Background construction $C \quad (\mathcal{M}_{\eta_{1k}}^c, \mathcal{N}_{\eta_{1k}}^c), F^+_v$

$\text{Res}_{\eta_{1k}}[N] = \langle \eta, e \rangle \quad \text{for} \quad N \cong M_{\eta_{1k}}$

$\sigma_{\eta_{1k}}[N] : N \to M_{\eta_{1k}}$

$\Sigma_{\eta_{1k}}^c$ comes from lifting $V$ via a conversion system $\text{lift}(F, M_{\eta_{1k}}, C, \Sigma^x) = \langle \eta^x, \langle \eta, e \rangle_{\eta \in h(F)}, \langle \eta^0, \eta \in h(F) \rangle \rangle$

Here:

(i) $\eta^x$ is a tree on $V_1$ models $P^+_\eta$, $F^+_\eta = E^x$

(ii) If $\eta \leq r$ and $\langle \eta, W \rangle_r$ does not chop then $\pi^+_\eta (\langle \eta, e \rangle) = \pi^+_\eta (\langle \eta, e \rangle) = \pi^+_\eta (\langle \eta, e \rangle)$

$F^+_\eta \quad \text{for} \quad V$

$P^+_\eta \quad \text{models} \quad \mathcal{M}_{\eta_{1k}}^c$

$\text{Res}_{\eta_{1k}}[M^x_{\eta_{1k}} \cup h_{\eta_{1k}}(F^+_\eta)] = \langle \eta_{10} \rangle$

$\Sigma_{\eta_{1k}}^c(F) = b \quad \text{if} \quad \text{lifting} \quad \text{lift}(F, M_{\eta_{1k}}, C) = (\eta^x, -)$

then $b = \Sigma^x(\eta^x)$.
Theorem: Suppose $c$ is a construction as above, $(P, \Sigma)$ is an $\aleph_1 / \aleph_0$ pair, let $M_{\aleph_1}^c, \mathbb{P}_{\aleph_1}$ be defined. Suppose $P$ iterates by $\Sigma$ via a normal tree $T$ to $\mathcal{C} = M_{\aleph_1}^c$. Then $\Sigma_{\aleph_1} \mathbb{P}_{\aleph_1}^c = \mathbb{P}_{\aleph_1}^c$.

Proof (Sketch): Let $U \subseteq V_{\aleph_1}$ be both $\Sigma_{\aleph_1}^c$ and $\mathbb{P}_{\aleph_1}^c$ of limit length. Let

$$\text{lift}(U, M_{\aleph_1}^c, \Sigma) = (U^+)$$

let $b = \Sigma^*(U^+) = \mathbb{P}_{\aleph_1}^c(U)$

Then $(T, U^b)$ is a pseudo-hull of $i_b^*(T)$. But $i_b^*(T)$ is a tree on $P$

(e.g. $P \subseteq b^b$, so is not moved by backgrounds)

by $i_b^*(\Sigma) = \Sigma \cap M_b^c$ by universal Baireness

(we made $b$ an assumption)

So $(T, U^b)$ is by $\Sigma \Rightarrow (T, U^b)$ is by $\Sigma$

(since $\Sigma$ has strong hull cond.)

(since $\Sigma$ normalizes $\text{sh}^b$)

To see the conclusion about pseudo-hull:

Assume there held for $V_{\aleph_1}^b \leq V_{\aleph_0}^b$

Had $W^*$, normal on $P$, $\mathcal{C}$-closed model

$M_{\aleph_0}^c \models \Sigma^*_{V_{\aleph_1}^b} M_{\aleph_1}^c = \mathbb{P}_{\aleph_1}^c$, for $(V_{\aleph_1}^b) \leq V_{\aleph_0}^b$
Let \( \text{lift}(u, M, n, C) = (w^u, y^0, l^0) \) if \( y^0 < l^0(u) \), \((y^0, l^0(u)) \)

Let \( S_y = M^u_y \). Let \( W^y = (W^*_y)_{y^0, l^0} \).

Construct by induction: (no dropping: \( W^*_y = i^u_y(S_y) \)).

A pseudo-hull embedding

\( \Phi_y \) from \( W_y = W^y(u^y, u^y + 1) \)

to \( W^*_y \)

Idea: If \( v < u \); and

\[ l^v = \text{lift}(u + v + 1, M, n, C) \]

\[ y^v = \text{lift}(f) \]

\( \lambda^v \) is the pseudo-hull embedding of \( W_y \)

into \( W^*_y \) obtained from the

normalization process

\( \lambda^v \) is the pseudo-hull embedding on

the background level of \( W^*_y \) into \( W^*_y \)

then "everything commutes", that is:

\[ \lambda^v \circ \Phi_y = \Phi_y \circ \lambda \]

"Embedding normalization commutes with the lifting process!"