Assume: $AD^+ + \neg AD_{\mathbb{R}}$

$+$ HPC Given a Suslin-co-Suslin set A, there is a (P, Σ) above A in $\leq_W$ (Hod pair capturing)

$+$ sh-MC

From last time: There is a pair (P, Σ) s.t.

(1) P is a lpm with last block (κ^P, δ^P)

(2) Σ is short-tree strategy of P

(Σ is a partial IS for P with strong hull condensation and ~~some~~ full normalization.) Also, if $\mathcal{T} \in dom(\Sigma)$, $b = \Sigma(\mathcal{T})$ is defined $\Rightarrow$ $\mathcal{T}$ according to Σ, $\overrightarrow{\mathcal{T}^\frown\lambda}$ short all $\lambda < lh(\mathcal{T})$ either b drops or else $i_b(\delta^P) > \delta(\mathcal{T})$)

(3) the dir lim $M(P, \Sigma)$ extends $HOD|\kappa$ when κ = the largest Suslin cardinal, and if $i: P \xrightarrow{\mathcal{T}} M(P, \Sigma)$ ($\mathcal{T}$ short) is the iteration map then $i(\kappa^P) = \kappa$. ($\Rightarrow \Sigma$ is κ-Suslin)

Why? Σ_1-reflect the statement "There is no such (P, Σ)" to some universe $N \models ZF^- + DC + HPC + \ldots +$ "No such (P, Σ)".

N is coded by set A which is Suslin-co-Suslin.

Let $(N^*, \delta_{N^*}, \Sigma_{N^*})$ be a Γ-Woodin mouse for some inductive-like scaled $\Gamma > N$. Want ~~to~~ that $(N^*, \delta_{N^*}, \Sigma_{N^*})$ captures a hod pair (P^*, Σ^*) s.t. $\Sigma^* \notin N$. & HPC

Do a HMC construction in N^*. Let $(P,\Sigma) = (N_{\gamma,k}, \Omega_{\gamma,k})$

$(N^*_{\gamma,k}, \Omega_{\gamma,k} \mid (\gamma,k) \leq_{lex} (\delta_{N^*}, 0))$

be least s.t. $\Sigma \notin N$. Or take (P,Σ) be the $\leq^*$-least s.t. $\Sigma \notin N$.

Then repeat the argument from the last time ⊠

SHORT TREE STRATEGY PREMICE

Fix (P,Σ) a short tree pair $\qquad$ P countable

P^+ is a Σ-pm if

$$P^+ = (|P^+|, \dot{B}^{P^+}, \dot{E}^{P^+}, \dot{F}^{P^+}, \dot{\Sigma}^{P^+})$$

and there is a sequence $(\delta_i)_{i<\eta}$ of ordinals s.t. TFH:

- $\dot{E}^{P^+}, \dot{F}^{P^+}$ code extenders as usual
- $\dot{\Sigma}^{P^+}$ codes Σ
- $\dot{B}^{P^+|\delta_i} = \emptyset$
- If $\dot{B}^{P^+|\gamma} \neq \emptyset$ then $\dot{B}^{P^+|\gamma}$ codes a branch b for $\mathcal{T} \in P^+|\gamma$ according to Σ, $\Sigma(\mathcal{T}) = b$ and $(\mathcal{T}, b)$ is the result of iterating P into the HMC $\leftarrow (*)$ of $P^+|\bar{\gamma}$ f.a. $\bar{\gamma} < \gamma$.
- δ_i's are cutpoints of P^+

(*) $P^+|\bar{\gamma}$ knows how to iterate its initial segments b.c. the Q-structures are given by $\dot{\Sigma}^{P^+|\bar{\gamma}}$

(*) Let (P, Σ) be a short tree pair s.t. letting
$i : P \xrightarrow{\Sigma\text{-short}} M_\infty(P, \Sigma)$ be the direct limit map, $i(\kappa^P) = \kappa$.

<u>Def</u> (1) For a set x, let

$L_p^\Sigma(x)$ = the union of all ctbly iterable Σ-pm over x that project to x and are sound

(2) $MC(\Sigma)$: for every real a / any $a \in HC$
$$\{b \subseteq a \mid b \in OD_\Sigma(a)\} = \mathcal{P}(a) \cap L_p^\Sigma(a)$$

(3) sh-MC : for every (P, Σ) as in (*), $MC(\Sigma)$ holds

<u>Def</u> Let (P, Σ) be as in (*). For P^+ a Σ-p.m. we say that P^+ is n-Σ-suitable if
- for all $i < n$: $P^+ \models \delta_i$ is Woodin + δ^P is a cardinal
- if γ is a strong cutpoint of P^+ then
$$P^+ | \gamma^{+P^+} = L_p^\Sigma(P^+ | \gamma)$$
- if $\gamma \neq \delta_i$ then $P^+ \models \gamma$ is not Woodin
- $P^+ = L_{p_\omega}^\Sigma(P^+ | \delta_{n-1})$

<u>End of Def</u>

Let $\mathbb{B}(P, \Sigma)$ = the set of all A s.t.
- A is OD
- if $(Q, \Lambda) \equiv (P, \Sigma)$ then $A_{(Q,\Lambda)} \subseteq \mathbb{R}$

Projection

For the pseudo-comparison
$(Q, \Lambda) \xrightarrow{\varphi} (R, \Psi)$
$(P, \Sigma) \xrightarrow{\psi} (R, \Psi)$
like the ps. comp of two M_n-like models

Either both φ, ψ maximal or else one both short and $\Lambda(\varphi), \Sigma(\psi)$ don't drop

Let (P,Σ) be as above. Let $A \in \mathbb{B}(P,\Sigma)$.

