MATH 2E WINTER 2018 SAMPLE WRITING

Purpose: Please have a look at sample writing. On these samples I would like to make clear how to write a mathematics text, and what kind of explanations I prefer.

In particular:

(a) Please always write down which integration formula you are going to use, in its general form. Then show how you are substituting the concrete data into it.

(b) Do not do too many calculations on top of your head. This generates mistakes. Also, I want to see what steps you are making.

These examples do not cover all topics that will be on the exam; that is, there will be questions on other topics on the exam, too.

1. Consider the wire in the shape of the portion C of the circle in the xz-plane above the xy-plane in $\mathbb{R}^3$ centered at $(0,0,1)$ which intersects the xy-plane in points $(1,0,0)$ and $(-1,0,0)$. The density function is given by $\sigma(x,y,z)=z$.

(a) Calculate the mass of the wire.

(b) Calculate the center of gravity of the wire. Calculate only those integrals which need to be calculated. If the moments can be determined without calculation based on geometric properties of the wire and properties of the density function, do not perform the calculation, but explain which of these properties and how determine the moments.

Sample solution. (a) The radius of the circle is the distance of the center from the point on the circle is the distance of the points $(0,0,1)$ from $(1,0,0)$, and this is $\sqrt{(0-1)^2+(0-0)^2+(1-0)^2}=\sqrt{2}$. Since the circle is in the xz-plane its y-coordinate is 0. Since its center is $(0,0,1)$, it has parametric expression

$$x(t)=\sqrt{2}\cdot\cos t \quad y(t)=0 \quad z(t)=\cdot(1+\sin t).$$

We are considering only the portion above the xy-plane. The line connecting the center $(0,0,1)$ with $(1,0,0)$ has angle $-\pi/4$ against axis x, and the line connecting the center $(0,0,1)$ with $(-1,0,0)$ has angle $5\pi/4$. Therefore we have the following bounds for the parameter t for C:

$$-\pi/4\le t\le 5\pi/4.$$

The formula for mass is the line integral of the density function along C:

$$M=\int_C \sigma(x,y,z)\mathrm{d}s=\int_{-\pi/4}^{5\pi/4} z(t)\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,\mathrm{d}t$$

Now

$$x'(t)=-\sqrt{2}\cdot\sin t \quad y'(t)=0 \quad z'(t)=\sqrt{2}\cdot\cos t$$

so

$$x'(t)^2+y'(t)^2+z'(t)^2=2\sin^2 t+2\cos^2 t=2(\sin^2 t+\cos^2 t)=2\cdot 1=2$$

and $\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}=\sqrt{2}$. By substituting to the above formula, we have

$$\begin{aligned} M &= \int_{-\pi/4}^{5\pi/4}(1+\sqrt{2}\sin t)\sqrt{2}\ \mathrm{d}t = \sqrt{2}[t-\sqrt{2}\cdot\cos t]_{-\pi/4}^{5\pi/4} \\ &= \sqrt{2}\left(5\pi/4-(-\pi/4)-\sqrt{2}\cdot\cos(5\pi/4)+\sqrt{2}\cdot\cos(-\pi/4)\right) \\ &= \sqrt{2}\left(6\pi/4-\sqrt{2}\cdot(-\sqrt{2}/2)+\sqrt{2}\cdot(\sqrt{2}/2)\right) \\ &= \sqrt{2}(3\pi/2-(-1)+1)=\sqrt{2}(3\pi/2+2) \end{aligned}$$

(b) The formula for M_{xy} is:

$$M=\int_C z\sigma(x,y,z)\mathrm{d}s=\int_{-\pi/4}^{5\pi/4} z(t)\cdot z(t)\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\ \mathrm{d}t$$

We insert our concrete data and obtain:

$$\begin{aligned} M_{xy} &= \int_{-\pi/4}^{5\pi/4}(1+\sqrt{2}\sin t)^2\sqrt{2}\ \mathrm{d}t=\sqrt{2}\int_{-\pi/4}^{5\pi/4}(1+2\sqrt{2}\cdot\sin t+2\sin^2 t)\ \mathrm{d}t \\ &= \sqrt{2}\int_{-\pi/4}^{5\pi/4}(1+2\sqrt{2}\cdot\sin t+1-\cos 2t)\ \mathrm{d}t \\ &= \sqrt{2}[t-2\sqrt{2}\cdot\cos t+t-1/2\cdot\sin 2t]_{-\pi/4}^{5\pi/4} \end{aligned}$$

