<u>Recall Thm</u> (Neeman, Woodin) Let $n \geq 0$ and there is no $\underset{\sim}{\Sigma}^1_{n+2}$ definable ω_1-sequence of distinct reals. TFAE

(1) $\underset{\sim}{\Pi}^1_n$-det + Π^1_{n+1}-det

(2) $\forall x \in \mathbb{R}$ $M^\#_{n-1}(x)$ exists and is ω_1-iterable, and $M^\#_n$ exists and is ω_1-iterable

(3) $M^\#_n$ exists and is ω_1-iterable

Today we do (1) $\Rightarrow$ (3) odd levels

<u>Recall</u> Let $n \geq 1$ and assume $\forall x \in \mathbb{R}$ $M^\#_{n-1}(x)$ exists. Assume $Det(\Sigma^1_{n+1})$. Then there is a p.c. inner model with a Woodin cardinal.

(*) <u>Throughout we assume</u>: $n \geq 1$ and $M^\#_{n-1}(x)$ exists f.a. $x \in \mathbb{R}$. (For the appropriate n.)

<u>Def</u> N is pre-n-suitable iff there is $\delta < \omega_1^V$ s.t.

(1) $N \models ZFC^- + \delta$ is the largest cardinal and

$$N = M_{n-1}(N|\delta) \mid \delta^{+M_{n-1}(N|\delta)}$$

(2) $N^\#_{n-1}$ is not $(n-1)$-small above δ and $M_{n-1}(N|\delta) \models \delta$ is Woodin

(3) f.e. $\gamma < \delta$ $M^{\#}_{n-1}(N|\gamma)$ is not fully sound or $M_{n-1}(N|\gamma) \not\models \gamma$ is Woodin

(4) f.e. $\eta < \delta$ $N_{n-1}(N|\delta) \models N|\delta$ is η-iterable

Lemma If $Det(\Sigma^1_{n+1})$ then there is a pre-n-suitable pm.

Def Short tree: Standard. We require $Q(T) \trianglelefteq M_{n-1}(M(T))$ if T is of limit length.

Now focus on odd levels:

Lemma Assume (4) For $2n-1$ and let N be pre $(2n-1)$-suitable pm. Then the statement "N is short tree iterable" is Π^1_{2n}-definable uniformly in any code of N.

Point: $(2n-2)$-small Q-structures give Π^1_{2n-1}-iterability.

Def Let N be a pre-$(2n-1)$ suitable pm. We say N is $(2n-1)$-suitable iff
N is short tree iterable and whenever T

is a short tree on N with $lh(\mathcal{T}) = \lambda + 1$ and $[0, \lambda]_{\mathcal{T}}$ is non-dropping then $M^{\mathcal{T}}_{\lambda}$ is pre-$(2n-1)$-suitable.

Lemma (*) for $2n-1$. If N is pre-$(2n-1)$-suitable then N is $(2n-1)$-suitable.

From now on assume for $n \geq 1$

- $Det(\underset{\sim}{\Pi}^1_{2n-1})$ + Π^1_{2n}-det
- (*) for $2n-1$
- There is no $\underset{\sim}{\Sigma}^1_{2n+1}$-definable ω_1-sequence of pairwise distinct reals

We show: $M^{\#}_{2n-1}$ exists.

Determinacy Transfer Theorem (Kechris, Woodin)
Let $n \geq 1$. $Det(\underset{\sim}{\Pi}^1_{2n-1}) + Det(\Pi^1_{2n}) \Rightarrow$

$$\Rightarrow Det(\partial^{(2n-1)}(<\omega^2 - \Pi^1_1))$$

Reference: Cabal

<u>Corollary</u> Let $n \geq 1$. Assume $\text{Det}(\underset{\sim}{\Pi}^1_{2n-1})$ + $\text{Det}(\Pi^1_{2n})$.
Assume

$Q = \{ x \in \mathbb{R} \mid M_{2n-2}(x) \models \varphi(x, E, \vec{\gamma}) \}$

where E is the extender sequence of $M_{2n-2}(x)$ and $\vec{\gamma}$ is a sequence of indiscernibles for $M_{2n-2}(x)$.
Then Q is determined.

Follows from the above Thm + Martin's paper "The largest countable ...."

Fix $x \in \mathbb{R}$ coding some ~~...~~ $(2n-1)$-suitable pm. Let
N_x = common pseudo-iterate of all $(2n-1)$-suitable pm N which are coded by $x_N \leq_T x$.

