Recall Thin (Neeman, Woodin) let \( n \geq 0 \) and

\[
\text{there is no } \Sigma^1_{n+2} \text{ definable } \omega_1 \text{ - sequence of distinct reals.} \quad \text{THE}
\]

(1) \( \Pi^1_n \text{-det} + \Pi^1_{n+1} \text{-det} \)

(2) \( \forall x \in \mathbb{R} \ M^\#_{m-1}(x) \text{ exists and is } \omega_1 \text{-iterable}, \quad \text{and} \quad M^\#_m \text{ exists and is } \omega_1 \text{-iterable} \)

(3) \( \exists M^\#_n \text{ exists and is } \omega_1 \text{-iterable} \)

Today we do (1) \( \Rightarrow \) (3) odd levels

Recall let \( m \geq 1 \) and assume \( \forall x \in \mathbb{R} \ M^\#_{m-1}(x) \text{ exist} \).

Assume \( \text{Det}(E_{m+1}) \). Then there is a \( \mu \)-c. inner model with a Woodin cardinal.

\[ (\exists) \quad \text{Transcendent we assume} \quad \text{m} \geq 1 \quad \text{and} \]

\[ (\ast) \quad M^\#_{m-1}(x) \text{ exists f.a. } x \in \mathbb{R} \quad (\text{For the appropriate } \alpha) \]

\[ \text{Det} \quad N \text{ is pre- } \mu \text{-suitable if there is } d < \omega_1 \text{ s.t.} \]

(1) \( N \models \text{ZF} + \exists d \text{ the largest cardinal and} \)

\[
N = M_{m-1}(N(d) | d^+ M_{m-1}(N \upharpoonright d))
\]

(2) \( M^\#_{m-1} \text{ is not } (n-1) \text{-small above } d \)

and \( M_{m-1}(N(d)) \subseteq d \text{ is Woodin} \)
(3) if $\gamma < \delta$ then $M^\#_{\omega_1} \neq M_\gamma^\# \neq \omega_1$ is Woodin.

(4) if $\gamma < \delta$ then $M_{\omega_1} \models \text{N}1 \sigma \models \gamma$ – iterable.

Lemma: If $\text{Def}(E_{\omega_1})$ then there is a pre-$n$-suitable pm.

Def: Short tree: Standard. We require $\omega_1 \leq M_{\omega_1}(M(\gamma))$ if $\gamma$ is of fixed length.

Now focus on odd levels:

Lemma: (4) For $2n-1$ and let $N$ be a pre $(2n-1)$ - suitable pm. Then the statement "$N$ is short tree iterable" is $\Pi^1_{2n-1}$ - definable uniformly in any code of $N$.

Point: $(2n-2)$ - small $Q$ - structures give $\Pi^1_{2n-1}$ - iterability.

Def: Let $N$ be a pre $(2n-1)$ suitable pm. We say $N$ is $(2n-1)$ - suitable off

$N$ is short tree iterable and whenever $\gamma$
is a short tree on \( N \) with \( \text{lh}(T) = \omega + 1 \)
and \( \langle 0, \omega \rangle \) non-dropping then \( M^\sigma_{\omega+1} \)
is pre-\( (2n-1) \)-suitable.

\textbf{Lemma (4):} for \( 2n-1 \), \( N \) is pre-\( (2n-1) \)-suitable
then \( N \) is \( (2n-1) \)-suitable.

From now on assume \( n \geq 1 \)

• \( \text{Det} (N_{2n+1}^{\omega_1}) + \Pi^0_{2n} \)-det

• (4) for \( 2n-1 \)

• There is an \( \Sigma^1_n \)-definable \( \omega_1 \)-sequence of pairwise distinct reals.

\textbf{We show:} \( \Pi^\#_{2n+1} \) exists.

\textbf{Determinacy Transfer Theorem} (Kechris, Woodin)

Let \( n \geq 1 \). \( \text{Det}(N_{2n+1}^{\omega_1}) + \text{Det}(N_{2n}^{\omega_1}) = \)
\[ \Rightarrow \text{Det}(\mathcal{E}^{2n-1})(\omega^2 - \Pi^0_4) \]

\textbf{Reference:} Cabal
Corollary \( \text{Let } n \geq 1. \text{ Assume } \text{Det}(\Sigma^n_{2^{n-1}}) + \text{Det}(\Pi^n_{2^n}). \)

Assume

\[ Q = \{ x \in \mathbb{R} \mid M_{2^{n-2}}(x) = \varphi(x_1, E, y) \} \]

where \( E \) is the extend set sequence of \( M_{2^{n-2}}(x) \)
and \( \varphi \) a sequence of indiscernibles for \( M_{2^{n-2}}(x) \).
Then \( Q \) is determined.

Follows from the above Theorem + Hartin’s paper
"The largest countable ..."

Fix \( x \in \mathbb{R} \) coding some \( (2n-1) \)-suitable pm. Let

\( N_x = \) common pseudo-iterate of all \( (2n-1) \)-suitable
pm \( N \) which are coded by \( x_N \leq_T x \).

