1. A semicircular wire is in the shape of the lower half of a circle of radius $a$ centered at the origin. The composition of the wire is such that the linear density of the wire is equal to $y^2$. Find the mass of this wire.

\( \rho(x, y) = y^2 \). A parameterization of this curve is \( \gamma(t) = \langle a \cos t, a \sin t \rangle \) with \( \pi \leq t \leq 2\pi \) so \( \gamma'(t) = \langle -a \sin t, a \cos t \rangle \).

\[
m = \int_C \rho \, ds = \int_{\pi}^{2\pi} \rho(\gamma(t)) \|\gamma'(t)\| \, dt = \int_{\pi}^{2\pi} a^2 \sin^2 t \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} \, dt = a^3 \int_{\pi}^{2\pi} \sin^2 t \, dt
\]

\[
= \frac{a^3}{2} \int_{\pi}^{2\pi} (1 - \cos(2t)) \, dt = \frac{a^3}{2} \left[ t - \frac{1}{2} \sin(2t) \right]_{\pi}^{2\pi} = \frac{a^3}{2} \left( 2\pi - \pi \right) = \frac{a^3 \pi}{2}
\]

2. An object is traveling in the force field

\[
F(x, y) = \frac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}
\]

along a circle centered at the origin with radius $a$ starting at the $x$-axis (i.e., at the point \( (a, 0) \)) and travelling counterclockwise for one eighth of the circle (ending at the point \( (a\sqrt{2}, a\sqrt{2}) \)). Compute the work done by the force field $F$.

It would be lovely if this were a conservative vector field, but if \( F = \langle M, N \rangle \), then

\[
N_x = \left( x^2 + y^2 \right)^{-1/2} - x^2 \left( x^2 + y^2 \right)^{-3/2} \quad \text{and} \quad M_y = \left( x^2 + y^2 \right)^{-1/2} - y^2 \left( x^2 + y^2 \right)^{-3/2}.
\]

These are not equal, so we need to directly compute the integral.

A parameterization of this curve is \( \gamma(t) = \langle a \cos t, a \sin t \rangle \) and we run \( 0 \leq t \leq \frac{\pi}{4} \). From this,

\[
F(\gamma(t)) = \left( \frac{a \sin t}{\sqrt{a^2 \cos^2 t + a^2 \sin^2 t}}, \frac{a \cos t}{\sqrt{a^2 \cos^2 t + a^2 \sin^2 t}} \right) = \langle \sin t, \cos t \rangle
\]

\( \gamma'(t) = \langle -a \sin t, a \cos t \rangle \) so evaluating the integral directly:

\[
\int_C F \cdot dr = \int_{\pi/4}^{\pi/4} F(\gamma(t)) \cdot \gamma'(t) \, dt = \int_{\pi/4}^{\pi/4} \langle \sin t, \cos t \rangle \cdot \langle -a \sin t, a \cos t \rangle \, dt = \int_{\pi/4}^{\pi/4} -a \sin^2 t + a \cos^2 t \, dt
\]

\[
= a \left[ \frac{\pi}{4} \right]_{0}^{\pi/4} + \cos(2t) \left. \right|_{0}^{\pi/4} - \frac{1 - \cos(2t)}{2} \left. \right|_{0}^{\pi/4} = a \left[ \frac{1}{2} \sin(2t) \right]_{0}^{\pi/4} = a \left[ \frac{1}{2} \right] = \frac{a}{2}
\]

3. An object is traveling in 3-dimensional space in the space in the force field given by
\[ \mathbf{F}(x, y, z) = \langle yz + ze^x, xz, xy + e^x \rangle \]

The trajectory of the point follows the hyperbola in the \( y/z \)-plane (i.e., \( x = 0 \)) given by the equation \( y^2 - z^2 = 1 \). The start point is on the \( y \)-axis and the end point is the intersection of the hyperbola with the line \( y = 2 \) that lies in the \( y/z \)-plane above the \( y \)-axis. Compute the work done by the field \( \mathbf{F} \).

The hyperbola and line are depicted. We can algebraically solve for the points of intersections: The equation intersects the \( y \)-axis when \( z = 0 \) so \( y^2 = 1 \) or \( y = \pm 1 \). The hyperbola intersects the line \( y = 2 \) when \( z^2 = 3 \) or \( z = \pm \sqrt{3} \). Given the graph, it is clear that we have a start point of \((0, 1, 0)\) and an endpoint of \((0, 2, \sqrt{3})\).

