Math 2E Midterm 2 Solution

1. A semicircular wire is in the shape of the lower half of a circle of radius a centered at the origin. The composition of the wire is such that the linear density of the wire is equal to y^2. Find the mass of this wire.

$\rho(x,y) = y^2$. A parameterization of this curve is $\boldsymbol{\gamma}(t) = \langle a\cos t, a\sin t\rangle$ with $\pi \le t \le 2\pi$ so $\boldsymbol{\gamma}'(t) = \langle -a\sin t, a\cos t\rangle$.

$$m = \int_C \rho\, ds = \int_\pi^{2\pi} \rho(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|\,dt = \int_\pi^{2\pi} a^2\sin^2 t\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = a^3\int_\pi^{2\pi}\sin^2 t\,dt$$

$$= \frac{a^3}{2}\int_\pi^{2\pi}(1-\cos(2t))\,dt = \frac{a^3}{2}\left[t - \frac{1}{2}\sin(2t)\right]_\pi^{2\pi} = \frac{a^3}{2}(2\pi-\pi) = \frac{a^3\pi}{2}$$

2. An object is traveling in the force field

$$\mathbf{F}(x,y) = \frac{\langle y,x\rangle}{\sqrt{x^2+y^2}}$$

along a circle centered at the origin with radius a starting at the x-axis (i.e., at the point $(a,0)$) and travelling counterclockwise for one eighth of the circle (ending at the point $\left(a/\sqrt{2}, a/\sqrt{2}\right)$). Compute the work done by the force field $\mathbf{F}$.

It would be lovely if this were a conservative vector field, but if $\mathbf{F} = \langle M, N\rangle$, then $N_x = \left(x^2+y^2\right)^{-1/2} - x^2\left(x^2+y^2\right)^{-3/2}$ and $M_y = \left(x^2+y^2\right)^{-1/2} - y^2\left(x^2+y^2\right)^{-3/2}$. These are not equal, so we need to directly compute the integral.

A parameterization of this curve is $\boldsymbol{\gamma}(t) = \langle a\cos t, a\sin t\rangle$ and we run $0 \le t \le \frac{\pi}{4}$. From this,

$$F(\boldsymbol{\gamma}(t)) = \left\langle \frac{a\sin t}{\sqrt{a^2\cos^2 t + a^2\sin^2 t}}, \frac{a\cos t}{\sqrt{a^2\cos^2 t + a^2\sin^2 t}}\right\rangle = \langle \sin t, \cos t\rangle$$

$\boldsymbol{\gamma}'(t) = \langle -a\sin t, a\cos t\rangle$ so evaluating the integral directly:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/4} F(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt = \int_0^{\pi/4}\langle \sin t, \cos t\rangle\cdot\langle -a\sin t, a\cos t\rangle\,dt = \int_0^{\pi/4} -a\sin^2 t + a\cos^2 t\,dt$$

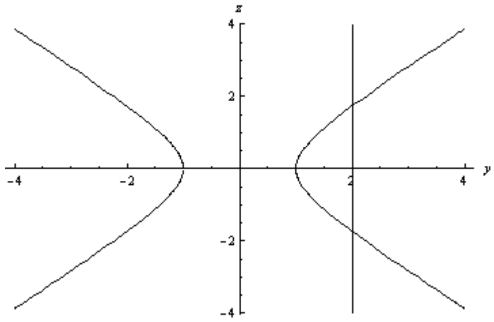
$$= a\int_0^{\pi/4}\frac{1+\cos(2t)}{2} - \frac{1-\cos(2t)}{2}\,dt = \frac{a}{2}\int_0^{\pi/4}(1+\cos(2t)) - (1-\cos(2t))\,dt = \frac{a}{2}\int_0^{\pi/4} 2\cos(2t)\,dt$$

$$= \frac{a}{2}\left[\sin(2t)\right]_0^{\pi/4} = \frac{a}{2}[1-0]_0^{\pi/4} = \frac{a}{2}$$

3. An object is traveling in 3-dimensional space in the space in the force field given by

$$\mathbf{F}(x,y,z)=\left\langle yz+ze^{x}, xz, xy+e^{x}\right\rangle$$

The trajectory of the point follows the hyperbola in the y/z-plane (i.e., $x=0$) given by the equation $y^2-z^2=1$. The start point is on the y-axis and the end point is the intersection of the hyperbola with the line $y=2$ that lies in the y/z-plane above the y-axis. Compute the work done by the field $\mathbf{F}$.



The hyperbola and line are depicted. We can algebraically solve for the points of intersections: The equation intersects the y-axis when $z=0$ so $y^2=1$ or $y=\pm 1$. The hyperbola intersects the line $y=2$ when $z^2=3$ or $z=\pm\sqrt{3}$. Given the graph, it is clear that we have a start point of $(0,1,0)$ and an endpoint of $\left(0,2,\sqrt{3}\right)$.