<u>Note</u>: N_x is ctbl in V.

M_{2n-2} ~~...~~ $(N_x \mid \delta_{N_x})$ is well-defined (i.e. is a proper class model with $2n-1$ Woodins) by fullness preservation.

Given a formula φ let

$\mathcal{Q}^{\varphi}_{m} = \{ x \in \mathbb{R} \mid M_{2n-2}(N_x \mid \delta_{N_x}) \models \varphi(E, \gamma_1 \ldots \gamma_m) \}$

where E is the extender sequence of $M_{2n-2}(N_x \mid \delta_{N_x})$ and $\gamma_0 \ldots \gamma_m$ are indiscernibles.

<u>Claim</u> $\mathcal{Q}^{\varphi}_{m}$ is Turing-invariant and determined.

<u>Notation</u> $Th(M_{2n-2}(N_x \mid \delta_x)) =$

$= \{ \varphi \mid M_{2n-2}(N_x \mid \delta_x) \models \varphi(E, \gamma_1 \ldots \gamma_m),\ m < \omega,$
φ a formula, $E, \gamma_1 \ldots \gamma_m$ as above $\}$

<u>Claim</u> $Th(M_{2n-2}(N_x \mid \delta_x))$ stabilizes on a cone. Let x_0 be its base.

Denote this theory by T, and let x_T be a real coding T.

Let $z \geq_T x_T$ be in the cone above. We construct M_z with the following properties

(1) $M_z \cap On = \omega_1^V$ $z = M_z$

(2) $M_z \prec_{\Sigma^1_{2n}} V$

(3) M is closed under $a \mapsto M^{\#}_{2n-2}(a)$

(4) $M_z \models ZFC$

Claim $M_z \models M^{\#}_{2n-1}$ exists

Proof We show $(K^c)^{M_z}$ is not $(2n-1)$-small

Pf If so:

Case 1 If there is a Woodin cardinal in $(K^c)^{M_z}$ let w.l.o.g. δ be a largest one. Compare $(K^c)^{M_z}$ with $M^{\#}_{2n-2}((K^c)^{M_z} | \delta)$ in M_z. By universality $(K^c)^{M_z}$ is $(2n-1)$-small

Case 2 No Woodin cardinal in $(K^c)^{M_z}$.
Let N be a $(2n-1)$-suitable pm. Compare $(K^c)^{M_z}$ with N in M_z and note that N is $(2n-1)$-suitable in M_z by correctness.

$$(K^c)^{M_z} \rightsquigarrow R$$
$$\trianglelefteq \text{ universality}$$
$$N \underset{T}{\rightsquigarrow} N^{*\prime}$$

Assume T is short, and compare R with $\underbrace{M^{\#}_{2n-2}(N' | \delta_{N'})}_{\text{Not } 2n-1\text{-small}}$

By universality, R wins $\Rightarrow$ is not $(2n-1)$-small.

In M_z: Let $x \in \mathbb{R} \cap M_z$ code x_0, T, $(M^{\#}_{2n-1})^{M_z}$

$N^* = (M^{\#}_{2n-1} | \delta_0^{+M^{\#}_{2n-1}})^{M_z}$

where δ_0 = the ~~largest~~ least Woodin in $(M^{\#}_{2n-1})^{M_z}$.

So N^* is $2n-1$-suitable in V by correctness

In fact $(N_x)^{M_z} = (N_x)^V$ and

$(M^{\#}_{2n-2}(N_x | \delta_{N_x}))^{M_z} = (M^{\#}_{2n-2}(N_x | \delta_{N_x}))^V$

By correctness and closure under sharps.

N_x is a non-dropping iterate of N^*

In fact

$M^{\#}_{2n-2}(N_x | \delta_x)$ is a non-dropping iterate of

$M^{\#}_{2n-2}(N^* | \delta_0) = (M^{\#}_{2n-1})^{M_z}$.

Hence $M_{2n-2}(N^* | \delta_0)$, $M_{2n-2}(N^* | \delta_0)$

have the same theory, so the theory of

$(M^{\#}_{2n-1})^{M_z}$ is constant for $z \geq_T x_0 \oplus x_+$.

Let $N = (M^{\#}_{2n-1})^{M_z}$ for $z \geq_T x_0 \oplus x_+$