2016 - 07 - 22 11 : 15 JOHN STEEL

Given $\mathcal{T}$ on M and $\mathcal{U}$ on $M^{\mathcal{T}}_{\theta}$ where $lh(\mathcal{T}) = \theta + 1$ both normal. Assume $\mathcal{U}$ has a last model P.

$$P \xrightarrow{\sigma} \text{last model of } W(\mathcal{T}, \mathcal{U})$$

We define $W(\mathcal{T}, \mathcal{U})$, normal. Define by induction on γ: $W(\mathcal{T}, \mathcal{U} \restriction (\gamma+1)) \overset{def}{=} W_\gamma$

Also define for $\nu <_{\mathcal{U}} \gamma$ a partial map

$$\varphi_{\nu,\gamma} : lh(W_\nu) \to lh(W_\gamma)$$

(Also: $W(\mathcal{T}, F) = \mathcal{T} \restriction \alpha + 1 {}^\frown \langle F \rangle$ if F not total on $M^{\mathcal{T}}_{\beta} | \lambda(E^{\mathcal{T}}_{\beta})$

$\operatorname{dom} \varphi = \beta + 1$)

And

$$\pi^{\nu,\gamma}_{\xi} : \cancel{M^{W_\gamma}_{\varphi_{\nu,\gamma}(\xi)}} \; M^{W_\nu}_{\xi} \longrightarrow M^{W_\gamma}_{\varphi_{\nu,\gamma}(\xi)} \quad \text{for } \xi \in \operatorname{dom}(\varphi_{\nu,\gamma})$$

Let $R_\gamma = M^{W_\gamma}_{z(\gamma)}$ = last model of W_γ.

We will have $\sigma_\gamma : M^{\mathcal{U}}_{\gamma} \longrightarrow R_\gamma$

- $W_0 = \mathcal{T}$.
- Given W_γ: let $\sigma_\gamma(E^{\mathcal{U}}_{\gamma}) = F_\gamma$

$$\nu = \mathcal{U}\text{-pred}(\gamma + 1)$$

$$W_{\gamma+1} = W(W_\nu, F_\gamma)$$

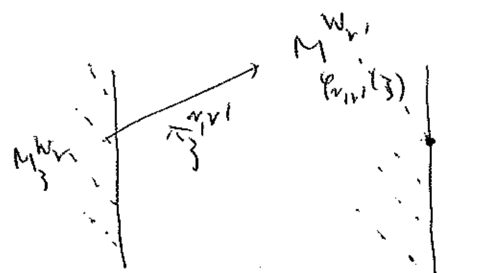
$$\varphi_{\nu,\gamma+1} = \varphi^{W_\gamma, F_\gamma}$$

$$\pi_\xi^{\nu,\gamma+1} = \pi_\xi^{W_\gamma, F_\gamma}$$

$$\begin{tikzcd}
M^{\mathcal{U}}_\gamma \arrow[r, "\sigma_\gamma"] \arrow[d, "i^{\mathcal{U}}_{\gamma,\gamma+1}"'] & R_\gamma = M^{W_\gamma}_{z(\gamma)} \arrow[d, "\pi^{\gamma,\gamma+1}_{z(\gamma)}"] \\
M^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"'] & R_{\gamma+1}
\end{tikzcd}$$

- Non-dropping case
- Standard commuting

Let λ be a limit $< lh(\mathcal{U})$. Define W_λ. Let $b = [0,\lambda)_{\mathcal{U}}$. We have for $\nu <_{\mathcal{U}} \nu' \in b$



$$R_\nu \xrightarrow{\pi^{\nu,\nu'}_{z(\nu)}} R_{\nu'}$$

So we can define $\varphi_{\nu,\lambda}(\xi)$ for each ξ s.t. $\varphi_{\nu,\nu'}(\xi) \in \operatorname{dom} \varphi_{\nu',\eta}$ for $\nu' < \eta \in b$

$lh(W_\lambda)$ = the set of all such $\varphi_{\nu,\lambda}(\xi)$

$$\pi_\xi^{\nu,\lambda} : M^{W_\nu}_\xi \longrightarrow M^{W_\lambda}_{\varphi_{\nu,\lambda}(\xi)}$$

From taking direct limit of

$$\pi^{\nu',\eta}_{\varphi_{\nu,\nu'}(\xi)} : M^{W_{\nu'}}_{\varphi_{\nu,\nu'}(\xi)} \longrightarrow M^{W_\eta}_{\varphi(\nu,\eta)(\xi)}$$

$$E^{W_\lambda}_{\varphi_{\gamma,\lambda}(\xi)} = \pi^{\gamma,\lambda}_{\xi}(E^{W_\gamma}_{\xi})$$

This was the case when $\mathcal{U}$ had a last model.

Now if $lh(\mathcal{U}) = \lambda$ limit:

$$W(\mathcal{T},\mathcal{U}) = \bigcup_{\gamma<\lambda} W_\gamma \restriction (\alpha_\gamma + 1)$$

$\uparrow$

α_γ = least s.t. F_γ is on the $M^{W_\gamma}_{\alpha_\gamma}$-sequence

$$W_\gamma \restriction (\alpha_\gamma+1) = W_\eta \restriction (\alpha_{\gamma+1}) \quad \forall \gamma \geq \eta$$

<u>Branches of $W(\mathcal{T},\mathcal{U})$</u>: $a \rightsquigarrow (c, b)$

$\uparrow$ cofinal in $lh(W(\mathcal{T},\mathcal{U}))$

$\uparrow$ branch of $\mathcal{T}$

$\uparrow$ branch of $\mathcal{U}$ cofinal in λ

Given b let η be least s.t. E_b is an extender over $R_0 | \eta$. Let τ be least s.t. $M^{W_0}_{\tau} | \eta = R_0 | \eta$

$$c = [0, \tau)_{\mathcal{T}}$$

Then $br^W(c, b)$ = determined by downward closure on $\mathcal{U}$ of

$$\{\varphi_{0,b}(\xi) \mid \xi \in c\}$$

If τ is a limit ordinal then the definition $a = br(b, c)$ makes sense for any c cofinal in $\mathcal{T}$.

