“Θ is Woodin in HOD” directly from mice

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This note describes some work of Schindler, Sargsyan, Steel and the author, mostly done at the time of the 1st Irvine Conference on Descriptive Inner Model Theory and Hod Mice, July 2016. The note is intended as a supplement to the talk given by the author at that conference (on July 27). It is not intended for publication; a modified version will be prepared for publication at a later date, and a very similar argument appears in Schindler and Sargsyan's paper [3]. The note gives more details than were provided in the talk. Moreover, in the talk I made a claim for which I eventually realized (in conversation with Ralf Schindler) that I did not have a proof. This is explained and retracted in Remark 2.1. The note includes a correction to this oversight.

1 Introduction

Suppose $M_1^\#$ exists and is fully iterable. Let δ^{M_1} be the Woodin cardinal of M_1. Let κ^{M_1} be the least inaccessible κ of M_1 such that $\kappa > \delta^{M_1}$. Let Λ be the restriction of the iteration strategy for M_1 to finite stacks of non-dropping normal trees in $M_1|\kappa^{M_1}$. We give an inner model theoretic proof that:

– $M_1[\Lambda]$ satisfies “δ^{M_1} is Woodin” and

– $M_1[\Lambda]$ is iterable as a hybrid mouse.

Let $x \in \mathbb{R}$ be such that $M_1^\# \leq_T x$. Let κ_x be the least inaccessible of $L[x]$. Let $G \subseteq \mathrm{Col}(\omega, <\kappa_x)$ be $L[x]$-generic. Let $\delta_\infty = \omega_2^{L[x,G]}$. We also give an inner model theoretic proof that $\mathrm{HOD}^{L[x,G]}$ satisfies “δ_∞ is Woodin”.

The fact that $\mathrm{HOD}^{L[x,G]}$ satisfies “δ_∞ is Woodin” was proven by Woodin from less assumptions, using the methods of [2]. But the argument we give here proceeds directly from the mouse existence assumptions, without using the theory of [2]. We will actually establish a little more than is necessary to prove the facts stated above. This is in order to ensure that the proof generalizes readily to higher contexts. In particular, the proof adapts to the context of Schindler and Sargsyan's paper [3], as explained there. Moreover, this argument is the only one known to work in that context, as Steel observed that the determinacy assumptions used in [2] do not hold there.

As we attempt to indicate below, parts of the argument are due (some independently) to Schindler, Sargsyan, and the author, and a key idea (regarding Claim 13) was suggested by John Steel. The unattributed claims are due to the author (though may have also been independently observed by others). Conversations with Nam Trang were useful to the author.

The argument relies on some analysis of the direct limit system for $\mathrm{HOD}^{L[x,G]}$. We briefly recall some facts and notation regarding the $L[x,G]$ system; see [1] for details. The reader can verify that everything in this recollection is established without using the fact that δ_∞ is Woodin in $\mathrm{HOD}^{L[x,G]}$.

Given an M_1-like premouse P, we write δ^P for the Woodin of P and κ^P for the least inaccessible κ of P such that $\delta^P < \kappa$. Let $\mathcal{V}$ be an iteration tree on an M_1-like premouse P. We say that $\mathcal{V}$ is *relevant* iff $\mathcal{V} = \vec{\mathcal{T}} = \langle \mathcal{T}_i \rangle_{i \leq n}$ is a stack of normal trees $\mathcal{T}_i$, where $n < \omega$, $M_0^{\mathcal{T}_0} = P$, and for each $i < n$, $\mathcal{T}_i$ has successor length and does not drop on its main branch $b^{\mathcal{T}_i}$ and $M_0^{\mathcal{T}_{i+1}} = M_\infty^{\mathcal{T}_i}$. If P is fully iterable for relevant trees then there is a unique iteration strategy for such trees (this uses the Zipper Lemma together with the fact that if X is any proper class of ordinals then $\mathrm{Hull}^P(X)$ is cofinal in δ^P), which we denote Σ_P; so $\Sigma_P(\vec{\mathcal{T}})$ is just the unique cofinal wellfounded branch of $\mathcal{T}_n$. A *relevant iterate* of P is a model of the form $M_\infty^{\vec{\mathcal{T}}}$, where $\vec{\mathcal{T}}$ is as above and has a final model, and the iterate is *non-dropping* iff $b^{\vec{\mathcal{T}}}$ does not drop. If Σ_P exists then Λ_P denotes the restriction of Σ_P to trees in $P|\kappa^P$.

