Assume: $AD^+ + 7ADR$

$\text{+ HPC}$ Given a Suslin-co-Suslin set $A$, there is a $(P, \Sigma)$ above $A$ in $\mathcal{N}$ (not pair compatible)

$\text{+ Sh-MC}$

From last time: There is a pair $(P, \Sigma)$ s.t.

1. $P$ is a b.p.m. with last block $(\mathfrak{b}, \mathfrak{d})$

2. $\Sigma$ is short-tree strategy of $P$

   $\Sigma$ is a partial $\mathcal{N}$ for $P$ with strong hill condensation and more full normalisation. Also if $\Sigma$ extends $(\mathcal{E}_i)$,

   $l = \Sigma(9)$ is defined. As $l \leq 3$

   (1) $\Sigma(3)$ short all $\mathcal{E}_i$'s.

   (2) $\Sigma(3)$ drops on all $(\mathfrak{d}_i, \mathfrak{b}_i)$. (2)

3. The claim $M(P, \Sigma)$ extends $\text{HOD} + \kappa$

   $\kappa$ is the largest Suslin cardinal, and if $i : \mathfrak{b} \rightarrow M(P, \Sigma)$

   is the induction map then $i(\mathfrak{b}) = \mathfrak{b}$. $(\Rightarrow) \Sigma$ is $\kappa$-Suslin

Why? $\Sigma_1$-reflect the statement "There is no such $(P, \Sigma)$" to some uncounte $\mathcal{N} = \mathcal{E}^+ + DC + \text{HPC} + \ldots + "\text{No such} (P, \Sigma)"$

$\mathcal{N}$ is coded by $\text{Set} A$ which is Suslin-co-Suslin

Let $(N^p, \mathfrak{d}, \Sigma, N^p, \Sigma')$ be a $\mathcal{N}$-Woodin universe for some

inductive-like scaled $\mathcal{N} > N$. Wants to show

$(N^p, \mathfrak{d}, N, \Sigma)$ captures a good pair $(P^*, \Sigma^*)$ s.t.

$\Sigma^* \notin N$, $\Sigma^* \text{ HPC}$
Do a HMC construction in $\mathbb{N}^*$. Let $(p, \mathcal{E}) = (N_{\omega_1}, R_{\omega_1})$

$$(\mathcal{N}_{\omega_1}, R_{\omega_1}, 1 \mathcal{E}_{\omega_1} \subseteq (\mathbb{E}_{\omega_1})^{\omega_1})$$

be least $\xi \in \mathcal{E} \cap \mathbb{N}$. Or take $(p, \mathcal{E})$ be the $\xi^*$-least $\xi \in \mathcal{E} \cap \mathbb{N}$. Then repeat the argument from the last time $\mathbb{Q}$

**SHORT TREE STRATEGY PREMISE**

Fix $(p, \mathcal{E})$ a short tree pair $\quad$ $P$ countable

$p^+ \vdash K \text{ is prem } \forall$

$$p^+ = (1 p^+ 1, B^{p^+} \subseteq \mathcal{E}^{p^+}, \mathcal{E}^{p^+})$$

and there is a sequence $(d_i)_{i \in \mathfrak{I}}$ of ordinals s.t. TFH:

- $\mathcal{E}^{p^+}, \mathcal{E}^{p^+}$ code extenders as usual
- $\mathcal{E}^{p^+}$ codes $\Sigma$
- $B^{p^+} \cap d_i = \emptyset$
- if $B^{p^+} \cap d_i$ then $B^{p^+} \cap d_i$ codes a branch $b$ for $\Sigma \in p^+ \times \mathcal{E}^{p^+}$ according to $\mathcal{E}$, $\mathcal{E}(\mathcal{E}) = b$ and $(9, 8)$, is the result of iterating $P$ into the HML $\mathcal{C}^{(*)}$ of $p^+ \mathcal{E}$ f. a. $\delta \prec \mathcal{E}$.
- $d_i$'s are cutpoints of $p^+$

$(*)$ $p^+ \mathcal{E}$ knows how to create its initial segment b.c. the $\mathfrak{E}$-structures are given by $\mathfrak{E} \subseteq p^+ \mathcal{E}$
Let $(P,\Sigma)$ be a short tree pair s.t. letting

\[ q : P \rightarrow M_0(P,\Sigma) \text{ be the direct limit map, } i(q(P)) = k. \]

Def (1) For a set $x$, let

\[ L^\Sigma(x) = \text{the union of all } \Sigma \text{-pm over } x \text{ that project to } x \]
and are sound.

(2) MC($\Sigma$) : for every real $a$ in $\mathcal{M}$, $a \in HC$

\[ \forall b \leq a \exists \delta \in OD, \delta(\varepsilon) = \delta(a) \wedge L^\Sigma(\delta) \]

(3) sh-MC : for every $(P,\Sigma)$ as in $\mathcal{M}$, $MC(\Sigma)$ holds.