- We say that P^+ is A-~~iterable~~ if short tree iterable iff there is a partial IS Σ^+ on P^+ extending Σ such that for trees $\mathcal{T}$ above P:

  $\Sigma^+(\mathcal{T})$ is defined iff $\mathcal{Q}(\mathcal{T})$ exists and $\mathcal{Q}(\mathcal{T}) \trianglelefteq L_p^{\Sigma}(\mathcal{M}(\mathcal{T}))$

  Similar definition for the trees of the form $\mathcal{T}^\frown\mathcal{U}$ where
  - $\mathcal{T}$ is on P according to Σ
  - $\mathcal{U}$ is above $i^{\mathcal{T}}(P)$

- P^+ is A-iterable iff P^+ is short tree iterable and there are terms $\tau^{P^+}_{A,k}$ for $Coll(\omega,(\delta^{+k}_{n-1})^{P^+})$ in P^+ s.t. given a tree $\mathcal{T}$ on P^+ with last model $(\mathcal{Q},\mathcal{Q}^+)$ (this also makes sense for maximal trees):

  $\boxed{\text{Assume } n=1}$

  (1) if $\mathcal{T}$ (according to Σ) is short and $b=\Sigma(\mathcal{T})$ does not drop then $i^{\mathcal{T}}_b(\tau^{P^+}_{A,k}) = \tau^{\mathcal{Q}^+}_{A,k}$ all k.

  (2) if $\mathcal{T}$ is maximal then there is a branch b s.t. $i^{\mathcal{T}}_b(\tau^{P^+}_{A,k}) = \tau^{\mathcal{Q}^+}_{A,k}$ all k

  (3) In (1) & (2): $(\tau^{\mathcal{Q}^+}_{A,k})^g = A_{(\mathcal{Q},\Sigma_{\mathcal{Q}})} \cap \mathcal{Q}^+[g]$

  whenever $g \subseteq Coll(\omega,(\delta_0^{+k})^{\mathcal{Q}^+})$-generic $/ \mathcal{Q}^+$

- P^+ is strongly A-iterable iff P^+ is A-iterable and for any two trees $P^+ \overset{\vec{\mathcal{T}}}{\underset{\vec{\mathcal{U}}}{\rightrightarrows}} \mathcal{Q}^+$ which are A-iterations we have:

if $\vec{\mathcal{T}}, i^{\vec{\mathcal{U}}}$ exist then for all k:

$$i^{\vec{\mathcal{T}}} \restriction H^{P^+}_{A,k} = i^{\vec{\mathcal{U}}} \restriction H^{P^+}_{A,k}$$

where

$$H^{P^+}_{A,k} = \mathrm{Hull}^{P^+}_{\Sigma_1}\left(\{\tau^{P^+}_{A,k}\} \cup \gamma^{P^+}_{A,k}\right)$$

$$\gamma^{P^+}_{A,k} = \sup\left(\mathrm{Hull}^{P^+}_{\Sigma_1}\left(\{\tau^{P^+}_{A,k}\}\right) \cap \delta^P\right)$$

<u>Note:</u> For normal trees $\mathcal{T}, \mathcal{U}$ in place of stacks $\vec{\mathcal{T}}, \vec{\mathcal{U}}$ this is automatic

Let $\mathcal{F}$ = the set of all $(P, \Sigma, P^+, \vec{A})$ s.t.
(P, Σ) is as in $(*)$, P^+ is 1-suitable,
$\vec{A} \in \mathbb{B}(P, \Sigma)^{<\omega}$ and P^+ is strongly A-iterable

$(P, \Sigma, P^+, \vec{A}) \leq^{\mathcal{F}} (Q, \Lambda, Q^+, \vec{B})$ iff

- $\vec{A} \subseteq \vec{B}$
- (Q, Λ, Q^+) is an $\vec{A}$-iterate of (P, Σ, P^+) and
- if $\vec{\mathcal{T}}$ witnesses this: $\Sigma_{\vec{\mathcal{T}}} = \Lambda$

<u>WANT TO PROVE</u>

1) $V = L^{\Sigma}_{P}(\mathbb{R})$ for any short tree pair (P, Σ) as in $(*)$
2) For all $A \in \mathbb{B}(P, \Sigma)$ there is a P^+ which is strongly A-iterable
3) Let $M_\infty = \mathrm{dir\,lim}(\mathcal{F}, \leq^{\mathcal{F}})$. Then $M_\infty | \Theta = \mathrm{HOD} | \Theta$