We have that $x \in \mathbb{R}$ and $M_1^\# \leq_T x$ and

$$G \text{ is } (L[x], \mathrm{Col}(\omega, <\kappa_x))\text{-generic},$$

where κ_x is the least inaccessible of $L[x]$. Let $\mathcal{F}^x$ denote the collection of all non-dropping relevant iterates P of M_1 such that $P|\delta^P \in \mathrm{HC}^{L[x,G]}$. Given non-dropping iterates P, Q of M_1 such that Q is an iterate of P, we write $i_{P,Q}$ for the iteration map $P \to Q$. The model M_∞^+ is the direct limit of the premice in $\mathcal{F}^x$ under the the iteration maps. We also write i_{P,M_∞^+} for the direct limit map $P \to M_\infty^+$. We write $\delta_\infty = \delta^{M_\infty^+}$ and $\kappa_\infty = \kappa^{M_\infty^+}$. Let $\Sigma = \Lambda_{M_\infty^+}$. For $P \in \mathcal{F}^x$ and $s \in \mathrm{OR}^{<\omega}$, say that (P,s) is *stable* iff whenever $Q \in \mathcal{F}^x$ is a relevant iterate of P, we have $i_{P,Q}(s) = s$. For $P \in \mathcal{F}^x$, say that P is *x-good* iff x is extender algebra generic over P.

Let $\mathcal{F}_{\mathrm{cov}}$ denote the covering direct limit system (with nodes (P,s) where P is M_1-like, $P|\delta^P \in \mathrm{HC}^{L[x,G]}$, $s \in \mathrm{OR}^{<\omega}$, $\delta^P < \max(s)$ and P is strongly s-iterable in $L[x,G]$; see [1]). Of course if $P \in \mathcal{F}^x$ and (P,s) is stable and $\delta^P < \max(s)$ then $(P,s) \in \mathcal{F}_{\mathrm{cov}}$. Let M_∞ be the direct limit of $\mathcal{F}_{\mathrm{cov}}$. So $M_\infty|\delta_\infty \in \mathrm{HOD}^{L[x,G]}$. Let $(P,s) \in \mathcal{F}_{\mathrm{cov}}$. Then

$$s^- = s \backslash \{\max(s)\},$$
$$H_s^P = \mathrm{Hull}^{P|\max(s)}(\gamma_s^P \cup \{s^-\}),$$

where

$$\gamma_s^P = \sup(\mathrm{Hull}^{P|\max(s)}(\{s^-\}) \cap \delta^P).$$

We write

$$\pi_{(P,s),(Q,t)} : H_s^P \to H_t^Q,$$
$$\pi_{(P,s),\infty} : H_s^P \to M_\infty$$

for the maps defined in [1] (in particular, $\pi_{(P,s),\infty}$ is the direct limit map). We also define the maps $*$ and σ as in [1]:

$$(s^-)^* = \pi_{(P,s),\infty}(s^-),$$

and $\sigma : M_\infty \to M_\infty^+$ is the unique map such that

$$\sigma(\pi_{(P,s),\infty}(x)) = i_{P,M_\infty^+}(x)$$

whenever $P \in \mathcal{F}^x$ and (P,s) is stable and $x \in H_s^P$. So we have $\sigma \restriction \delta_\infty = \mathrm{id}$; we show below that in fact, $\sigma = \mathrm{id}$. (In the present context we obviously have $M_\infty = M_\infty^+$, because $M_\infty = L[M_\infty|\delta_\infty] = M_\infty^+$, but we give a more generalizable proof of this in what follows.)

