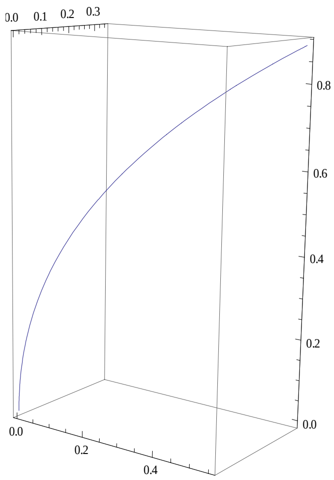
Math 2E Lec A Solution
Quiz 2 (Redux)

1) *(4 pts)* Find the arc-length for the curve $\mathbf{r}(t) = \left\langle \frac{t}{\sqrt{2}}, -\ln(\cos(t)), \frac{t}{\sqrt{2}} \right\rangle$ between the points $(0,0,0)$ and $\left(\frac{\pi}{4\sqrt{2}}, -\ln\left(\frac{1}{\sqrt{2}}\right), \frac{\pi}{4\sqrt{2}}\right)$.

For your reference, here's a graph of this curve:



By looking at the *x* or *z* component of the points, we see that these points correspond to $t = 0$ and $t = \frac{\pi}{4}$

To calculate the arc length, we need:

$\mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{2}}, \tan t, \frac{1}{\sqrt{2}} \right\rangle$

$\|\mathbf{r}'(t)\| = \sqrt{\frac{1}{2} + \tan^2 t + \frac{1}{2}} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$. With $t \in \left[0, \frac{\pi}{4}\right]$ $\sec t > 0$ so $\|\mathbf{r}'(t)\| = \sec t$

Finally,

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^{\frac{\pi}{4}} \sec t \, dt = \int_0^{\frac{\pi}{4}} \sec t \frac{\sec t + \tan t}{\sec t + \tan t} dt = \int_0^{\frac{\pi}{4}} \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t} dt$$

Let $u = \sec t + \tan t$ then $du = (\sec t \tan t + \sec^2 t) dt$. Our new bounds will be $u(0) = \sec 0 + \tan 0 = 1 + 0$ and $u\left(\frac{\pi}{4}\right) = \sec\frac{\pi}{4} + \tan\frac{\pi}{4} = \sqrt{2} + 1$.

$$s = \int_0^{\frac{\pi}{4}} \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t} dt = \int_1^{1+\sqrt{2}} \frac{1}{u} du = \ln|u|\Big|_1^{1+\sqrt{2}} = \ln\left(1 + \sqrt{2}\right)$$

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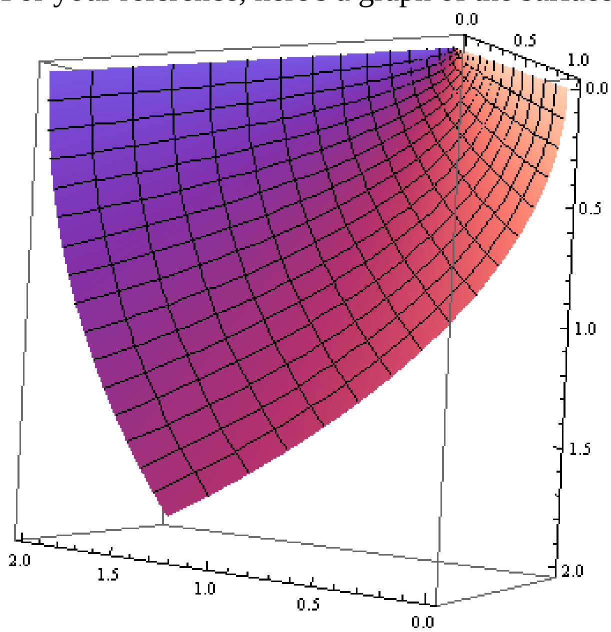
2) *(6 pts)* Find the surface area of the surface with the parametric equation $\mathbf{r}(u,v) = \left\langle u^2, uv, \frac{1}{2}v^2 \right\rangle$ with $0 \le u \le 1$ and $0 \le v \le 2$. Leave your answer in terms of an integral with explicit bounds.

For your reference, here's a graph of the surface:



$$A = \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA$$

$\mathbf{r}_u = \langle 2u, v, 0 \rangle$ $\qquad$ $\mathbf{r}_v = \langle 0, u, v \rangle$ so

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = v^2\,\mathbf{i} - 2uv\,\mathbf{j} + 2u^2\,\mathbf{k} = \left\langle v^2, -2uv, 2u^2 \right\rangle$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{\left(v^2 + 2u^2\right)^2} = \left|v^2 + 2u^2\right| = v^2 + 2u^2$$

$$A = \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA = \int_0^1 \int_0^2 \left(v^2 + 2u^2\right) dv\, du$$

You could leave your integral in this form, or solve:

$$\int_0^1 \int_0^2 \left(v^2 + 2u^2\right) dv\, du = \int_0^1 \left[\frac{1}{3}v^3 + 2u^2 v\right]_{v=0}^{v=2} du = \int_0^1 \left(\frac{8}{3} + 4u^2\right) du = \left[\frac{8}{3}u + \frac{4}{3}u^3\right]_{u=0}^{u=1} = \frac{8}{3} + \frac{4}{3} = 4$$