MATH 13 WINTER 2017 MIDTERM SAMPLE WRITING

Here is some sample writing for midterm. This should be understood as a guide only. There are many other correct ways of writing mathematical text and argue. The purpose of this sheet is to give you, additionally to samples of mathematical writing and proofs given in the lecture, some examples how to write mathematical text and how to build mathematical arguments. The numbering below refers to Midterm Practice Problems sheet.

Problem 1(d). The set is described as follows:

$$\{n \in \mathbb{N} \mid (\exists p \in \mathbb{N})(\exists q \in \mathbb{N})(P(p) \wedge P(q) \wedge n = p + q)\}$$

where

$$P(x) : x \neq 1 \wedge (\forall u \in \mathbb{N})(\forall v \in \mathbb{N})(x = u \cdot v \implies (u = 1 \vee v = 1))$$

Thus, we use the abbreviation $P(x)$ in the description of the set and then write below in symbolic language what $P(x)$ means.

Problem 2(b). Here is the statement in symbolic language:

$$(\forall r \in \mathbb{R})(\exists x \in \mathbb{R})\,(8x + r = 0 \wedge (\forall y \in \mathbb{R})(8y + r = 0 \implies y = x))$$

Negation:

$$(\exists r \in \mathbb{R})(\forall x \in \mathbb{R})\,(8x + r \neq 0 \vee (\exists y \in \mathbb{R})(8y + r = 0 \wedge y \neq x))$$

Problem 3(e). Given three consecutive integers, let a be the smallest of them. These integers are then

$$a, a + 1, a + 2$$

We first observe that one of them is divisible by 3. Denote by i the remainder when a is divided by 3. If $i = 0$ then a is divisible by 3. If $i = 1$ then $a + 2$ is divisible by 3. If $i = 2$ then $a + 1$ is divisible by 3. We now discuss some cases.

Case 1. a is divisible by 3. So there is some $k \in \mathbb{Z}$ such that $a = 3k$. If a is even then k is even, as the product of two odd integers is odd (proved in the lecture), so $a = 3k$ could not be even if k were odd. So there is some $\ell \in \mathbb{Z}$ such that $k = 2\ell$. Then $a = 3k = 3 \cdot 2\ell = 6\ell$. It follows that $a(a + 1)(a + 2) = 6\ell(a + 1)(a + 2)$, so this product is divisible by 6. If a is odd then $a + 1$ is even so there is some $\ell \in \mathbb{Z}$ such that $a + 1 = 2\ell$. Then $a(a + 1)(a + 2) = 3k \cdot 2\ell \cdot (a + 2) = 6k\ell(a + 2)$ and is thus divisible by 6.

Case 2. The remainder i of a divided by 3 is 1. Then $a+2$ is divisible by 3 and the argument in Case 1 can be repeated for $a+2$ in place of a. The only difference is that if $a + 2$ is odd then $a + 1$ is even so we use $a + 1$ as an even factor in this case.

Case 3. The remainder i of a divided by 3 is 2. Then $a+1$ is divisible by 3 and the argument in Case 1 can be repeated for $a + 1$ in place of a.

Problem 4(b). Assume $A \subseteq B \cap C$. If $(x, y) \in A \times A$ then $x, y \in A$ by the definition of Cartesian product. Since $A \subseteq B$, we have $x \in B$. Since $A \subseteq C$, we have $y \in C$. It follows that $(x, y) \in B \times C$. Thus, we proved:

$$(\forall x, y)((x, y) \in A \times A \implies (x, y) \in B \times C).$$

By the definition of inclusion, this implies that $A \times A \subseteq B \times C$.

Problem 4(e). There are sets A, B such that the statement false.

Argument 1. Consider sets A, B such that $A \cap B$ is nonempty (notice this means that both A, B are nonempty) and $A \setminus B$ is nonempty either. Pick $x \in A \setminus B$ and $y \in A \cap B$. Then $\{x, y\} \subseteq A$, so $\{x, y\} \in \mathcal{P}(A)$. On the other hand, $x \notin B$, so $\{x, y\} \nsubseteq B$. It follows that$\{x, y\} \notin \mathcal{P}(B)$. Taking everything together and using the definition of $\setminus$ we conclude:

$$\{x, y\} \in \mathcal{P}(A) \setminus \mathcal{P}(B). \tag{1}$$

Now, since $y \in B$, we have $\{x, y\} \nsubseteq A \setminus B$. That is,

$$\{x, y\} \notin \mathcal{P}(A \setminus B). \tag{2}$$

By the definition if inclusion, (1) and (2) tell us that $\in \mathcal{P}(A) \setminus \mathcal{P}(B) \nsubseteq \mathcal{P}(A \setminus B)$.

Argument 2. We can argue by giving a concrete counterexample. Consider sets $A = \{1, 2, 4\}$ and $B = \{1, 3\}$. Then $\{1, 2\} \subseteq A$, so $\{1, 2\} \in \mathcal{P}(A)$. On the other hand, $2 \notin B$, so $\{1, 2\} \nsubseteq B$. It follows that$\{1, 2\} \notin \mathcal{P}(B)$. Taking everything together and using the definition of $\setminus$ we conclude:

$$\{1, 2\} \in \mathcal{P}(A) \setminus \mathcal{P}(B). \tag{3}$$

Now, since $1 \in B$, we have $1 \notin A \setminus B$, hence $\{1, 2\} \nsubseteq A \setminus B$. That is,

$$\{1, 2\} \notin \mathcal{P}(A \setminus B). \tag{4}$$

By the definition if inclusion, (3) and (4) tell us that $\in \mathcal{P}(A) \setminus \mathcal{P}(B) \nsubseteq \mathcal{P}(A \setminus B)$.