models \delta$ Woodin
- t is $(W, \mathbb{B}_\delta)$-generic
- $H = W[t]$
- W is M-definable

~~Hence~~

Moreover, if M is tame:

- $E^M \restriction [\omega_1^M, On^M)$ is M-definable
- $\mathscr{E}^M$ is M-definable "on a tail" → see (*) below
- $H \mathbin{/\!\!\!\!\!/} = W^*$ = premouse over $(W|\delta, t)$
- $M = H[\mathscr{E}^M]$
- extenders on sequences of W, H, M are related via generic extensions/reductions

If M is non-tame:

- $M = H[x]$ some $x \subseteq \omega_1^M$
- E^M = the outcome of $*$-translation from W

Note Structure of HOD^M below ω_2^M is unclear e.g. for $L[M_1^\#]$.

(*) M-definable "on a tail". In M:
say a pm N is a candidate iff the domain of N is HC^M. For $\alpha < \omega_1$ say $M|\omega_1 \overset{>\alpha}{\sim} N$ iff $M|\alpha, N|\alpha$ are interdefinable from parameters and $E^M \restriction [\alpha, \omega_1^M) = E^N \restriction [\alpha, \omega_1^M)$ and $\rho(M|\alpha) = \omega = \rho(N|\alpha)$

parameters must be in their common universe

Let $\wp^M = \{N \mid \exists \alpha > \omega_1^M \quad N \overset{>\alpha}{\sim} M|\omega_1\}$

"$\wp^M$ is M-definable on a tail" means that

$\wp^M$ is M-definable

<u>Proof for Thm 1</u> $M \models ZF^-$, ρ = the largest cardinal of M

WMA we know $M|\rho$ and $\rho \geq \omega_1^M$. Want
$E^M \upharpoonright [\rho, O_m^M)$

<u>Def</u> N a pm. For $M|\rho \trianglelefteq N$ say N is <u>good</u> iff
N is sound + $\rho(N) = \rho$.

Given N good, define

N^{+con} = stack of all good P s.t. $N \trianglelefteq P$
and condensation holds for P

Let $X^M = \{\bar{M} \mid \exists P \triangleleft M \ \exists \pi : \bar{M} \to P$ elementary, $\bar{M} \in HC^M\}$ in M

<u>Lemma</u> $X^M = X^{M|\omega_1^M}$

N^{+Hulls} = stack of all good P s.t. in M
every countable elementary substructure
of P is in X^M + N is a cutpoint of P.

Case 1 ρ is regular in M

Claim Let $N = M|\rho$. Then $N^{+Con} = M$

This is because by condensation, any two ~~element~~ pm in N^{+Con} are initial segment of each other.

Case 2 ρ is a cutpoint in M.

Claim (Woodin) $M = N^{+Hull}$ $(N = M|\rho)$

Now do a comparison with countable collapses in V, and recall we assume M is (ω_1+1)-iterable in V. Since δ is a cutpoint, the comparison proves that the collapses are initial segment of each other.