Let P be a transitive proper class. By (the) *indiscernibles* for P we mean Silver indiscernibles (if they exist), and we denote this class by I^P. By *elementarily-indiscernibles* for P we mean an infinite class I of ordinals such that for all $n < \omega$ and formulas φ and $\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1} \in I$, if $\alpha_i < \alpha_{i+1}$ and $\beta_i < \beta_{i+1}$ for $i + 1 < n$, then

$$P \models \varphi(\alpha_0, \ldots, \alpha_{n-1}) \iff P \models \varphi(\beta_0, \ldots, \beta_{n-1}).$$

2 The argument

Write $I = I^{L[x]}$. Claims 1 and 3 were observed independently by the author and Schindler.

Claim 1. *For any $P, Q \in \mathcal{F}^x$ with P being x-good and Q a relevant iterate of P, we have*

$$I^P = I^Q = I = I^{L[x,G]}.$$

Therefore $i_{P,Q} \restriction I = \mathrm{id}$ and (P,s) is stable for every $s \in I^{<\omega}$.

Proof. The fact that $I^P = I = I^{L[x,G]}$ is standard, and likewise the fact that $I^Q = i_{P,Q}``I^P$. But we can further iterate Q to some x-good $P' \in \mathcal{F}^x$. Then $I^{P'} = I^P$, which implies that $I^Q = I^P$. The rest follows. □

In the next claim we prove that $M_\infty = M_\infty^+$. This fact of course already follows from the fact that $M_\infty|\delta_\infty = M_\infty^+|\delta_\infty$, but we give a proof here which is more generalizable. The claim resolves a question raised implicitly in [1] (see the paragraph preceding [1, Lemma 3.38]).

Claim 2. $\sigma = \mathrm{id}$ *and therefore $M_\infty = M_\infty^+$.*

Proof. Let $y \in M_\infty^+$. Let $P \in \mathcal{F}^x$ and $\bar{y} \in P$ be such that P is x-good and $i_{P,M_\infty^+}(\bar{y}) = y$. Let $s \in I^{<\omega} \backslash \{\emptyset\}$ be such that $\bar{y} \in H_s^P$. Then (P,s) is stable by Claim 1, so $(P,s) \in \mathcal{F}_{\mathrm{cov}}$, and

$$\sigma(\pi_{(P,s),\infty}(\bar{y})) = i_{P,M_\infty^+}(\bar{y}) = y,$$

so $y \in \mathrm{rg}(\sigma)$, as required. □

From now on we just write M_∞ (not M_∞^+). Let

$$G' \text{ be } (M_\infty, \mathrm{Col}(\omega, <\kappa_\infty))\text{-generic}.$$

We write $\mathcal{F}^{M_\infty}$ for the set of non-dropping relevant iterates P of M_∞ such that $P \in \mathrm{HC}^{M_\infty[G']}$. We write $(M_\infty)^{M_\infty}$ for the direct limit of the premice in $\mathcal{F}^{M_\infty}$ under the iteration maps. Let

$$j : M_1 \to M_\infty,$$

$$k : M_\infty \to (M_\infty)^{M_\infty}$$

be the iteration maps. By [1], we have

$$\mathrm{HOD}^{L[x,G]} = M_\infty[* \restriction \delta_\infty] = M_\infty[\Sigma],$$

$$V_{\delta_\infty}^{M_\infty[\Sigma]} = V_{\delta_\infty}^{M_\infty},$$

$$\delta_\infty = \omega_2^{L[x,G]} = (\kappa_x^+)^{L[x]},$$

and in particular, δ_∞ is regular in $L[x,G]$, and hence, in $M_\infty[\Sigma]$.

Claim 3. *Let $P \in \mathcal{F}^x$ be x-good and let $Q \in \mathcal{F}^{M_\infty}$. Then*

$$I = I^{M_\infty} = I^Q = I^{(M_\infty)^{M_\infty}}.$$

Therefore $i_{P,M_\infty} \restriction I = i_{Q,(M_\infty)^{M_\infty}} \restriction I = \mathrm{id}$.