Def Let $(P,\Sigma)$ be as in $\mathcal{M}$. For $P^+ \Sigma$-p.m., we say that $P^+$ is $\Sigma$-suitable if

- for all $\varepsilon$ in $P^+$, $P^+ \varepsilon = \delta$ is Woodin + $\delta$ is a cardinal.
- if $\varepsilon$ is a strong endpoint of $P^+$ then

\[ P^+ | P^+ = L^\Sigma(P^+ | \varepsilon) \]
- if $\varepsilon \in \delta$, then $P^+ \varepsilon$ is not Woodin.
- $P^+ = L^\Sigma(P^+ | \delta_{m-1})$  

End of Def

Let $B(P,\Sigma)$ = the set of all $A \in L$

- $A$ is OD
- If $(P,\Sigma) \vDash (P,\Sigma)$ then $PA(P,\Sigma) \in R$

Either both $P,\Sigma$ maximal or else neither short and

$$2(\varepsilon), \Sigma(x) \text{ don't drop}$$
Let $(\pi, \sigma)$ be as above. Let $A \in \mathbb{R}(\pi, \sigma)$.

- We say that $P^+$ is $A$-iterable if short tree iterable off then is a partial is $\sigma^+$ on $P^+$ extending $\sigma$ such that for there $A$-iterate $P$:
  \[\sigma^+ (\pi) \text{ is defined if } Q (\pi) \text{ exists and } Q (\pi) \sqsubseteq \mathcal{L}_{\pi} (\pi (\pi))\]
  Similar definition for the tree of the form $\pi^+ \pi$ when
  - $\pi$ is $P$ according to $\sigma$
  - $\sigma$ is above $\pi (\pi)$

- $P^+$ is $A$-iterable off $P^+$ is short tree iterable and there are terms $\tau_{A, k}^+$ for $\text{Coll}(\omega, (\sigma + k)^+) \in P^+$ s.t.
  given a tree $T$ on $P^+$ with least model $< (\pi, \rho^+) >$ (this always makes sense for maximal trees):
  1. if $T$ (according to $\sigma$) is short and $b = \sigma^+ (\sigma)$ does not drop then $\tau_{b}^+ (\tau_{A, k}^+) = \tau_{A, k}^+$ all $k$.
  2. if $T$ is maximal then there is brand $b$ s.t.
     \[\tau_{b}^+ (\tau_{A, k}^+) = \tau_{A, k}^+ \quad \text{all } k\]
  3. in (1), (2):
     \[\sigma_{A, k}^+ \equiv A (\omega, \sigma_{A, k}^+) \uplus Q^+ (\sigma_{A, k}^+)\]
     whenever $\sigma \leq \text{coll}(\omega, (\sigma + k)^+) \equiv \text{generic } Q^+$

- $P^+$ is strongly $A$-iterable if $P^+$ is $A$-iterable and for any two trees $P^+ \Rightarrow Q^+$ which are $A$-iterations we have:

\[\tau_{P^+} \Rightarrow \tau_{Q^+}\]
If $\tilde{\tau}, \tilde{\omega}$ exist then for all $z$:
\[ z_{A_k}^{p^+} \in \tilde{\omega} \cup \tilde{\tau} \cap \text{Hull}_{A_k}^{p^+} \]

where
\[ \text{Hull}_{A_k}^{p^+} = \text{Hull}_{A_k}^{p^+} (\tilde{\tau} \cup \tilde{\omega}^{p^+}) \]
\[ \tilde{\omega}^{p^+}_A = \sup (\text{Hull}_{A_k}^{p^+} (\tilde{\tau} \cup \tilde{\omega}^{p^+}) \cap \delta^+ \rho) \]

Note: For maximal trees $\pi_1 \tilde{\omega}$ in place of stacks $\pi_1 \tilde{\omega}$
this is automatic.

Let $\mathcal{G}$ be the set of all $(p_1 \xi, p^+, \tilde{\omega})$ s.t.
\[ (p_1 \xi) \text{ as in } G, p^+ \text{ is } (q, \rho) \text{-suitable}, \]
\[ \tilde{\omega} \in \mathcal{B}_G(p_1 \xi) < \omega \text{ and } p^+ \text{ is strongly } A \text{-iterable} \]

\[ (p_1 \xi, p^+, \tilde{\omega}) \leq^G (\alpha, \lambda, q^+, \tilde{\omega}) \cap \mathcal{G} \]

- \[ \tilde{\omega} \leq \tilde{\omega}^* \]
- \[ (\alpha, \lambda, q^+) \text{ is an } \tilde{\omega} \text{-iterate of } (p_1 \xi, p^+) \text{ and} \]
- \[ \text{if } \tilde{\omega} \text{ witnesses this: } \tilde{\omega}^*_\xi = \lambda \]

\textbf{Want to prove:}

1) \[ V = L_{\xi}^{\mathcal{G}} (\mathbb{R}) \text{ for any short tree pair } (p_1 \xi) \text{ as in } G \]

2) \[ \text{For all } A \in \mathcal{B}_G(p_1 \xi) \text{ there is a } p^+ \text{ which is strongly } A \text{-iterable} \]

3) \[ \text{Let } M \omega = \text{lim } (\mathcal{F}, \leq^G) \text{. Then } M \omega \models \text{HOD } \Theta \]