Math 2E Midterm #1

Solution

1. Compute the volume of the parallelepiped determined by vectors $\langle 1,-1,2\rangle$, $\langle 1,1,1\rangle$ and $\langle 1,2,-1\rangle$.

$$V = \left\|\langle 1,1,1\rangle \cdot \left(\langle 1,-1,2\rangle \times \langle 1,2,-1\rangle\right)\right\| = \left|\langle 1,1,1\rangle \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}\right| = \left\|\begin{matrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{matrix}\right\|$$

$$= \left\|\begin{matrix} -1 & 2 \\ 2 & -1 \end{matrix}\right| - \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + \left|\begin{matrix} 1 & -1 \\ 1 & 2 \end{matrix}\right\| = |-3+3+3| = 3$$

2. Compute the arc length of the line in 3-dimensional space; the line is given parametrically by:

$$\begin{aligned} x(t) &= t\cos t - \sin t \\ y(t) &= t\sin t + \cos t \\ z(t) &= t^2 \end{aligned}$$

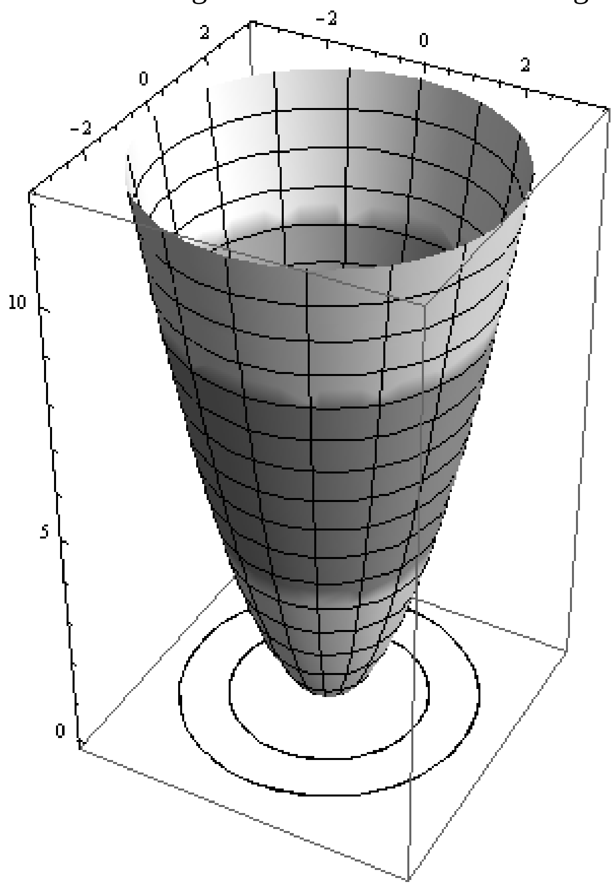
where t ranges from 0 to 2π.

$\mathbf{r}(t) = \langle t\cos t - \sin t, t\sin t + \cos t, t^2\rangle$ so $\mathbf{r}'(t) = \langle -t\sin t, t\cos t, 2t\rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{t^2\sin^2 t + t^2\cos^2 t + 4t^2} = \sqrt{t^2\left(\sin^2 t + \cos^2 t\right) + 4t^2} = \sqrt{5t^2} = |t|\sqrt{5}$. In this range t is strictly positive, so $\|\mathbf{r}'(t)\| = t\sqrt{5}$.

So the arc length is $s = \int_0^{2\pi} t\sqrt{5}\, dt = \frac{\sqrt{5}}{2}\left[t^2\right]_{t=0}^{t=2\pi} = 2\pi^2\sqrt{5}$

3. Compute the surface area of the surface given by the equation $z = x^2 + y^2$ above the region in the xy-plane between two circles centered at the origin, one of radius 2 and one of radius 3.

We are looking for the area of the shaded region of the following shape:



We can apply the surface area formula using a function:

$$A = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA \text{ with } f_x(x,y) = 2x \text{ and } f_x(x,y) = 2y$$

so $A = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA = \iint_R \sqrt{4(x^2 + y^2) + 1}\, dA$.

The region we're integrating on is an annulus (the area between the circles on the plane), so we'd benefit from performing the integral in polar coordinates:

$A = \int_0^{2\pi} \int_2^3 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_2^3 \sqrt{4r^2+1}\, r\, dr\right) = 2\pi \int_2^3 \sqrt{4r^2+1}\, r\, dr$. We'll do a substitution to complete this integral:

Let $u = 4r^2 + 1$ so $du = 8r\, dr$. Our endpoints become $u(2) = 17$ and $u(3) = 37$.

$$A = \frac{\pi}{6}\left[u^{3/2}\right]_{u=17}^{u=37} = \frac{\pi}{6}\left(37^{3/2} - 17^{3/2}\right)$$

4. Evaluate the integral – change the coordinates suitably.

$$\iint_A \frac{\sin(x+y)}{y-2x}\,dx\,dy$$

The region A is the parallelogram bounded by the lines:

$$\begin{aligned} y &= -x \\ y &= \pi - x \\ y &= 2x+1 \\ y &= 2x+3 \end{aligned}$$

We'll let $u = y + x$ and $v = y - 2x$. Under these new variables u varies from 0 to π and v varies from 1 to 3.

Solving for the inverse functions, we find that $u = y + x \Leftrightarrow x = u - y$. Substituting this into the other equation and solving we find that $v = y - 2(u-y) \Leftrightarrow y = \frac{1}{3}v + \frac{2}{3}u$ so $x = u - \frac{v+2u}{3} = \frac{1}{3}u - \frac{1}{3}v$. So the inverse functions are $x(u,v) = \frac{1}{3}u - \frac{1}{3}v$ and $y(u,v) = \frac{2}{3}u + \frac{1}{3}v$.

The Jacobian matrix is $J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$. We need the absolute value of the determinant $|\det J| = \left|\frac{1}{9} + \frac{2}{9}\right| = \frac{1}{3}$.

$$\iint_A \frac{\sin\overbrace{(x+y)}^{u}}{\underbrace{y-2x}_{v}}\underbrace{dx\,dy}_{dA} = \int_1^3\int_0^\pi \frac{\sin(u)}{v}\underbrace{\overbrace{\frac{1}{3}}^{|\det J|}\,du\,dv}_{dA} = \frac{1}{3}\left(\int_1^3 \frac{1}{v}\,dv\right)\left(\int_0^\pi \sin(u)\,du\right) = \frac{1}{3}(\ln 3)\left[-\cos u\right]_{u=0}^{u=\pi} = \frac{2}{3}(\ln 3)$$