Sample solution: Book, Exercises for Section 15.6, problem 52.

I give only main points, please fill in the details.

The surface is the portion of paraboloid $z = x^2+y^2$ inside the cylinder $x^2+y^2 = 4$. The density is given by $\rho(x, y, z) = z$.

Draw a picture as usual. The intersection of the paraboloid with the cylinder is in the plane $z = 4$ (why?). This intersection projects into the xy-plane as the circle $x^2 + y^2 = 4$. The surface is inside the cylinder, so it consists of the portion of paraboloid $z = x^2 + y^2$ above the disk in xy-plane $x^2 + y^2 \leq 4$. Denote this disk by A. The mass M is given by

$$M = \int\int_S \rho(x, y, z)\, \mathrm{d}S$$

Since the paraboloid is given by a function $f(x, y) = x^2+y^2$, we may use evaluation formula (2) from the "brief guide". However, we use the evaluation formula (3) to demonstrate the method. This may look more complicated, but check that if we used formula (2) we would have to do change of coordinates, so the complexity of the entire computation in both cases is about the same.

We use polar coordinates to parametrize A. We thus get $\mathbf{r}(r, \theta) = \langle x(r, \theta), y(r, \theta), z(r, \theta)$ where $x(r, \theta) = r \cos\theta$ and $y(r, \theta) = r \sin\theta$. Since the disk A has radius 2, we have $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Then

$$z(r, \theta) = r^2 \cos^2\theta + r^2 \sin^2\theta = r^2.$$

We then compute the partial derivatives:

$$\begin{aligned} \mathbf{r}_r(r, \theta) &= \langle \cos\theta, \sin\theta, 2r\rangle \\ \mathbf{r}_\theta(r, \theta) &= \langle -r \sin\theta, r \cos\theta, 0\rangle \end{aligned}$$

Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \langle -2r^2 \cos\theta, -2r^2 \sin\theta, r\rangle$$

and

$$\| \mathbf{r}_r \times \mathbf{r}_\theta \| = r\sqrt{8r^2 + 1}.$$

Substituting into evaluation formula (3) from "brief guide" we get

$$\begin{aligned} M &= \int\int_S \rho(x, y, z)\, \mathrm{d}S = \int_0^{2\pi}\int_0^2 \rho(x(r, \theta), y(r, \theta), z(r, \theta) \| \mathbf{r}_r \times \mathbf{r}_\theta \| \,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^2 r^2 r\sqrt{8r^2 + 1}\, \mathrm{d}r\mathrm{d}\theta = 2\pi \int_0^2 r^2 r\sqrt{8r^2 + 1}\, \mathrm{d}r \end{aligned}$$

The last equality holds since the argument of integration does not depend on θ. Now if we substitute $r^2 = s$ in the last integral we get

$$M = \pi \int_0^4 s\sqrt{8s + 1}\, \mathrm{d}s$$

This integral can be solved by integration in parts: we let $u = s$ and $v' = \sqrt{8s+1}$. Then $u' = 1$ and $v = \frac{1}{12}(8s+1)^{3/2}$. We then get

$$M = \pi \int_0^4 s\sqrt{8s+1}\ \mathrm{d}s = \frac{\pi}{12}[s(8s+1)^{3/2}]_0^4 - \frac{\pi}{12}\int_0^4 (8s+1)^{3/2}\mathrm{d}s$$

The last integral is easy to evaluate.

Now compute the center of mass $(\bar{x}, \bar{y}, \bar{z})$. **Here is an important point:** The paraboloid is symmetric with respect to x, y and the density $\rho(x,y,z) = z$, i.e. it depends only on z. In this case $\bar{x} = 0 = \bar{y}$ (if you don't believe you can verify this by computation) so we only need to compute $\bar{z}$. As in Problem 50, we have $\bar{z} = M_z/M$ where

$$M_z = \int\int_S z\rho(x,y,z)\ \mathrm{d}S$$

Using our previous work we again use formula (3) from "brief guide" with $g(x,y,z) = z\rho(x,y,z) = z^2$ and get

$$M_z = \int_0^{2\pi}\int_0^2 r^4 r\sqrt{8r^2+1}\ \mathrm{d}r\mathrm{d}\theta$$

Then substitution $s = r^2$ and the fact that the integrand does not depend on θ give the integral

$$M_z = \pi\int_0^4 s^2\sqrt{8s+1}\ \mathrm{d}s$$

which can be evaluated in parts setting $u = s^2$ and $v' = \sqrt{8s+1}$ similarly as in the computation of mass. In this case the computation is more lengthy.