Sample solution: Book, Exercises for Section 15.6, problem 44.

I give only main points, please fill in the details.

We want to compute the flux of $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$ over the surface S where S is the boundary of the region bounded by $z = 0$ and $z = -\sqrt{4 - x^2 - y^2}$. The book does not specify the orientation of S which is a mistake in the book. Let us orient S so that the normal vector $\mathbf{n}$ points outward.

The surface is the portion below the xy-plane of the sphere with radius 2 centered at the origin. I will present both methods of computing the flux.

Method 1: Direct evaluation. We first compute the normal vector $\mathbf{n}(x, y, z)$. The sphere is described by equation $x^2 + y^2 + z^2 = 4$, so the function $h(x, y, z)$ in the "guide" is given by $h(x, y, z) = x^2 + y^2 + z^2 - 4$. Then

$$\mathbf{n}(x, y, z) = \pm\frac{\nabla h(x, y, z)}{\| \nabla h(x, y, z) \|} = \pm\frac{\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}}$$

Notice that $\mathbf{n}(x, y, z)$ has the same direction as the vector $\langle x, y, z\rangle$. Since h has continuous partial derivatives, it suffices to check the orientation of the vector in a single point. In the point $\langle 0, 0, -2\rangle$ which lies on S the value of $\langle x, y, z\rangle$ is $\langle 0, 0, -2\rangle$ (in this case it is the computation of the unit normal vectore is easy – it is $\langle 0, 0, -1\rangle$, but in the case of other surfaces the denominator in the fraction above might be complicated.) This vector points outwards, so we have so in this case our choice of the sign is "+". We thus have:

$$\mathbf{n}(x, y, z) = \frac{\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}}$$

and

$$\mathbf{F}(x, y, z) \cdot \mathbf{n}(x, y, z) = (x, y, z) \cdot \frac{\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}} = \frac{x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}} = 2.$$

Hence

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \int\int_S 2\, \mathrm{d}S$$

Now the relevant potion of sphere is given by function

$$f(x, y) = -\sqrt{4 - x^2 - y^2}.$$

The intersection with the xy-plane is given by equation $4 - x^2 - y^2 = 0$, which is the circle $x^2 + y^2 = 4$; denote the disk surrounded by this circle by D. So the surface S is the graph of $f(x, y)$ for $\langle x, y\rangle$ ranging over D. We compute the partial derivatives:

$$f_x(x, y) = \frac{x}{\sqrt{4 - x^2 - y^2}} \quad \text{and } f_y(x, y) = \frac{y}{\sqrt{4 - x^2 - y^2}}.$$

Then

$$1 + f_x^2(x, y) + f_y^2(x, y) = 1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} = \frac{4}{4 - x^2 - y^2}$$

and

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int\int_D 2\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\mathrm{d}y = \int\int_D \frac{4}{\sqrt{4-x^2-y^2}}\,\mathrm{d}x\mathrm{d}y$$

Now integration over the disk D works best with polar coordinates, so we change the coordinates to polar. We get $\mathbf{r}(r,\theta)=\langle x(r,\theta),y(r,\theta)\rangle$ where

$$x(r,\theta)=r\cos\theta,\quad y(r,\theta)=r\sin\theta,$$

$0\le r\le 2$ and $0\le\theta\le 2\pi$. Then $|J|=r$ where J is the Jacobian hence after the change of coordinates we get the integral

$$\int\int_S \mathbf{F}\cdot\mathrm{n}\,\mathrm{d}S = \int_0^{2\pi}\int_0^2 \frac{4r}{\sqrt{4-r^2}}\,\mathrm{d}r\mathrm{d}\theta = \int_0^2 \frac{8\pi r}{\sqrt{4-r^2}}\,\mathrm{d}r.$$

Substitution $s=r^2$ gives

$$\int_0^2 \frac{8\pi r}{\sqrt{4-r^2}}\,\mathrm{d}r = \int_0^4 \frac{4\pi}{\sqrt{4-s}}\,\mathrm{d}s = 8\pi[-\sqrt{4-s}]_0^4 = 16\pi.$$

Method 2: Using formula (5) in the "guide". Recall formula (5) reads

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \pm\int\int_A \mathbf{F}(x(u,v),y(u,v),z(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,\mathrm{d}u\mathrm{d}v$$

where $\mathbf{r}(u,v)=\langle x(u,v),y(u,v),z(u,v)\rangle$ is a parametrization of S with the parameters u,v ranging over a region A.

We use spherical coordinates to parametrize the sphere. We get:

$$\mathbf{r}(\varphi,\theta)=\langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle.$$

The region is the half-sphere below the xy-plane, so $0\le\theta\le 2\pi$ and $\pi/2\le\varphi\le\pi$. Then

$$\begin{aligned}\mathbf{r}_\varphi(\varphi,\theta) &= \langle 2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi\rangle\\ \mathbf{r}_\theta(\varphi,\theta) &= \langle -2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, 0\rangle\end{aligned}$$

and

$$\mathbf{r}_r\times\mathbf{r}_\theta = \langle 4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi\rangle.$$

Furthermore

$$\mathbf{F}(x(\varphi,\theta),y(\varphi,\theta),z(\varphi,\theta)) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle$$

hence

$$\begin{aligned}\mathbf{F}(x(\varphi,\theta),y(\varphi,\theta),z(\varphi,\theta))\cdot(\mathbf{r}_\varphi\times\mathbf{r}_\theta) &= 8\sin^3\varphi\cos^2\theta+8\sin^3\varphi\sin^2\theta+8\sin\varphi\cos^2\varphi\\ &= 8\sin^3\varphi+8\sin\varphi\cos^2\varphi = 8\sin\varphi\end{aligned}$$

Then

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \pm\int_0^{2\pi}\int_{\pi/2}^{\pi} 8\sin\varphi\,\mathrm{d}\varphi\mathrm{d}\theta = \pm 16\pi$$

Finally we have to check the orientation of $\mathbf{r}_\varphi \times \mathbf{r}_\theta$. Since the partial derivatives are continuous for $0 < \varphi < \pi$ and $0 < \theta < 2\pi$ we may choose $\varphi = \pi/2$ and $\theta = \pi$; the cross product has the value $\langle -4, 0, 0\rangle$ in the point $\langle -2, 0, 0\rangle$ on the sphere, so it points outwards and therefore points the same direction as $\mathbf{n}$. Hence the integral has to be taken with positive value, i.e.

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \; \mathrm{d}s = 16\pi$$

Compare the two methods and try to see the advantages and disadvantages of each one.