PRODUCING $M_n^\#$ FROM OPTIMAL DETERMINACY HYPOTHESES

JOINT WITH R. SCHINDLER AND W. H. WOODIN

Theorem (Martin, Harrington, Neeman, Woodin) Let $n \geq 0$ and assume there is no $\underset{\sim}{\Sigma}^1_{n+2}$-definable ω_1-sequence of pairwise distinct reals. TFAE

(1) $\underset{\sim}{\Pi}^1_n$-det & Π^1_{n+1}-det

(2) $\forall x \in \mathbb{R}$ $M^\#_{n-1}(x)$ exists and is ω_1-iterable and $M^\#_n$ exists and is ω_1-iterable.

(3) $M^\#_n$ exists and is ω_1-iterable

Corollary Let $n \geq 0$. TFAE

(1) $\underset{\sim}{\Pi}^1_{n+1}$-det

(2) $\forall x \in \mathbb{R}$ $M^\#_n(x)$ exists

Lemma Let $n \geq 1$. Assume $M^\#_{n-1}(x)$ exists for all $x \in \mathbb{R}$ and $Det(\Sigma^1_{n+1})$. Then for a Turing cone of y, $M_{n-1}(y) | \delta_y \models OD$-det

where $\delta_y = \begin{cases} \text{the least Woodin of } M_{n-1}(y), & n > 1 \\ \text{least } y\text{-indiscernible in } L[y] = M_0(y) & \text{if } n = 1 \end{cases}$

($n = 1$ is due to Kechris + Solovay)

Corollary Under the same hypothesis, $\omega_1^{M_{n-1}(y)}$ is measurable in $HOD^{M_{n-1}(y)}|\delta_y$ for a cone of reals y.

Theorem Let $n \geq 1$. Same hypothesis as in the lemma. Then there is a proper class model with n Woodin cardinals.

Pf Fix x in the cone above.

Case 1 $(K^c)^{M_{n-1}(x)|\delta_x}$ has no Woodin cardinals and is fully iterable.

Isolate $K^{M_{n-1}(x)|\delta_x}$: Fix singular α s.t. in $M_{n-1}(x)|\delta_x$: $\alpha^{+K} = \alpha^+$. Let g be coll(ω, α)-generic / $M_{n-1}(x)|\delta_x$. Then

$$M_{n-1}(x)[g]|\delta_{x\oplus y} = M_{n-1}(x\oplus y)|\delta_{x\oplus y}$$

and in this model $\omega_1 = \alpha^{+K} = \alpha^{+HOD}$, which contradicts the measurability of ω_1 in HOD.

Case 2 $(K^c)^{M_{n-1}(x)|\delta_x}$ has a Woodin cardinal, say δ. WLOG let δ be the largest one (otherwise we are done). ~~Let K~~ Consider $L[E]((K^c)^{M_{n-1}|\delta_x}|\delta)$ in the model $M_{n-1}(x)$.

Case 3 $(K^c)^{M_{n-1}(x)|\delta_x}$ has no Woodins and is not iterable.

Say this is witnessed by a tree T. Then T is maximal. Consider $(L[E](M(T)))^{M_{n-1}(x)}$. $\boxtimes$

NOW FOCUS ON EVEN n

Def Let $N_0 = A$ where A is a countable set.

$N_{\alpha+1}$ = the union of all countable A-premice $M \trianglerighteq N_\alpha$ s.t.

(1) M is $(2n-1)$-small

(2) $\rho_\omega(M) \leq N_\alpha \cap O_n$ and M is sound above $N_\alpha \cap O_n$

(3) $N_\alpha \cap O_n$ is a cutpoint in M.

(4) M is ω_1-iterable.

$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$ if λ is limit

$L_p^{2n-1}(A) = N_{\omega_1^V}$.

Note: This is well-defined ~~and Δ^1_2~~ under $\mathrm{Det}(\Pi^1_{2n})$.

Assume
- Every Σ^1_{2n+2} sequence of pairwise distinct reals is countable
- $\mathrm{Det}(\Sigma^1_{2n+1})$

WTS: $M_{2n}^{\#}$ exists.

Step 1 Construct a model M_x for $x \in \mathbb{R}$ with the following property:

(1) $M_x \cap On = \omega_1^V$

(2) $x \in M_x$

(3) $M_x \prec_{\underset{\sim}{\Sigma}^1_{2n+2}} V$ (This uses Π^1_{2n+1}-uniformization)

(4) M_x is closed under $a \mapsto M_{2n-1}^{\#}(a)$

(5) $\forall \eta < \omega_1^V$ $\forall g$ generic for $coll(\omega, \eta)$ over V: $M[g] \prec_{\underset{\sim}{\Sigma}^1_{2n+2}} V[g]$ (uses generic absoluteness.)