Fact: Every cofinal branch a of $W(\mathcal{T},\mathcal{U})$ has the form $br(c,b)$ where b is cofinal in $\mathcal{U}$, and b, c are determined by a.

Given Σ a strategy for M:

$$\left(\begin{array}{c} \Sigma \text{ normalizes well } \langle \mathcal{T}, \mathcal{U}^\frown b \rangle \text{ is by } \Sigma \text{ iff} \\ W(\mathcal{T},\mathcal{U})^\frown \end{array} \right)$$

Σ normalizes well $\Rightarrow$ $a = \cancel{W}\ \Sigma(W(\mathcal{T},\mathcal{U}))$

$\Rightarrow$ $a = br(c,b)$ where $b = \Sigma_{\mathcal{T}}(\mathcal{U})$ and $c = \Sigma(\mathcal{T} \mid \sup(c))$

Want every $\Sigma_{\vec{\mathcal{T}}}$ has this property

2-normalizes well:

$$\begin{tikzcd}
 & & \cdot \\
M \arrow[r, "\mathcal{T}"'] \arrow[rru, "W(\mathcal{T},\mathcal{U})"] & \cdot \arrow[r, "\mathcal{U}"'] & \cdot \arrow[u, "\sigma"']
\end{tikzcd}$$

$$\left(\Sigma_{W(\mathcal{T},\mathcal{U})}\right)^{\sigma} = \Sigma_{\langle \mathcal{T},\mathcal{U}\rangle}.$$

Strong hull condensation: Let $\mathcal{T}, \mathcal{U}$ be normal trees on M. A pseudo-hull embedding of $\mathcal{T}$ into $\mathcal{U}$ is a

$$\langle u, \langle t^0_\beta \mid \beta < lh(\mathcal{T}) \rangle, \langle t^1_\beta \mid \beta+1 < lh(\mathcal{T}) \rangle, p \rangle$$

s.t. (a) $u : lh(\mathcal{T}) \to lh(\mathcal{U})$ s.t. ... (like $\varphi^{\mathcal{T},\mathcal{F}}$)

(b) $p : \{E \mid E \text{ used in } \mathcal{T}\} \to \{E \mid E \text{ used in } \mathcal{U}\}$ induces $\hat{p} : \mathcal{T}^{ext} \to \mathcal{U}^{ext}$

$\hat{p}(S) =$ downward closure of $p''\mathrm{rng}(S)$

(c) Let $v : lh(\mathcal{T}) \to lh(\mathcal{U})$ be given by

$$s^{\mathcal{U}}_{v(\beta)} = \hat{p}(s^{\mathcal{T}}_\beta)$$

Then

$$t^0_\beta : M^{\mathcal{T}}_\beta \to M^{\mathcal{U}}_{v(\beta)}$$

is total and elementary. For $\alpha <_T \beta$

$$t^0_\beta \circ \hat{\imath}^{\mathcal{T}}_{\alpha\beta} = \hat{\imath}^{\mathcal{U}}_{v(\alpha), v(\beta)} \circ t^0_\alpha$$

(d) For $\alpha + 1 < lh(\mathcal{T})$, $v(\alpha) \leq_{\mathcal{U}} u(\alpha)$

$$t^1_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha), u(\alpha)} \circ t^0_\alpha$$

and

$$p(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)} = t^1_\alpha(E^{\mathcal{T}}_\alpha)$$

and

$$t^0_\beta \restriction lh(E^{\mathcal{T}}_\alpha) + 1 = t^1_\alpha \restriction \underbrace{lh(E^{\mathcal{T}}_\alpha) + 1}_{\lambda(E^{\mathcal{T}}_\alpha)?} \quad \text{for } \alpha < \beta$$

(e) if $\beta = T\text{-pred}(\alpha+1)$ then

$\mathcal{U}\text{-pred}(u(\alpha)+1) \in [v(\beta), u(\beta)]_{\mathcal{U}}$

and for

$\beta^* = \mathcal{U}\text{-pred}(u(\alpha)+1)$:

$$t^0_{\alpha+1}\left([a,f]^{P}_{E^{T}_\alpha}\right) = \left[t^1_\alpha(a), \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ t^0_\beta(f)\right]_{E^{\mathcal{U}}_{u(\alpha)}}$$

$$\begin{tikzcd}
M^T_{\alpha+1} \arrow[rr, "t^0_{\alpha+1}"] & & M^{\mathcal{U}}_{v(\alpha+1)} \\
& M^{\mathcal{U}}_{u(\beta)} & \\
& & M^{\mathcal{U}}_{\beta^*} \arrow[ul] \arrow[uu, "E^{\mathcal{U}}_{u(\alpha)}"'] \\
M^T_\beta \arrow[uuu, "E^T_\alpha"] \arrow[rr] & & M^{\mathcal{U}}_{v(\beta)} \arrow[u]
\end{tikzcd}$$

Σ has strong hull condensation iff

$\mathcal{U}$ by Σ and T pseudo hull of $\mathcal{U}$

$\Rightarrow$ T by Σ