A Brief Guide on Surface Integrals

Most important point: You have to understand the surface integral of a function of a function over a surface S:

$$\int\int_S g(x, y, z)\ \mathrm{d}S. \tag{1}$$

All other surface integrals are just special cases of this one.

The most straightforward application of this integral are the following:

- Computing the mass of S given density per square unit. In this case we put $g(x, y, z) = \rho(x, y, z)$ in (1) where $\rho(x, y, z)$ is the density function.
- Computing the center of mass $\langle \bar{x}, \bar{y}, \bar{z} \rangle$ of S. Recall that $\bar{x} = M_x/M$ where M is the mass of S and M_x is given by the surface integral (1) where we put $g(x, y, z) = x\rho(x, y, z)$ where again $\rho(x, y, z)$ is the density function. Similarly we compute $\bar{y}$ and $\bar{z}$.
- Computing the electric charge of S given the charge density. In this case we let $g(x, y, z) = \rho(x, y, z)$ where $\rho(x, y, z)$ is the charge density.

Now recall there are two formulas for evaluating the integral in (1).

Case 1. The surface S is given as a graph of a function $f(x, y)$ over a region A in the xy-plane. Then

$$\int\int_S g(x, y, z)\ \mathrm{d}S = \int\int_A g(x, y, z)\sqrt{1 + f_x^2 + f_y^2}\ \mathrm{d}x\mathrm{d}y \tag{2}$$

where f_x, f_y are the partial derivatives of f.

Case 2. The surface S is given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ where the parameters u, v range over a region A in a 2-dimensional uv-plane. Then

$$\int\int_S g(x, y, z)\ \mathrm{d}S = \int\int_A g(x(u, v), y(u, v), z(u, v)) \parallel \mathbf{r}_u \times \mathbf{r}_v \parallel \mathrm{d}u\mathrm{d}v \tag{3}$$

Now consider the flux integral of a vector field $\mathbf{F}(x, y, z)$ over a surface S

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}S. \tag{4}$$

This is a special case of (1) where we put

$$g(x, y, z) = \mathbf{F}(x, y, z) \cdot \mathbf{n}(x, y, z)$$

Recall that the most straightforward applications of flux integral include:

- Computing rate of flow of a fluid through a surface S. In this case $\mathbf{F}(x, y, z)$ is the velocity field of the fluid.
- Computing the rate of heat flow through a surface S. In this case $\mathbf{F}(x, y, z)$ is the vector field of heat flow.

If the surface S is given by an equation $h(x, y, z) = 0$ then

$$\mathbf{n}(x, y, z) = \pm\frac{\nabla h(x, y, z)}{\parallel \nabla h(x, y, z) \parallel}.$$

If the surface S is given by a parametrization $\mathbf{r}(u, v)$ then

$$\mathbf{n}(x, y, z) = \pm\frac{\mathbf{r}_u \times \mathbf{r}_v}{\parallel \mathbf{r}_u \times \mathbf{r}_v \parallel}.$$

In either case the sign depends on the orientation of the surface S.

If the surface S is given by a parametrization $\mathbf{r}(u,v)$ where the parameters u, v range over a region A in a 2-dimensional uv-plane, the flux integral (4) may also be evaluated using the evaluation formula

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\;\mathrm{d}S = \pm\int\int_A \mathbf{F}(x(u,v), y(u,v), z(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\;\mathrm{d}u\mathrm{d}v \tag{5}$$

where the sign once again depends on the orientation of surface S. Notice that formula (5) makes the evaluation a sort of a routine, since once you get the parametrization $\mathbf{r}(u,v)$ for your surface then you just mechanically follow (5).

Gauss (Divergence) Theorem. $\mathbf{F}(x,y,z)$ is a vector field with continuous partial derivatives in an open region Q and ∂Q is the boundary of Q. Then

$$\int\int_{\partial Q} \mathbf{F}\cdot\mathbf{n}\;\mathrm{d}S = \int\int\int_Q \nabla\cdot\mathbf{F}\;\mathrm{d}V \tag{6}$$

The most straigthforward application of the divergence theorem is turning flux integrals over ∂Q into triple integrals over Q. This frequently simplifies the computation.

Stokes Theorem. $\mathbf{F}(x,y,z)$ is a vector field with continuous partial derivatives in an open region D. We have a surface S entirely contained in D and the boundary of S is a closed piecewise smooth curve C. Then

$$\oint_C \mathbf{F}\;\mathrm{d}\mathbf{r} = \int\int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\;\mathrm{d}S \tag{7}$$

The most straightforward application of Stokes theorem is turning line integrals into flux integrals that may be evaluated more easily.

Notice that the two above mentioned applications of the divergence and Stokes theorems are purely mechanical — the only thing you have to do is to evaluate the integrals on the right sides.