Proof. Because $I = I^{L[x,G]}$ and $M_\infty|\delta_\infty \in \mathrm{HOD}^{L[x,G]}$, I is a club class of elementarily-indiscernibles for M_∞, so $I \subseteq I^{M_\infty}$. But by Claim 1, $I = I^P$ so

$$I^{M_\infty} = i_{P,M_\infty}``I.$$

It follows that $I^{M_\infty} = I$ and $i_{P,M_\infty} \restriction I = \mathrm{id}$. The rest is similar. □

Claim 4. $* \restriction I = \mathrm{id}$.

Proof. For any $s \in I^{<\omega} \backslash \{\emptyset\}$ and x-good $P \in \mathcal{F}^x$,

$$(s^-)^* = \sigma((s^-)^*) = i_{P,M_\infty}(s^-) = s^-$$

by Claims 1 and 3. □

Let $\mathcal{F}_{\mathrm{cov}} \restriction I$ denote the restriction of $\mathcal{F}_{\mathrm{cov}}$ to pairs (P,s) with $s \in I^{<\omega}$.

Claim 5. *$\mathcal{F}_{\mathrm{cov}} \restriction I$ covers $\mathcal{F}_{\mathrm{cov}}$, in that for every $x \in M_\infty$ there is $(P,s) \in \mathcal{F}_{\mathrm{cov}}$ such that $x \in \mathrm{rg}(\pi_{(P,s),\infty})$.*

Proof. Use the previous claims and the fact that M_∞ is the hull of $I \cup \delta_\infty$. □

Claim 6. *Let $P, Q, R \in \mathcal{F}^x$ with P being x-good, Q an iterate of P, and R an iterate of Q. Let $s \in I^{<\omega} \backslash \{\emptyset\}$. So $(P,s), (Q,s), (R,s) \in \mathcal{F}_{\mathrm{cov}}$. Then*

$$\pi_{(Q,s),(R,s)} = i_{Q,R} \restriction H_s^Q,$$

$$\pi_{(Q,s),\infty} = i_{Q,M_\infty} \restriction H_s^Q.$$

Likewise for $Q, R \in \mathcal{F}^{M_\infty}$.

As far as the author knows, the following claim is due to Schindler (the lesser fact that $* \restriction \delta_\infty = k \restriction \delta_\infty$ is proved in [1]):

Claim 7. $* = k \restriction \mathrm{OR}$.

Proof. Let $\alpha \in \mathrm{OR}$. Let $s \in I^{<\omega}$ be such that $\alpha \in H_s^{M_\infty}$. Let $P \in \mathcal{F}^x$ be such that $\alpha \in \mathrm{rg}(\pi_{(P,s),\infty})$ and $(P, s \cup \{\alpha\})$ is stable and P is x-good. Let $\pi_{(P,s),\infty}(\bar{\alpha}) = \alpha$. Let $s^+ = s \cup \{\iota\}$ where $\iota \in I$ and $\max(s) < \iota$. By Claim 1, $(P, s^+ \cup \{\alpha\})$ is stable. Now $P|\iota$ satisfies "It is forced by $\mathrm{Col}(\omega, < \kappa_x)$ that $\pi_{(L[\mathbb{E}],s),\infty}(\bar{\alpha}) = \alpha$". Using Claim 1, therefore $M_\infty|\iota$ and M_∞ satisfy "It is forced by $\mathrm{Col}(\omega, < \kappa_\infty)$ that $\pi_{(L[\mathbb{E}],s),\infty}(\alpha) = \alpha^*$". So by Claim 6, $k(\alpha) = \alpha^*$, as required. $\square$

For $s \in I^{<\omega} \backslash \{\emptyset\}$, let $k_s = k \upharpoonright H_s^{M_\infty}$.