(6) $M_x \models ZFC$

<u>Construction of M_x</u>: Using $\underset{\sim}{\Pi}^1_{2n+1}$-uniformization let F be Π^1_{2n+1}-definable function uniformizing a universal Π^1_{2n+1}-set U. Wlog

$F(\ulcorner\varphi\urcorner \frown a)$ is witnessing $\exists x\, \varphi(x, a)$

$\uparrow$

Π^1_{2n+1}-formula

- Start with $W_0 = \{x\}$
- Odd successors: $W_{\alpha+1} = rud(W_\alpha \cup \{F(z) \mid z \in dom(F) \cap W_\alpha\})$

<u>Even Successors</u>: Let $\eta < \omega_1^V$. Define a $\mathrm{Coll}(\omega,\eta)$-name as follows:

$$\sigma^\eta(\tau) = \{ (p, \check{s}) \mid p \in \mathrm{Coll}(\omega,\eta),\ s \in \omega^{<\omega} \text{ and } p \Vdash_{\mathrm{Coll}(\omega,\eta)} \text{"}(\exists\sigma)\,\varphi(\tau,\sigma) \wedge \sigma \text{ extends } s\text{"}$$

τ: Coll(ω,η)-name for a real

where φ_F is a Π^1_{2n+1}-formula s.t.

$$\varphi_F(x,y) \Leftrightarrow F(x) = y \quad \}$$

<u>Note</u>: Det$(\underset{\sim}{\Pi}^1_{2n+1})$ yields that

$$(*) \quad V \prec_{\underset{\sim}{\Sigma}^1_{2n+2}} V[g]$$

Here g is generic for Coll(ω,η) over V where $\eta < \omega_1^V$, so Coll(ω,η) is Cohen forcing from the point of view of V. So the ~~the~~ proof of $(*)$ one can follow the argument from Woodin's paper "On the consistency strength of projective uniformisation / well-orderings".

$$W_{\alpha+2} = \mathrm{rud}\left(W_{\alpha+1} \cup \{\sigma^\eta(\tau) \mid \tau \in W_{\alpha+1} \wedge \eta < \omega_1^V\} \right)$$

$$M_x = \bigcup_{\alpha < \omega_1^V} W_\alpha$$

Step 2 Claim $M_x \models$ "$M_{2n}^{\#}$ exists"

Proof Assume not. Then K^{M_x} exists ($2n$-small)
Using weak covering: Let $\gamma \in M_x$ be singular.

$$\gamma^{+K^{M_x}} = \gamma^{+M_x}$$

Subclaim $\exists z \geq_T x$ s.t.

(a) $\gamma^{+M_x} = \omega_2^{Lp^{2n-1}(z)}$

(b) $K^{M_x} | \gamma^{+M_x} \in Lp^{2n-1}(z)$

(c) $K^{M_x} | \gamma^{+M_x}$ is fully iterable in $Lp^{2n-1}(z)$

See: Schindler: Coding into K by reasonable forcing

Consider $(K^c)^{Lp^{2n-1}(z)}$ in the sense of Mitchell-Schindler 2004, constructed relative to A s.t. $V = L[A]$. Make sure that $(K^c)^{Lp^{2n-1}(z)} \subseteq HOD^{Lp^{2n-1}(z)}$.

Case A $(K^c)^{Lp^{2n-1}(z)}$ does not have a Woodin cardinal below $\omega_2^{Lp^{2n-1}(z)}$.
Consider $K^* = (K^c | \omega_2)^{Lp^{2n-1}(z)}$. Then K^* is Q-structure iterable.

Subclaim $\omega_2^{Lp^{2n-1}(z)}$ is a successor cardinal in $(K^c)^{Lp^{2n-1}(z)}$.

Pf: Work in $Lp^{2n-1}(z)$: Compare K^* and $K^{M_x}|\gamma^{+M_x}$. If $\omega_2^{Lp^{2n-1}(z)}$ is limit, then use universality of $(K^c)^{Lp^{2n-1}(z)}$ and derive a contradiction.

Lemma Let $n \geq 1$ and assume $Det(\underset{\sim}{\Pi}^1_{2n}) \wedge$ $\wedge$ $Det(\Pi^1_{2n+1})$. Then for a Turing cone of x, $Lp^{2n-1}(x) \models OD\text{-}det$

Moschovakis' Coding Lemma gives
$HOD^{Lp^{2n-1}(x)} \models$ " $\omega_2^{Lp^{2n-1}(x)}$ is inaccessible "

in fact, " $\omega_2^{Lp^{2n-1}(x)}$ is Woodin "

(This is due to Woodin)

As $(K^c)^{Lp^{2n-1}(z)} \subseteq HOD^{Lp^{2n-1}(z)}$,

this is a contradiction.

Case B Consider $(K^c)^{Lp^{2n-1}(z)}|\delta$