Math 2E Lec A Solution
Quiz 4

1) *(4 pts)* Compute the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z)=\langle z,0,3x^2\rangle$ and C is the quarter-ellipse parameterized by $\boldsymbol{\gamma}(t)=\langle 2\cos t,3\sin t,1\rangle$ from $t=0$ to $t=\frac{\pi}{2}$.

$\mathbf{F}(\boldsymbol{\gamma}(t))=\langle 1,0,12\cos^2(t)\rangle$ and $\boldsymbol{\gamma}'(t)=\langle -2\sin t,3\cos t,0\rangle$ so

$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{\pi/2}\mathbf{F}(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt=\int_0^{\pi/2}\langle 1,0,12\cos^2(t)\rangle\cdot\langle -2\sin t,3\cos t,0\rangle\,dt$

$=-2\int_0^{\pi/2}\sin t\,dt=2[\cos t]_0^{\pi/2}=2(0-1)=-2$

2) *(6 pts)* Compute $\int_C \sin y\,dx+(2+x\cos y)\,dy$ where C is the part of the curve $y=\pi x^5$ running from $(0,0)$ to $(1,\pi)$.

First the intended approach: The vector field in this integral is $\mathbf{F}(x,y)=\langle \sin y,2+x\cos y\rangle=\langle M,N\rangle$. Note that $M_y=\cos y=N_x$ so the vector field could be conservative. We try to solve for the potential function:
$f(x,y)=\int M\,dx=\int \sin y\,dx=x\sin y+G(y)$. We now take the y partial:
$f_y(x,y)=x\cos y+G'(y)=N=2+x\cos y$ so $G'(y)=2$. Now $G(y)=\int 2\,dy=2y+c$, so $f(x,y)=x\sin y+2y+c$. The choice of constant doesn't affect our later calculations, so we can take $c=0$, leaving us with the potential function $f(x,y)=x\sin y+2y$. This vector field has a potential function, so it is conservative. Now we can apply the fundamental theorem of line integrals:
$\int_C \mathbf{F}\cdot d\mathbf{r}=f(1,\pi)-f(0,0)=(\sin\pi+2\pi)-0=2\pi$

If we hadn't noticed that the vector field was conservative, we would get an unreasonably tricky integral: $\boldsymbol{\gamma}(t)=\langle t,\pi t^5\rangle$ (where $0\le t\le 1$) so $\boldsymbol{\gamma}'(t)=\langle 1,5\pi t^4\rangle$ and
$\mathbf{F}(\boldsymbol{\gamma}(t))=\langle \sin(\pi t^5),2+t\cos(\pi t^5)\rangle$ so:

$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1\mathbf{F}(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt=\int_0^1\langle \sin(\pi t^5),2+t\cos(\pi t^5)\rangle\cdot\langle 1,5\pi t^4\rangle\,dt$

$=\int_0^1\left(\sin(\pi t^5)+5\pi t^4\left(2+t\cos(\pi t^5)\right)\right)dt=\int_0^1\left(\sin(\pi t^5)+10\pi t^4+5\pi t^5\cos(\pi t^5)\right)dt$

$=\int_0^1 10\pi t^4\,dt+\int_0^1\sin(\pi t^5)\,dt+\int_0^1 \underbrace{t}_{u}\underbrace{\left(5\pi t^4\cos(\pi t^5)\right)}_{dv}dt$ (integration by parts)

$=2\pi+\int_0^1\sin(\pi t^5)\,dt+\left[t\sin(\pi t^5)\right]_0^1-\int_0^1\sin(\pi t^5)\,dt=2\pi$