We note that if $F=\langle L,M,N\rangle$ then $L_z=y+e^x=N_x$, $L_y=z=M_x$ and $M_z=x=N_y$, so the vector field may be conservative. Solving for the potential function:

$f(x,y,z)=\int L\,dx=yzx+ze^x+G(y,z)$, so $f_y(x,y,z)=zx+G_y(y,z)=M$ so

$G_y(y,z)=0$. Solving, $G(y,z)=\int G_y(y,z)\,dy=H(z)$.

$f_z(x,y,z)=yx+e^x+H'(z)=N$ so $H'(z)=0$ so $H(z)=\int H'(z)\,dy=c$. We'll let

$c=0$ so $f(x,y,z)=yzx+ze^x$. We have a potential function, so the vector field is conservative. As such, we can solve the desired integral by using the Fundamental Theorem of Line Integrals: $\int_C \mathbf{F}\cdot d\mathbf{r}= f\left(0,2,\sqrt{3}\right)-f(0,1,0)=\sqrt{3}$

Alternately, we could parameterize this curve and then directly integrate.

$\boldsymbol{\gamma}(t)=\langle 0,\cosh t,\sinh t\rangle$, making $\boldsymbol{\gamma}'(t)=\langle 0,\sinh t,\cosh t\rangle$, We run $0\le t\le \cosh^{-1}2$.

$F(\boldsymbol{\gamma}(t))=\langle \cosh t\sinh t+\sinh t,0,1\rangle=\langle \sinh t(\cosh t+1),0,1\rangle$, so

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{\cosh^{-1}2} F(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt=\int_0^{\cosh^{-1}2}\langle \sinh t(\cosh t+1),0,1\rangle\cdot\langle 0,\sinh t,\cosh t\rangle\,dt=\int_0^{\cosh^{-1}2}\cosh t\,dt$$

$$=[\sinh t]_0^{\cosh^{-1}2}=\sinh\left(\cosh^{-1}2\right)\underbrace{=}_{\text{laborous arithmatic}}\sqrt{3}$$

4. Use Green's Theorem to compute the amount of work done by the force field

$$\mathbf{F}(x,y)=\left\langle -x(x+y)^{3/2},\, y(x+y)^{3/2}\right\rangle$$

that moves an object clockwise along the parallelogram determined by lines

$$y=1-x \qquad y=2-x$$
$$y=2x \qquad y=2x-2$$

We'll call $\mathbf{F}(x,y)=\langle M,N\rangle$. We first check to see if $\mathbf{F}$ is conservative; we compute $N_x=\frac{3}{2}y(x+y)^{1/2}$ and $M_y=-\frac{3}{2}x(x+y)^{1/2}$. These are not equal, so $\mathbf{F}$ is clearly not conservative.

This path is simple, closed and piecewise smooth, and the vector field $\mathbf{F}$ is suitably smooth (it has continuous partials) so we can apply Green's Theorem. Note that the path is negatively oriented (it travels clockwise), so Green's Theorem tells us:

$$\int_C \mathbf{F}\cdot d\mathbf{r}=-\iint_R \left(N_x-M_y\right)dA=-\frac{3}{2}\iint_R (x+y)^{1/2}(y+x)\,dA=-\frac{3}{2}\iint_R (x+y)^{3/2}\,dA$$

We are evaluating this integral on the region R, the parallelogram described above. We could attempt to partition the parallelogram into pieces that we can integrate, but it is easier to do a change of variables:

Let $u(x,y)=x+y$ and $v(x,y)=y-2x$. With this substitution, we have $1\le u\le 2$ and $-2\le v\le 0$. We must solve for the inverse functions to get the Jacobian Matrix:

$y=u-x$ so $v=(u-x)-2x=u-3x$ so $x(u,v)=\frac{u-v}{3}$ and thus

$y(u,v)=u-\frac{u-v}{3}=\frac{2u+v}{3}$.

Now we can calculate the Jacobian Matrix:

$J=\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}=\begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$ so the absolute value of the determinant is $\left|\det J\right|=\frac{1}{3}$.

$$\int_C \mathbf{F}\cdot d\mathbf{r}=-\frac{3}{2}\iint_R (x+y)^{3/2}\,dA=-\frac{3}{2}\int_1^2\int_{-2}^0 u^{3/2}\underbrace{\frac{1}{3}dv\,du}_{dA}=-\frac{1}{2}\left(\int_1^2 u^{3/2}\,du\right)\left(\int_{-2}^0 dv\right)=-\int_1^2 u^{3/2}\,du$$

$$=-\frac{2}{5}\left[u^{5/2}\right]_{u=1}^{u=2}=-\frac{2}{5}\left(2^{5/2}-1\right)$$