Claim 8. *Let $s \in I^{<\omega} \backslash \{\emptyset\}$. Then $k_s \in M_\infty$ and in fact, $k_s \in \mathrm{rg}(j)$.*

Proof. Let ι, s^+ be as before and let $\gamma = \gamma_s^{M_\infty}$. Over $M_\infty|\iota$, from the parameter s we can define (i) the natural surjection $\gamma \to H_s^{M_\infty}$, (ii) $k \upharpoonright \gamma$, (iii) $(M_\infty)^{M_\infty}|\delta^{(M_\infty)^{M_\infty}}$, and therefore, (iv) k_s. But $s^+ \in \mathrm{rg}(j)$, so $k_s \in \mathrm{rg}(j)$. $\square$

Claim 9. $\mathrm{rg}(j)$ *is closed under $*$ and $*^{-1}$.*

Proof. Let $\alpha \in \mathrm{OR}$ and let $s \in I^{<\omega}$ be such that $\alpha \in H_s^{M_\infty}$. By Claim 8, $k_s \in \mathrm{rg}(j)$, and by Claim 7, $\alpha^* = k_s(\alpha)$. Therefore $\alpha \in \mathrm{rg}(j)$ iff $\alpha^* \in \mathrm{rg}(j)$. $\square$

Let $H = \mathrm{Hull}^{M_\infty[\Sigma]}(\mathrm{rg}(j)) = \mathrm{Hull}^{M_\infty[\Sigma]}(I)$; note this hull is uncollapsed.

Claim 10. $H \cap \mathrm{OR} = \mathrm{rg}(j) \cap \mathrm{OR}$.

Proof. Let $\alpha \in H \cap \mathrm{OR}$. By Claim 9, it suffices to see that $\alpha^* \in \mathrm{rg}(j)$. Let $s \in I^{<\omega}$ be such that α is definable over $M_\infty[\Sigma]$ from s. Let $P \in \mathcal{F}^x$ be x-good and such that $(P, \{\alpha\})$ is stable. Then note that α is definable over P from s, and it easily follows that $\alpha^* \in \mathrm{rg}(j)$. $\square$

Letting Λ be the restriction of Σ_{M_1} to trees in $M_1|\kappa^{M_1}$, we therefore have:

Claim 11. *The transitive collapse of H is $M_1[\Lambda]$.*

2.1 Remark. The claim above was proved in [1], and it was also proved (implicitly) in [1] that $M_1[\Lambda]$ is iterable. However, the arguments given there do not generalize well to higher contexts. [1] We just gave a proof of Claim 11 which does generalize well. Schindler first found a proof for iterability which does generalize well; we give a variant of his proof to establish Claim 12 below.

During the 1st Irvine Conference on Descriptive Inner Model Theory and Hod Mice, I claimed that

$$M_1 \ \&\ M_1[\Lambda] \text{ have the same } < \delta^{M_1}\text{-sequences of ordinals,} \tag{1}$$

[1] The argument implicit in [1] is as follows. Let $\mathcal{T}$ be an iteration tree on M_∞ of successor length $\alpha + 1$, such that $[0, \alpha]_\mathcal{T}$ does not drop. Suppose that $\mathcal{T}$ copies to a putative tree $\mathcal{U}$ on $M_\infty[\Sigma]$. We claim that $M_\alpha^\mathcal{U}$ is wellfounded. To see this, we define an order-preserving

$$\varrho : \mathrm{OR}^{M_\alpha^\mathcal{U}} \to \mathrm{OR}.$$

Let $\delta = \delta(M_\alpha^\mathcal{T})$. So $M_\alpha^\mathcal{U}$ is δ-wellfounded and δ is the unique Woodin of $M_\alpha^\mathcal{U}$ and $M_\alpha^\mathcal{U}|\delta = M_\alpha^\mathcal{T}|\delta$. Let $\kappa < \delta_\infty$ and $f : \kappa^{<\omega} \to \mathrm{OR}$ with $f \in M_\infty[\Sigma]$. Let $a \in \delta^{<\omega}$. Let $f^* = * \circ f$. Then $f^* \in M_\infty$ by [1, Lemma 3.45]. Define

$$\varrho(i_{0,\alpha}^\mathcal{U}(f)(a)) = i_{0,\alpha}^\mathcal{T}(f^*)(a).$$

It is easy to see that ϱ works. By copying, it follows that $M_1[\Lambda]$ is iterable. Moreover, iterates of $M_1[\Lambda]$ are of the form $N[\Lambda_N]$, where N is an iterate of M_1; this is because Λ_N chooses unique wellfounded branches.

and used this to establish iterability (in a supposedly generalizable manner). However, I realized afterward, in conversation with Schindler, that I did not have a proof of line (1) above. So I retract this claim. The proof of Claim 12 does not rely on line (1). In the end, as a corollary to Claim 12, we will deduce in 2.2 a weakening of line (1).