We note that if \( \mathbf{F} = \langle L, M, N \rangle \) then \( L_x = y + e^x = N_z \), \( L_y = z = M_x \) and \( M_z = x = N_y \), so the vector field may be conservative. Solving for the potential function:

\[ f(x, y, z) = \int L \, dx = yzx + ze^x + G(y, z), \quad \text{so} \quad f_y(x, y, z) = zx + G_y(y, z) = M \quad \text{so} \]

\[ G_y(y, z) = 0. \]

Solving, \( G(y, z) = \int G_y(y, z) \, dy = H(z) \).

\[ f_z(x, y, z) = yx + e^x + H'(z) = N \quad \text{so} \quad H'(z) = 0 \quad \text{so} \quad H(z) = \int H'(z) \, dy = c. \]

We'll let \( c = 0 \) so \( f(x, y, z) = yzx + ze^x \). We have a potential function, so the vector field is conservative. As such, we can solve the desired integral by using the Fundamental Theorem of Line Integrals:

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 2, \sqrt{3}) - f(0, 1, 0) = \sqrt{3} \]

Alternately, we could parameterize this curve and then directly integrate.

\( \gamma(t) = \langle 0, \cosh t, \sinh t \rangle \), making \( \gamma'(t) = \langle 0, \sinh t, \cosh t \rangle \). We run \( 0 \leq t \leq \cosh^{-1} 2 \).

\( \mathbf{F}(\gamma(t)) = \langle \cosh t \sinh t + \sinh t, 0, 1 \rangle = \langle \sinh t (\cosh t + 1), 0, 1 \rangle \), so

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\cosh^{-1} 2} \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt = \int_0^{\cosh^{-1} 2} \langle \sinh t (\cosh t + 1), 0, 1 \rangle \cdot \langle 0, \sinh t, \cosh t \rangle \, dt = \int_0^{\cosh^{-1} 2} \cosh t \, dt \]

\[ = \left[ \sinh t \right]_0^{\cosh^{-1} 2} = \sinh \left( \cosh^{-1} 2 \right) = \sqrt{3} \]

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4. Use Green's Theorem to compute the amount of work done by the force field
\[ F(x, y) = \left( -x(x + y)^{3/2}, y(x + y)^{3/2} \right) \]
that moves an object clockwise along the parallelogram determined by lines
\[
\begin{align*}
y &= 1 - x \\
y &= 2 - x \\
y &= 2x \\
y &= 2x - 2
\end{align*}
\]
We'll call \( F(x, y) = \langle M, N \rangle \). We first check to see if \( F \) is conservative; we compute
\[
N_x = \frac{3}{2} y(x + y)^{1/2} \quad \text{and} \quad M_y = -\frac{3}{2} x(x + y)^{1/2}.
\]
These are not equal, so \( F \) is clearly not conservative.

This path is simple, closed and piecewise smooth, and the vector field \( F \) is suitably smooth (it has continuous partials) so we can apply Green's Theorem. Note that the path is negatively oriented (it travels clockwise), so Green's Theorem tells us:
\[
\int_c F \cdot dr = -\iint_R \left( N_x - M_y \right) dA = -\frac{3}{2} \iint_R (x + y)^{1/2} (y + x) dA = -\frac{3}{2} \iint_R (x + y)^{3/2} dA
\]
We are evaluating this integral on the region \( R \), the parallelogram described above. We could attempt to partition the parallelogram into pieces that we can integrate, but it is easier to do a change of variables:

Let \( u(x, y) = x + y \) and \( v(x, y) = y - 2x \). With this substitution, we have \( 1 \leq u \leq 2 \) and \(-2 \leq v \leq 0\). We must solve for the inverse functions to get the Jacobian Matrix:
\[
y = u - x \quad \text{so} \quad v = (u - x) - 2x = u - 3x \quad \text{so} \quad x(u, v) = \frac{u - v}{3} \quad \text{and thus} \quad y(u, v) = u - \frac{u - v}{3} = \frac{2u + v}{3}.
\]
Now we can calculate the Jacobian Matrix:
\[
J = \begin{bmatrix}
x_u & x_v \\
y_u & y_v
\end{bmatrix} = \begin{bmatrix}
1/3 & -1/3 \\
2/3 & 1/3
\end{bmatrix}
\]
so the absolute value of the determinant is \( \left| \det J \right| = 1/3 \).
\[
\int_c F \cdot dr = -\frac{3}{2} \iint_R (x + y)^{3/2} dA = -\frac{3}{2} \int_1^2 \int_{-2}^0 u^{3/2} \frac{1}{3} dv du = -\frac{1}{2} \left( \frac{1}{3} \int_1^2 u^{3/2} du \right) \left( \int_{-2}^0 dv \right) = -\frac{1}{2} \left( \frac{1}{3} \int_1^2 u^{3/2} du \right) \left( 2^{3/2} - 1 \right) = -\frac{2}{5} \left( 2^{5/2} - 1 \right)
\]