MATH 13 WINTER 2017 HOMEWORK 3

Due: Wednesday, February 22, 2017 Please turn in in the lecture.

If you work in a group of two please turn in only one paper, but put both names and Student ID's on the paper. If you work alone, turn in a paper with your name and Student ID only.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS: It is crucial that you write your arguments clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding. Also, try to write your arguments as efficiently as possible, meaning that you should judge what to write and what not. Writing irrelevant text makes the argument confused and difficult to understand. Leaving out important points makes the argument incomplete. So you need to judge what is relevant and what not. There is no recipe for this; the ability to recognize what to write and what not is a skill which needs to be developed, and this course (= Math 13) is intended to help you with this.

When preparing the homeworks, please follow the **Rules for homeworks** on the course webpage under **Course information and policies** and also the guidelines under **Grading**. In particular, keep in mind the **Aspects of grading** in the **Grading** section.

Let R, S be the binary relations defined as follows.

- R is a binary relation on $\mathbb{N}$ defined by
$$(a,b) \in R \iff a \mid b$$
- Given X a nonempty set, S is a binary relation on $\mathcal{P}(X)$ defined by
$$(A,B) \in S \iff A \subseteq B$$

(Recall that $a \mid b$ iff there is some $k \in \mathbb{N}$ such that $b = k \cdot a$.

1. (5pt) Prove the following.

(a) **(1pt)** $R \circ R = R$.
(b) **(1pt)** $R^{-1} \circ R = \mathbb{N} \times \mathbb{N}$.

Guideline. In either case you need to prove both inclusions, that is, "$\subseteq$" and "$\supseteq$". Follow the corresponding definitions.

Similarly, prove the following:

(c) **(1pt)** $S \circ S = S$.
(d) **(2pt)** $S^{-1} \circ S = \mathcal{P}(X) \times \mathcal{P}(X)$.

Hint. There is an analogy between (a),(c) and (b),(d). Try to understand this analogy; if you know how to do (a) and (b) then this analogy should help you to do (c) and (d).

2. (5pt) Determine the following images and inverse images. In either case give a proof that your solution is correct.

(a) **(1pt)** $R^{-1}[\{p\}]$ where p is a prime.
(b) **(1pt)** $R^{-1}[\{p \cdot q\}]$ where p, q are primes.

Prove the following.

(c) **(1pt)** If $x \in X$ then $S[\{\{x\}\}] = \{A \in \mathcal{P}(X) \mid x \in A\}$.
(d) **(2pt)** If Q is a binary relation from a set Y to a set Z and $A, B \subseteq Y$ then $Q[A \cup B] = Q[A] \cup Q[B]$.

3. (5pt) In each of the cases (a) – (c) determine whether the function f is injective and/or surjective. Always give a proof supporting your solution.

(a) **(1pt)** $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $f(a,b) = a + b$.
(b) **(1pt)** $f : \mathcal{P}(\{1,2,\ldots,n\}) \to \{0,1,2,\ldots,n\}$ is defined by
$$f(A) = \text{the number of elements of } A.$$
(c) **(1pt)** Given a set X, the function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by
$$f(A) = A^c.$$

Given a function $g : X \to Y$, the function $f : \mathcal{P}(Y) \to \mathcal{P}(X)$ is defined by
$$f(B) = g^{-1}[B]$$

Prove the following.

(d) **(1pt)** If g is not surjective then f is not injective.
(e) **(1pt)** If g is not injective then f is not surjective.