Let

$$\pi_0 : M_1 \to M_1 \subseteq M_1[\Lambda]$$

be the identity map. Let Ψ be the putative iteration strategy for $M_1[\Lambda]$ given by "inverse copying" trees $\mathcal{U}$ on $M_1[\Lambda]$ to trees $\mathcal{T}$ on M_1 via Σ_{M_1}. This makes sense because M_1 and $M_1[\Lambda]$ have the same universe below δ^{M_1}, which is regular in both models. Given a putative tree $\mathcal{U}$ on $M_1[\Lambda]$ via Ψ, and given $\alpha < \mathrm{lh}(\mathcal{U})$ such that $[0, \alpha]_{\mathcal{U}}$ does not drop, write $M^{\mathcal{U}}_\alpha = N^{\mathcal{U}}_\alpha[\Lambda^{\mathcal{U}}_\alpha]$ (so $N^{\mathcal{U}}_\alpha$ is putatively M_1-like and $\Lambda^{\mathcal{U}}_\alpha$ is some putative strategy). Given also the inverse copy $\mathcal{T}$ on M_1, let

$$\pi_\alpha : M^{\mathcal{T}}_\alpha \to N^{\mathcal{U}}_\alpha$$

be the copy map.

The following claim was first proved by Ralf Schindler, by a related but different argument (see [3]) to what we give. The author later found the proof below, mostly independently of Schindler's work.

Claim 12 (Schindler). *Let $\mathcal{T}, \mathcal{U}, \alpha, \pi_\alpha$ be as above. Then $N^{\mathcal{U}}_\alpha = M^{\mathcal{T}}_\alpha$ and $\pi_\alpha = \mathrm{id}$ and $\Lambda^{\mathcal{U}}_\alpha = \Lambda_{M^{\mathcal{T}}_\alpha}$; thus, $M^{\mathcal{U}}_\alpha = M^{\mathcal{T}}_\alpha[\Lambda_{M^{\mathcal{T}}_\alpha}]$, and in particular, $M^{\mathcal{U}}_\alpha$ is wellfounded.*

Proof. Because we are literally dealing with M_1, the fact that $\Lambda^{\mathcal{U}}_\alpha$ is correct follows from the fact that $\pi_\alpha = \mathrm{id}$. But the proof to follow will actually work more generally, by lifting $\Lambda^{\mathcal{U}}_\alpha$ to a strategy which we know is correct.

Let $P = M^{\mathcal{T}}_\alpha$. Modifying earlier notation, let $M_\infty = (M_\infty)^{P[G]}$ and $M_\infty[\Sigma] = \mathrm{HOD}^{P[G]}$ where G is $\mathrm{Col}(\omega, <\kappa^P)$-generic over P. We have the natural maps i_{M_1,M_∞}, $i_{M_1,P}$ and i_{P,M_∞}, which commute. Let

$$H^{M_\infty[\Sigma]}_{M_1} = \mathrm{Hull}^{M_\infty[\Sigma]}(\mathrm{rg}(i_{M_1,M_\infty})),$$

$$H^{M_\infty[\Sigma]}_{P} = \mathrm{Hull}^{M_\infty[\Sigma]}(\mathrm{rg}(i_{P,M_\infty})),$$

$$H^{P[\Lambda_P]}_{M_1} = \mathrm{Hull}^{P[\Lambda_P]}(\mathrm{rg}(i_{M_1,P})).$$