Here we use the equatlity $2\sin^2 t=1-\cos 2t$. We calculate, and also use calculations from above:

$$\begin{aligned} M_{xy} &= \sqrt{2}\left(3\pi/2-2\sqrt{2}\cdot(-\sqrt{2}/2)+2\sqrt{2}\cdot(\sqrt{2}/2)+3\pi/2-(1/2)\sin(5\pi/2)+(1/2)\sin(-\pi/2)\right) \\ &= \sqrt{2}(3\pi+2+2-1/2-1/2)=\sqrt{2}(3\pi+3)=3\sqrt{2}(\pi+1) \end{aligned}$$

Then

$$\bar{z}=\frac{M_{xy}}{M}=\frac{3\sqrt{2}(\pi+1)}{\sqrt{2}(3\pi/2+2)}=\frac{3(\pi+1)}{3\pi/2+1}$$

The other two moments:

- $M_{xz}=0$ because C lies entirely in the xz-plane. Hence $\bar{y}=0$.
- $M_{yz}=0$ because C lies in the xz-plane, and in this plane both the density function and C are symmetric about axis z. Hence $\bar{x}=0$.

Then the center of gravity is $(0,0,\frac{3(\pi+1)}{3\pi/2+1})$.

2. Use Green's theorem to calculate the work done by the force field

$$\mathbf{F}(x,y)=\ln(1+y^2)\mathbf{i}+\frac{2x^2y}{1+y^2}\mathbf{j}$$

which pushes an object along the path C making one loop clockwise where C is the boundary of the square with corners $(0,0),(0,1),(1,1)$ and $(1,0)$.

Sample solution. We need to evaluate the integral

$$\int_C x\ln(1+y^2)\ \mathrm{d}x+\frac{2x^2y}{1+y^2}\ \mathrm{d}y$$

where C is oriented clockwise, that is the orientation is negative. We have

$$P(x,y) = x\ln(1+y^2) \qquad Q(x,y) = \frac{2x^2y}{1+y^2}.$$

Then

$$\frac{\partial P}{\partial y}(x,y) = \frac{2xy}{1+y^2} \qquad \frac{\partial Q}{\partial x}(x,y) = \frac{4xy}{1+y^2}.$$

We need to check the assumptions of Green's theorem. The partial derivatives are clearly continuous in $\mathbb{R}^2$ and C is clearly piecewise smooth and closed. C is also simple, as we do only one loop. Green's theorem for positively oriented curve K reads

$$\int_K P\ \mathrm{d}x + Q\ \mathrm{d}y = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ \mathrm{d}A.$$

where D is the region inside K. Since our C is negativelyh oriented, we need to change the sign on the right side. Therefore

$$\int_C P\ \mathrm{d}x + Q\ \mathrm{d}y = \int\int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\ \mathrm{d}A.$$

where D is the region inside C. Since D is the square with corners $(0,0), (0,1), (1,1)$ and $(1,0)$, the bounds for x and y are:

$$0 \le x \le 1 \quad 0 \le y \le 1.$$

Then

$$\begin{aligned}
\int\int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\ \mathrm{d}A &= \int_0^1\int_0^1 \left(\frac{2xy}{1+y^2} - \frac{4xy}{1+y^2}\right)\ \mathrm{d}y\mathrm{d}x = \int_0^1\int_0^1 \frac{-2xy}{1+y^2}\ \mathrm{d}y\mathrm{d}x \\
&= -\left(\int_0^1 x\mathrm{d}x\right)\cdot\left(\int_0^1 \frac{2y}{1+y^2}\ \mathrm{d}y\right) \\
&= -(1/2)[x^2]_0^1 \cdot [\ln(1+y^2)]_0^1 \\
&= -(1/2)\cdot 1\cdot(\ln 2 - \ln 1) = -(1/2)\ln 2.
\end{aligned}$$