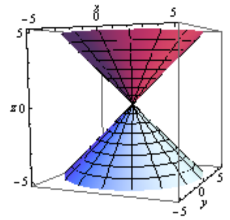
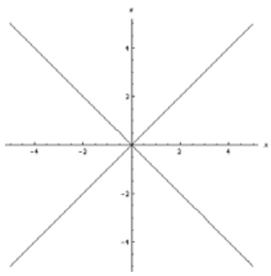
Math 2E Lec A   Solution
Quiz 6

1) *(10 pts)* Evaluate the integral $\iint_X z\, dS$ where the surface X is $z^2 = x^2 + y^2$ with $0 \le z \le 4$.

The surface is a cone, shown below:



To parameterize, we first look at the z-trace, which is a circle centered at the origin with radius z. This suggests that this is a surface of rotation. Now examining the $y = 0$ trace:



We want to rotate this shape about the z-axis to get our surface of rotation. Indeed, we don't need both of these lines, as we'd get the same shape by rotating just one of the lines.

A parameterization for the positive slope line is $\boldsymbol{\gamma}(u) = \langle u, 0, u \rangle$. We then rotate this about the z-axis to get the full surface $\mathbf{r}(u,\theta) = \langle u\cos\theta, u\sin\theta, u \rangle$. The surface that we are integrating on is $0 \le \theta \le 2\pi$ and $0 \le u \le 4$.

We are integrating a scalar valued function $f(x,y,z) = z$. We calculate

$$\mathbf{r}_u \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -u\sin\theta & u\cos\theta & 0 \end{vmatrix} = \left\langle -u\cos\theta, -u\sin\theta, u\cos^2\theta + u\sin^2\theta \right\rangle = \left\langle -u\cos\theta, -u\sin\theta, u \right\rangle$$

so $\left\|\mathbf{r}_u \times \mathbf{r}_\theta\right\| = \sqrt{u^2\cos^2\theta + u^2\sin^2\theta + u^2} = \sqrt{2u^2} = u\sqrt{2}$ as $u > 0$. Putting it together:

$$\iint_X f\, dS = \iint_D f(\mathbf{r}(u,\theta)) \underbrace{\left\|\mathbf{r}_u \times \mathbf{r}_\theta\right\| dA}_{dS} = \sqrt{2}\int_0^{2\pi}\int_0^4 u^2\, du\, d\theta = \sqrt{2}\left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^4 u^2\, du\right)$$

$$= \frac{2\pi\sqrt{2}}{3}\left[u^3\right]_0^4 = \frac{128\pi\sqrt{2}}{3}$$

After looking through the quiz, it appears that most people used a different approach for the parameterization. I prefer the approach above, but for completeness, here is the approach used by most of the students:

Because our surface only includes portions where $z > 0$, we can view the surface as the graph of the function $z = f(x, y) = \sqrt{x^2 + y^2}$. As this is a function, we can parameterize this surface using $\mathbf{r}(x, y) = \left\langle x, y, \sqrt{x^2 + y^2} \right\rangle$. With this view, we want to run our parameterization within the circle $16 = x^2 + y^2$ in the x/y plane, a region that we'll call D. As we are integrating within a circular region we will need to shift to polar coordinates to compute the integral.

With this parameterization, we still need to calculate $\left\| \mathbf{r}_x \times \mathbf{r}_y \right\|$, but we have a special form for this that works only for functions:

$$\left\| \mathbf{r}_x \times \mathbf{r}_y \right\| = \sqrt{(f_x)^2 + (f_y)^2 + 1} = \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^2 + 1} = \sqrt{\frac{x^2 + y^2}{x^2 + y^2} + 1} = \sqrt{2} \text{ so}$$

$$\iint_X f \; dS = \iint_D f(\mathbf{r}(x, y)) \underbrace{\left\| \mathbf{r}_x \times \mathbf{r}_y \right\| dA}_{dS} = \sqrt{2} \iint_D \sqrt{x^2 + y^2} \; dA = \sqrt{2} \int_0^{2\pi} \int_0^4 \sqrt{r^2} \; \underbrace{r dr d\theta}_{dA} =$$

$$\sqrt{2} \int_0^{2\pi} \int_0^4 r^2 \, dr d\theta = \sqrt{2} \left(\int_0^{2\pi} 1 d\theta \right) \left(\int_0^4 r^2 \, dr \right) = \frac{2\pi\sqrt{2}}{3} \left[r^3 \right]_0^4 = \frac{128\pi\sqrt{2}}{3}$$