Then as before,

$$H^{M_\infty[\Sigma]}_{M_1} \cap M_\infty = \mathrm{rg}(i_{M_1,M_\infty}),$$

$$H^{M_\infty[\Sigma]}_{P} \cap M_\infty = \mathrm{rg}(i_{P,M_\infty}).$$

So the transitive collapse of $H^{M_\infty[\Sigma]}_{M_1}$ is $M_1[\Lambda]$, and the transitive collapse of $H^{M_\infty[\Sigma]}_{P}$ is $P[\Lambda_P]$ (note their strategies lift to $M_\infty[\Sigma]$). So by commutativity, the transitive collapse of $H^{P[\Lambda_P]}_{M_1}$ is $M_1[\Lambda]$. Let

$$i^+_{M_1,M_\infty} : M_1[\Lambda] \to M_\infty[\Sigma]$$

be the uncollapse map; so $i_{M_1,M_\infty} \subseteq i^+_{M_1,M_\infty}$. Likewise for $i^+_{M_1,P}$ and i^+_{P,M_∞}. Let E be the $(\delta^{M_1}, \delta^{M_\infty})$-extender derived from i_{M_1,M_∞}, let $E_{M_1,P}$ be the (δ^{M_1}, δ^P)-extender derived from $i_{M_1,P}$, and E_{P,M_∞} the $(\delta^P, \delta^{M_\infty})$-extender derived from i_{P,M_∞}. Then

$$\begin{aligned} E &= E_{P,M_\infty} \circ E_{M_1,P}, \\ P &= \mathrm{Ult}(M_1, E_{M_1,P}), \\ M_\infty &= \mathrm{Ult}(P, E_{P,M_\infty}) = \mathrm{Ult}(M_1, E), \end{aligned}$$

and $i_{M_1,P}$, i_{P,M_∞}, i_{M_1,M_∞} are the ultrapower maps. Because $i_{M_1,M_\infty} \subseteq i^+_{M_1,M_\infty}$, the commuting factor map

$$\varrho : \mathrm{Ult}(M_1[\Lambda], E) \to M_\infty[\Sigma]$$

is the identity. Similarly for the other ultrapowers, so

$$\begin{aligned} M_\infty[\Sigma] &= \mathrm{Ult}(M_1[\Lambda], E), \\ P[\Lambda^P] &= \mathrm{Ult}(M_1[\Lambda], E_{M_1,P}), \\ M_\infty[\Sigma] &= \mathrm{Ult}(P[\Lambda^P], E_{P,M_\infty}), \end{aligned}$$

and $i^+_{M_1,M_\infty}$, $i^+_{M_1,P}$, i^+_{P,M_∞} are the ultrapower maps. But $E_{M_1,P}$ is the branch extender of $\mathcal{U}$, so

$$N^{\mathcal{U}}_\alpha[\Lambda_\alpha] = M^{\mathcal{U}}_\alpha = \mathrm{Ult}(M_1[\Lambda], E_{M_1,P}) = P[\Lambda^P]$$

and $i^{\mathcal{U}} = i^+_{M_1,P}$ is the ultrapower map. So $N^{\mathcal{U}}_\alpha = P = M^{\mathcal{T}}_\alpha$ and $\Lambda_\alpha = \Lambda_P = \Lambda_{M^{\mathcal{T}}_\alpha}$. Further,

$$\pi_\alpha : P \to N^{\mathcal{U}}_\alpha$$

is such that $\pi_\alpha \restriction \delta^P = \mathrm{id}$ and $i^{\mathcal{U}} = \pi_\alpha \circ i^{\mathcal{T}}$. But since

$$i^{\mathcal{T}} = i_{M_1,P} \subseteq i^+_{M_1,P} = i^{\mathcal{U}},$$

we have $\pi_\alpha \restriction \mathrm{rg}(i^{\mathcal{T}}) = \mathrm{id}$. But then

$$\delta^{N^{\mathcal{U}}_\alpha} \cup I^{N^{\mathcal{U}}_\alpha} \subseteq \mathrm{rg}(\pi_\alpha),$$

so $\mathrm{rg}(\pi_\alpha) = N^{\mathcal{U}}_\alpha$, so $\pi_\alpha = \mathrm{id}$. □

Now that we have proved Claim 12, we don't need line (1). However, from Claim 12 we can deduce that the following weakening of line (1) holds, and if we had known this weakening in advance, we could have used it in the role initially desired for line (1).

