MATH 2E WINTER 2018 HOMEWORK 3

Due: Thursday, February 15, 2018 Please turn in in the discussion.

If you work in a group of two please turn in only one paper, but put both names and Student ID's on the paper. If you work alone, turn in a paper with your name and Student ID only.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS: It is crucial that you write your solutions clearly and that each solution clearly shows how you arrive at the results. This is the point of homeworks – to practice understanding of the material, clear writing, and the ability to express your understanding. Also, try to write your solutions as efficiently as possible, meaning that you should judge what to write and what not. Writing irrelevant text makes the solution confused and difficult to understand and grade. Leaving out important points makes the solution incomplete. So you need to judge what is relevant and what not. Often there is no recipe for this; the ability to recognize what to write and what not is a skill which needs to be developed, and this course (= Math 2E) is intended to help you with this.

The single integrals in this assignment are quite easy to evaluate, so please try to evaluate to the end. Do not use calculators. Instead, leave the results in the symbolic form, for instance if the result is of the form $\pi(1-3\sqrt{2})$, leave it in this form.

1. (5pt) The vector field $\mathbf{F}(x, y)$ in the plane is given by the potential function

$$f(x, y) = \frac{1}{2}x^2y^2.$$

(a) Calculate the function $\mathbf{F}(x, y)$.
(b) Sketch the values $\mathbf{F}(x, y)$ for the following points on the square with with corners $(\pm 1, \pm 1)$:
   – $x = \pm 1$ and $y \in \{0, \pm 1/2, \pm 1\}$
   – $y = \pm 1$ and $x \in \{0, \pm 1/2, \pm 1\}$
(c) Consider the circle C centered in the origin with radius $a > 0$. Calculate all points $(x, y) \in C$ such that
   (i) The magnitude of $\mathbf{F}(x, y)$ is the least possible.
   (ii) The magnitude of $\mathbf{F}(x, y)$ is the largest possible.
   Also, in each of this cases calculate what the magnitude is.

2. (5pt) Consider the portion of a spiral in the plane the equation of which is expressed in polar coordinates as follows:

$$r = e^{\theta}$$

where $0 \leq \theta \leq 2\pi$. Draw a sketch of this spiral. Assume the spiral consists of material where the density function is given by

$$\rho(x, y) = \frac{1}{r} \qquad \left(= \frac{1}{\sqrt{x^2 + y^2}} \right).$$

Calculate the mass, the moments M_x and M_y, and the center of gravity of the spiral.

Remark. You may make a use of the following facts:

$$\begin{aligned} \frac{\mathsf{d}}{\mathsf{d}t}\left(\frac{1}{2}e^t(\sin t + \cos t)\right) &= e^t \cos t \\ \frac{\mathsf{d}}{\mathsf{d}t}\left(\frac{1}{2}e^t(\sin t - \cos t)\right) &= e^t \sin t \end{aligned}$$

3. (5pt) Now consider a situation analogous as in Problem 2, this time in the 3-dimensional space, where the spiral also develops into the z-direction. In cylindrical coordinates the equations of the spiral are:

$$r = e^{\theta} \qquad\qquad z = r$$

where $0 \leq \theta \leq 2\pi$. Try to draw a sketch of this portion of the spiral. This time the density function is given by

$$\rho(x, y, z) = \frac{1}{rz}$$

Calculate the mass, the moments M_{yz}, M_{xz} and M_{xy}, and the center of gravity of this portion of the spiral.

4. (5pt) Consider the ellipsoid in the 3-dimensional space with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a, b, c > 0$, and assume additionally that $a \neq b$.

In the center of the coordinate system there is an object with mass M, and an object with mass m is moving on the surface of the ellipsoid at height $c/2$ above the xy-plane. Assume the moving object starts in the xz-plane and finishes in the yz-plane. (Try to visualize the situation, and possibly sketch a picture.)

Calculate the work done by the gravitational field generated by the object with mass M.

Remark. The substitution $u = \sin^2 t$ may be useful at certain stage in the evaluation of the relevant integral.