MATH 13 WINTER 2017 PRACTICE PROBLEMS

FOR FINAL EXAM

1. Problems on compositions of relations, images and inverse images. Assume R is a binary relation from A to B and S is a binary relation from B to C.

(a) Assume $X \subseteq A$. Prove that $S \circ R[X] = S[R[X]]$.
(b) Assume $X, Y \subseteq A$. Prove that $R[X \cup Y] = R[X] \cup R[Y]$.
(c) Assume $X, Y \subseteq A$. Is $R[X \cap Y] \subseteq R[X] \cap R[Y]$? Is $R[X] \cap R[Y] \subseteq R[X \cap Y]$? In either case give a proof supporting your conclusion.
(d) Prove that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

2. Assume $f : A \to B$ be a function.

(a) Assume $Y \subseteq B$. Prove that $f[f^{-1}[Y]] = Y$.
(b) Assume $X \subseteq A$. Prove that $X \subseteq f^{-1}[f[X]]$. Give an example of a set $X \subseteq A$ such that $X \neq f^{-1}[f[X]]$.
(c) Assume $X, Y \subseteq B$. Prove that

$$f^{-1}[X \setminus Y] = f^{-1}[X] \setminus f^{-1}[Y]$$

Given example of a function $f : A \to B$ and sets $X, Y \subseteq A$ such that $f[X \setminus Y] \neq f[X] \setminus f[Y]$.
(d) Assume f is injective and $X, Y \subseteq A$. Prove that

$$f[X \setminus Y] = f[X] \setminus f[Y]$$

3. Assume $f : A \to B$ and $g : B \to C$ are functions.

(a) Assume $g \circ f$ is injective. Prove that f is injective. Does it follow that g is injective? If so, prove this. If not, give an example of a function $g : B \to C$ which is not injective, but the composition $g \circ f$ is injective.
(b) Assume $g \circ f$ is surjective. Prove that g is surjective. Does it follow that f is surjective? If so, prove this. If not, give an example of a function $f : A \to B$ which is not surjective, but the composition $g \circ f$ is surjective.
(c) Assume g is injective. Prove that for any two functions $h : A \to B$ and $h' : A \to B$ the following holds.

$$g \circ h = g \circ h' \implies h = h'$$

(d) Assume for any two functions $h : B \to C$ and $h' : B \to C$ the following holds.

$$h \circ f = h' \circ f \implies h = h'$$

Prove that f is surjective.

4. The next exercises are on induction and the well-ordering principle.

(a) Prove by induction on $n \geq k$ that the set

$C^n_k =$ the collection of subsets of $\{1, \ldots, n\}$ which have k elements

has

$$\frac{n!}{k!(n-1)!}$$

elements.

(b) Prove by induction on $k \leq n$ that the set

$$V_k^n = \text{the collection of all functions } f : \{1, \ldots, k\} \to \{1, \ldots, n\}$$

has n^k elements.

(c) Prove by induction on n that if A is a set with n elements then the set

$$P_n^A = \text{the collection of all bijections } f : \{1, \ldots, n\} \to A$$

has $n!$ elements.

(d) Prove using the well-ordering principle that every number $n \in \mathbb{N} \setminus \{1\}$ is divisible by a prime.

5. The next exercises are on equivalence relations and partitions.

(a) Consider the binary relation $\sim$ on $\mathbb{R}^2$ defined by

$$(x, y) \sim (x', y') \iff \max(|x|, |y|) = \max(|x'|, |y'|)$$

Here $\max(r, s)$ is the larger of the numbers r, s.

(i) Prove that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

(ii) Determine the equivalence classes $[(0,0)]_\sim$ and $[(0,1)]_\sim$. Describe in words what geometric objects are these equivalence classes. Draw a picture.

(iii) Describe in words what is the partition $\mathbb{R}^2/\sim$.

(b) Consider the binary relation $\sim$ on $\mathbb{R} \times [0, \infty)$ defined by

$$(x, y) \sim (x', y') \iff \mathsf{distance}((x, y), \text{axis } x) = \mathsf{distance}((x', y'), \text{axis } x)$$

Here recall that the distance of a point (x, y) from a line ℓ is defined as the distance of points $(x, y), (x^*, y^*)$ where (x^*, y^*) is the intersection of the line ℓ with the line perpendicular to ℓ which contains the point (x, y). (Draw the picture!)

(i) Prove that $\sim$ is an equivalence relation on $\mathbb{R} \times [0, \infty)$.

(ii) Determine the equivalence classes $[(0,0)]_\sim$ and $[(0,1)]_\sim$. Describe in words what geometric objects are these equivalence classes. Draw a picture.

(iii) Describe in words what is the partition $\mathbb{R} \times [0, \infty)/\sim$.

(c) Consider the set $V = \mathbb{R}^2 \setminus \{(0,0)\}$. Thus, V is the set of all points of the plane except the origin. We view such points as non-zero vectors. Consider the binary relation $\sim$ on V defined as follows.

$$(x, y) \sim (x', y') \iff (x', y') = (\alpha \cdot x, \alpha \cdot y) \text{ for some } \alpha \in \mathbb{R}^+$$

(i) Prove that $\sim$ is an equivalence relation on V.

(ii) Determine the equivalence classes $[(1,0)]_\sim$, $[(1,1)]_\sim$ and $[(0,1)]_\sim$. Describe in words what geometric objects are these equivalence classes. Draw a picture.

(iii) Describe in words what is the partition $V/\sim$.

(d) Consider the set $F = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$. Consider the binary relation $\sim$ on F defined by

$$f \sim g \iff \text{there exist only finitely many } r \in \mathbb{R} \text{ such that } f(r) \neq g(r).$$

(i) Prove that $\sim$ is an equivalence relation on F.

(ii) Determine the equivalence classes $[c_0]_\sim$, where c_0 is the constant function with value 0, and $[\mathsf{id}_\mathbb{R}]_\sim$ where $\mathsf{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}$ is the idenity function defined by $\mathsf{id}_\mathbb{R}(r) = r$.

6. Recall that two sets are equinumeros iff there is a bijection between them, and a set is countable iff it is equinumeros with $\mathbb{N}$.

(a) Let
$$A = \{a + b\sqrt{n} \in (0,1) \mid a, b \in \mathbb{Q} \ \wedge \ n \in \mathbb{N}\}$$
Prove that A is countable.

(b) Let
$$A = \{x \in \mathbb{R} \mid \sin(\alpha \cdot x) = 0 \ \wedge \ \alpha \in \mathbb{Q}\}$$
Prove that A is countable.

(c) Let $r, s \in \mathbb{R}^+$. Prove that the intervals $(0, r)$ and $(0, s)$ are equinumeros by constructing an explicit bijection.

(d) Consider the following geometric objects in plane.
- S is the “empty square” consisting of lines $(0,0), (1,0), (1,1), (0,1)$, and
- T is the “empty square” consisting of lines $(0,0), (2,0), (2,2), (0,2)$.

Prove that S, T are equinumeros by constructing an explicit bijection.