1. (4pt) Let $\mathbb{R}^* = \mathbb{R} - \{0\}$, i.e. $\mathbb{R}^*$ is the set of all real numbers without 0. Recall that $(\mathbb{R}^*, \cdot)$ is a group where $\cdot$ is the usual multiplication. Recall also that $\mathbb{Q}$ is the set of all rational numbers. Let

$$H = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} - \{0\}.$$ 

Show that $(H, \cdot)$ is a subgroup of $(\mathbb{R}^*, \cdot)$.

2. (6pt) Let $A$ be a nonempty set. Recall that $(S_A, \circ)$ is the group of all permutations (bijections $b : A \to A$) with the operation “map composition”. Let $K \subseteq A$ and

$$H = \{b \in S_A \mid b[K] = K\}.$$ 

Show that $(H, \circ)$ is a subgroup of $(S_A, \circ)$.

3. (3+3pt) Let

$$G = \text{the set of all maps } f : \mathbb{Z} \to \mathbb{Z}_3.$$ 

Define an operation $*$ on $G$ as follows:

$$(f * g)(x) = f(x) +_3 g(x).$$

Show that $(G, *)$ is a group.

Let $\Phi : G \to \mathbb{Z}_3$ be a map defined by

$$\Phi(f) = f(0) +_3 f(1) +_3 f(2).$$

Show that $\Phi : (G, *) \to (\mathbb{Z}_3, +_3)$ is a homomorphism.

4. (2+2pt) Consider the group $(\mathbb{Z}_{24}, +_{24})$.

(a) Find all elements of $\mathbb{Z}_{24}$ that are generators of the group $(\mathbb{Z}_{24}, +_{24})$.

(b) Find all elements $a \in \mathbb{Z}_{24}$ such that $\langle a \rangle$ has 6 elements.

5. (4+3pt) Consider the following permutation $\sigma \in S_{10}$:

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
5 & 3 & 7 & 10 & 9 & 1 & 2 & 4 & 6 & 8
\end{pmatrix}
$$
(a) Decompose $\sigma$ into disjoint cycles and compute the order of $\sigma$.

(b) Express $\sigma$ as a product of transpositions. Determine the signum of $\sigma$.

6. (5+7pt) Determine the values and kernels of the following homomorphisms.

(a) $(1+4pt)$ $h : \mathbb{Z} \to \mathbb{Z}_9$ is a homomorphism such that $h(1) = 5$. Find $h(100)$ and Ker$(h)$.

(b) $(2+5pt)$ $h' : \mathbb{Z}_{12} \to S_5$ is a homomorphism such that $h'(5) = (1, 4)(2, 3, 5)$. Determine $h'(9)$ and Ker$(h)$.

7. (4pt) Let $\sigma = (1, 5, 4, 2)(3, 4, 6)$. Find the index of $\langle \sigma \rangle$ in $S_6$.

8. (3+4pt) Present a proof of two of the following results: Proposition 2.10, 2.12, Theorem 3.2, Proposition 3.3, Proposition 4.2.