1) (10 pts) Evaluate the integral \( \iint_X z \, dS \) where the surface \( X \) is \( z^2 = x^2 + y^2 \) with \( 0 \leq z \leq 4 \).

The surface is a cone, shown below:

To parameterize, we first look at the \( z \)-trace, which is a circle centered at the origin with radius \( z \). This suggests that this is a surface of rotation. Now examining the \( y = 0 \) trace:

We want to rotate this shape about the \( z \)-axis to get our surface of rotation. Indeed, we don't need both of these lines, as we'd get the same shape by rotating just one of the lines.

A parameterization for the positive slope line is \( \gamma(u) = (u,0,u) \). We then rotate this about the \( z \)-axis to get the full surface \( r(u,\theta) = (u \cos \theta, u \sin \theta, u) \). The surface that we are integrating on is \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq u \leq 4 \).

We are integrating a scalar valued function \( f(x,y,z) = z \). We calculate

\[
\mathbf{r}_u \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -u \sin \theta & u \cos \theta & 0 \end{vmatrix} = (u \cos \theta, -u \sin \theta, u \cos^2 \theta + u \sin^2 \theta) = (u \cos \theta, -u \sin \theta, u)
\]

so \( \|\mathbf{r}_u \times \mathbf{r}_\theta\| = \sqrt{u^2 \cos^2 \theta + u^2 \sin^2 \theta + u^2} = \sqrt{2u^2} = u\sqrt{2} \) as \( u > 0 \). Putting it together:

\[
\iint_X f \, dS = \iiint_D \frac{\|\mathbf{r}_u \times \mathbf{r}_\theta\|}{ds} dA = \sqrt{2} \int_0^{2\pi} \int_0^4 u^2 \, du \, d\theta = \sqrt{2} \left( \int_0^{2\pi} 1 \, d\theta \right) \left( \int_0^4 u^2 \, du \right)
\]

\[
= \frac{2\pi \sqrt{2}}{3} \left[ u^3 \right]_0^4 = \frac{128\pi \sqrt{2}}{3}
\]
After looking through the quiz, it appears that most people used a different approach for the parameterization. I prefer the approach above, but for completeness, here is the approach used by most of the students:

Because our surface only includes portions where \( z > 0 \), we can view the surface as the graph of the function \( z = f(x, y) = \sqrt{x^2 + y^2} \). As this is a function, we can parameterize this surface using \( \mathbf{r}(x, y) = \left(x, y, \sqrt{x^2 + y^2}\right) \). With this view, we want to run our parameterization within the circle \( 16 = x^2 + y^2 \) in the x/y plane, a region that we'll call \( D \). As we are integrating within a circular region we will need to shift to polar coordinates to compute the integral.

With this parameterization, we still need to calculate \( \|\mathbf{r}_x \times \mathbf{r}_y\| \), but we have a special form for this that works only for functions:

\[
\begin{align*}
\|\mathbf{r}_x \times \mathbf{r}_y\| &= \sqrt{\left(f_x\right)^2 + \left(f_y\right)^2 + 1} \\
&= \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + 1} \\
&= \sqrt{\frac{x^2 + y^2}{x^2 + y^2} + 1} = \sqrt{2}
\end{align*}
\]

\[
\iint_X f \, dS = \iint_D f\left(\mathbf{r}(x, y)\right) \left|\mathbf{r}_x \times \mathbf{r}_y\right| \, dA = \sqrt{2} \iint_D \sqrt{x^2 + y^2} \, dA = \sqrt{2} \int_0^{2\pi} \int_0^4 r^2 \, rdr \, d\theta = \sqrt{2} \int_0^{2\pi} \left[ \frac{r^3}{3} \right]_0^4 \, d\theta = \frac{128\pi \sqrt{2}}{3}
\]