2.2 Corollary. *Let $\kappa < \delta^{M_1}$ and $f : \kappa^{<\omega} \to \mathrm{OR}$ with $f \in M_1[\Lambda]$. Let $\mathcal{T}$ be an iteration tree on M_1 of length $\alpha + 1$, such that $[0, \alpha]_{\mathcal{T}}$ does not drop. Let $b \in i^{\mathcal{T}}(\kappa)^{<\omega}$ and E_b be the measure over $\kappa^{<\omega}$ derived from $i^{\mathcal{T}}$ and b (so $E_b \in M_1$). Then there is an E_b-measure one set $A \in M_1$ such that $f \restriction A \in M_1$.*

Proof. Otherwise $\mathcal{T}$ and its copy $\mathcal{U}$ on $M_1[\Lambda]$, contradicts Claim 12. □

We now return to our earlier notation with M_∞ and $M_\infty[\Sigma]$ being in the sense of $L[x, G]$. In order to prove the final claim below, we need the following fact, due to the author and Steel:

2.3 Fact. M_∞ is a normal iterate of M_1.

The basic idea for proving the final claim was suggested to the author by Steel. The claim was also proved independently by Sargsyan, by a similar argument.

Claim 13. *δ_∞ is Woodin in $M_\infty[\Sigma]$, and therefore δ^{M_1} is Woodin in $M_1[\Lambda]$.*

Proof. Suppose not and let $Q \trianglelefteq M_\infty[\Sigma]$ be the Q-structure for δ_∞. (Here we take $M_\infty[\Sigma]$ arranged in a fine-structural strategy premouse hierarchy above δ_∞.) Using the fact above, there is a limit length normal tree $\mathcal{T}$ on M_1 such that $M(\mathcal{T}) = M_\infty|\delta^{M_\infty}$. Clearly $\mathcal{T} \in L[x, G]$. Let $b = \Sigma_{M_1}(\mathcal{T})$. Let $\mathcal{U}$ on $M_1[\Lambda]$ be the copy of $\mathcal{T}$. Then by Claim 12, $i^{\mathcal{U}}_b(\bar{Q}) = Q$, where $\bar{Q} \trianglelefteq M_1[\Lambda]$ is the Q-structure for $M_1|\delta^{M_1}$, (where $M_1[\Lambda]$ is also arranged as a strategy premouse). Working in $L[x, G][J]$ where J is $(L[x, G], \mathrm{Col}(\omega, \delta_\infty))$-generic, we have a tree searching for a pair (R, c) such that R is a strategy premouse extending $M_1|\delta^{M_1}$, δ^{M_1} is inaccessible in $\mathcal{J}(R)$, c is a non-dropping $\mathcal{T}$-cofinal branch, and considering $\mathcal{T}$ as a tree on $\mathcal{J}(R)$, then $i^{\mathcal{T}}_c(R) = Q$. So there is some such $(c_0, R_0) \in L[x, G][J]$. But the proof of the Zipper Lemma shows that there is a unique such pair (c, R). (Given two such pairs (c, R) and (c', R'), we consider $\mathcal{T}$ as a tree $\mathcal{U}$ on $\mathcal{J}(R)$ and as a tree $\mathcal{U}'$ on $\mathcal{J}(R')$. But $\mathcal{J}(R)$ and $\mathcal{J}(R')$ agree below δ^{M_1}, an inaccessible of both models, and $\mathcal{U}, \mathcal{U}'$ are based below δ^{M_1}. Therefore the models of $\mathcal{U}$ agree sufficiently with the models of $\mathcal{U}'$ that we can run the proof of the Zipper Lemma, even with trees on two different base models.) Therefore $(c_0, R_0) = (b, \bar{Q}) \in L[x, G]$, so δ_∞ has cofinality ω in $L[x, G]$, a contradiction. □

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