MATH 13 WINTER 2017 HOMEWORK 6

Due: Wednesday, March 15, 2017 Please turn in at the lecture.

If you work in a group of two please turn in only one paper, but put both names and Student ID's on the paper. If you work alone, turn in a paper with your name and Student ID only.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS: It is crucial that you write your arguments clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding. Also, try to write your arguments as efficiently as possible, meaning that you should judge what to write and what not. Writing irrelevant text makes the argument confused and difficult to understand. Leaving out important points makes the argument incomplete. So you need to judge what is relevant and what not. There is no recipe for this; the ability to recognize what to write and what not is a skill which needs to be developed, and this course (= Math 13) is intended to help you with this.

When preparing the homeworks, please follow the **Rules for homeworks** on the course webpage under **Course information and policies** and also the guidelines under **Grading**. In particular, keep in mind the **Aspects of grading** in the **Grading** section.

1. **(5pt)** Let A be a set. Define a binary relation $\sim$ on $\mathcal{P}(A)$ by

$$X \sim Y \iff \text{There exists a bijection } f : X \to Y$$

(a) **(2pt)** Prove that $\sim$ is an equivalence relation on $\mathcal{P}(A)$.
(b) **(1pt)** Calculate $[\varnothing]_\sim$.
(c) **(2pt)** Consider the case where $A = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. Describe all equivalence classes with respect to $\sim$. How many such equivalence classes are there?

2. **(5pt)** Let A, B be nonempty set and $f : A \to B$. Define a binary relation $\sim_f$ on A as follows:

$$a \sim_f b \iff f(a) = f(b)$$

(a) **(2pt)** Prove that $\sim_f$ is an equivalence relation on A.

For each of the equivalence relations I_A, congruence modulo n and "equal distance from the origin in the plane" discussed in the lecture can be represented as $\sim_f$ for a suitable f. The following exercises make this precise.

(b) **(1pt)** Find a set B and a function $f : A \to B$ such that $\sim_f$ is the equivalence relation I_A where recall that

$$a I_A b \iff a = b$$

(c) **(1pt)** Assume $A = \mathbb{Z}$ and $n \in \mathbb{N}$. Find a set B and a function $f : A \to B$ such that $\sim_f$ is the relation congruence modulo n, that is, $\sim_f$ should satisfy the condition

$$a \sim_f b \iff a \equiv b \mod n$$

(d) **(1pt)** Assume $A = \mathbb{R}^2$. Find a set B and a function $f : A \to B$ such that $\sim_f$ is the equivalence relation E on $\mathbb{R}^2$ defined by

$$(x, y)E(x', y') \iff \mathsf{distance}((x, y), (0, 0)) = \mathsf{distance}((x', y'), (0, 0)).$$

3. **(5pt)** Prove that the following sets are countable. For this, refer to appropriate theorems from the lecture.

(a) **(2pt)** $A = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$.
(b) **(2pt)** $B =$ the set of all finite sequences of numbers in $\mathbb{N}$.
Here use the fact that $B = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$; first prove by induction on $n \in \mathbb{N}$ that $\mathbb{N}^n$ is countable.
(b) **(1pt)** $C =$ the set of all finite subsets of $\mathbb{N}$.
The fastest approach here is by constructing a surjection $h : B \to C$.