**Note:** \( N_x \) is \( c^b \)-bl in \( V \).

\( M_{2^{n-2}}(x_1, d_{N_x}) \) is well-defined (i.e. as a
proper class model with \( 2n-1 \) Woodin)
by fullness preservation.
Given a formula $\phi$ let

$$Q^\phi_w = \{ x \in R \mid M_{2u-2}(N_x \Delta x) \vdash \phi(E, x_1 \ldots x_m) \}$$

where $E$ is the extension universe of $M_{2u-2}(N_x \Delta x)$ and $x_1 \ldots x_m$ are indiscernibles.

Claim $Q^\phi_w$ is Turing-invariant and determined.

Notation $Th(M_{2u-2}(N_x \Delta x)) =$

$$= \{ \phi \mid \exists \xi M_{2u-2}(N_x \Delta x) \vdash \phi(E, x_1 \ldots x_m), m \in \omega, \phi \text{ a formula, } E, x_1 \ldots x_m \text{ as above} \}$$

Claim $Th(M_{2u-2}(N_x \Delta x))$ stabilizes on a cone but $x_1$ to be its base.

Denote this theory by $T$, and let $x_T$ be a real coding $T$. 

Let $x_T \geq x_T$ be in the cone above. We construct $M_2$ with the following properties:

1. $M_2 \cap \omega^m = \omega^m$ and $x_T \in M_2$
2. $M_2 \not\models \exists x_1 \forall x_2$
3. $M_2$ is closed under $a \mapsto M_{2u-2}^{+}(a)$
4. $M_2 = ZFC$
Claim \( M_2 = \mathcal{P}^{\mathcal{M}_2}_{\kappa-1} \text{ exists} \)

Proof \( \text{We show } (\mathcal{P}^c)^{\mathcal{M}_2} \text{ is not } (2n-1) \text{-small} \)

If so:

Case 1: If there is a Woodin cardinal in \( (\mathcal{P}^c)^{\mathcal{M}_2} \), let w.l.o.g. \( \xi \) be a largest one. Compare \( (\mathcal{P}^c)^{\mathcal{M}_2} \) with \( \mathcal{M}^{\mathcal{M}_2}_{\kappa-2} \) \( (\mathcal{P}^c)^{\mathcal{M}_2}_{\kappa-1} \) in \( \mathcal{M}_2 \).

By universality \( (\mathcal{P}^c)^{\mathcal{M}_2} \text{ is not } (2n-1) \text{-small} \)

Case 2: No Woodin cardinal in \( (\mathcal{P}^c)^{\mathcal{M}_2} \).

Let \( N \) be a \( (2n-1) \text{-suitable pm} \). Compare \( (\mathcal{P}^c)^{\mathcal{M}_2} \) with \( N \) in \( \mathcal{M}_2 \) and note that \( N \) is \( (2n-1) \text{-suitable in } \mathcal{M}_2 \) by correctness.

\( (\mathcal{P}^c)^{\mathcal{M}_2} \rightarrow R \text{ by universality} \)

\( N \rightarrow N^* \text{ by universality} \)

Assume \( T \) is short, and compare \( R \) with \( \mathcal{M}^{\mathcal{M}_2}_{\kappa-2} (N/1, \xi) \)

Not \( 2n-1 \text{-small} \)

By universality, \( R \text{ wins } \rightarrow \text{ is not } (2n-1) \text{-small} \).
In $M_2$: Let $x \in R^nM_2$ code $x \cup \{1, 2, \ldots, n\}$. Then $\mathcal{N}^* = (M_{2n-1}^\# | \delta_0 + M_{2n-1}^\#)M_2$

when $\delta_0 = $ the least Woodin in $(M_{2n-1}^\#)^{\mathcal{M}_2}$.

So $\mathcal{N}^* \in \mathcal{M}_2$ - witness in $\mathcal{V}$ by correctness.

In fact $(\mathcal{N}_x)^{M_2} = (\mathcal{N}_x)^{\mathcal{V}}$ and

$$(M_{2n-2}^\# (\mathcal{N}_x | \delta_{\mathcal{N}_x}))M_2 = (M_{2n-2}^\# (\mathcal{N}_x | \delta_{\mathcal{N}_x}))^{\mathcal{V}}$$

By correctness and closure under sharps,

$\mathcal{N}_x$ is a non-dropping iterate of $\mathcal{N}^*$.

In fact

$M_{2n-2}^\# (\mathcal{N}_x | \delta_{\mathcal{N}_x})$ is a non-dropping iterate of

$M_{2n-2}^\# (\mathcal{N}_x | \delta_0) = (M_{2n-1}^\#)^{M_2}$.

Hence $M_{2n-2}^\# (\mathcal{N}_x | \delta_0)$, $M_{2n-2}^\# (\mathcal{N}_x | \delta_0)$

have the same theory, so the theory of

$(M_{2n-2}^\#)^{M_2}$ is constant for $2 \geq x_0 \oplus x_+$. Let

$\mathcal{N} = (M_{2n-1}^\#)^{M_2}$ for $2 \geq x_0 \oplus x_+$. 