MATH 13 FALL 2019
FINAL EXAM PRACTICE PROBLEMS

Instructions: If you plan to come to a review session please attempt to solve as many problems as possible and record the spots where you encounter difficulties. At the review session please describe the difficulties in a concrete way so that I am able to give you specific answers.

If you just ask me to solve the problems on the board without your own prior attempts to solve them, the review session will not be helpful for you.

1. Consider binary relations R, S, T. Prove the following:

(a) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.
(b) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.
(c) $T \circ (R \cup S) = (T \circ R) \cup (T \circ S)$.
(d) $T \circ (R \cap S) \subseteq (T \circ R) \cap (T \circ S)$.
(e) Use the relations $T = R$ and $S = R^{-1}$ on $\mathbb{Z}$ where R is the relation "$<$" to demonstrate that "$\subseteq$" in (d) cannot be replaced with "$=$". Do the same thing with relations on $\mathbb{N}$ where R is the divisibility relation and S, T are defined from R as above.

2. Consider binary relations R, S, T and sets X, Y. Prove the following:

(a) $(S \circ R)[X] = S[R[X]]$.
(b) $R[X \cup Y] = R[X] \cup R[Y]$.
(c) $X \subseteq R^{-1}[R[X]]$. Use the divisibility relation as R to demonstrate that "$\subseteq$" cannot be replaced with "$=$" here.
(d) Assume R is a function. Prove that $R^{-1}[X \cap Y] = R^{-1}[X] \cap R^{-1}[Y]$.
(e) Consider a set A and the binary relation $R \subseteq A \times \mathcal{P}(A)$ defined by

$$\langle x, X \rangle \in R \iff x \in X$$

Given $X, Y \in \mathcal{P}(A)$, calculate $R^{-1}[\{X\}]$ and $R^{-1}[\{X, Y\}]$.

3. Prove that the following relations are equivalence relations on the respective sets. Identify the equivalence classes of these relations. Recall that we identify the Euclidean plane $\mathbb{E}$ with $\mathbb{R} \times \mathbb{R}$.

(a) E_1 is the binary relation on $\mathbb{E}$ defined by

$$\langle \langle x, y \rangle, \langle x', y' \rangle \rangle \in E_1 \iff \max\{|x|, |y|\} = \max\{|x'|, |y'|\}$$

Identify the equivalence classes in geometric terms: What geometric objects in the plane are they?

(b) E_1 is the binary relation on $\mathbb{E}$ defined by

$$\langle \langle x, y \rangle, \langle x', y' \rangle \rangle \in E_1 \iff \frac{y}{x} = \frac{y'}{x'}$$

Identify the equivalence classes in geometric terms: What geometric objects in the plane are they?

4.

(a) Consider functions $f : A \to B$, $g : A \to B$ and $h : B \to C$. Assume h is injective. Prove:

$$h \circ f = h \circ g \implies f = g$$

(b) Consider functions $f : A \to B$ and $g : B \to C$. Prove:

$$g \circ f \text{ is injective } \implies f \text{ is injective.}$$

(c) Give an example of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is injective but g is not injective.

(d) Consider functions $f : A \to B$ and $g : B \to C$. Prove:

$$g \circ f \text{ is surjective } \implies g \text{ is surjective.}$$

(e) Give an example of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is surjective but f is not surjective.

(f) Consider a function $f : A \to B$. Assume

$$g \circ f = h \circ f \implies g = h$$

holds for all functions $g : B \to C$, $h : B \to C$. Prove that f is surjective.

5. For each of the following functions determine whether they are injective. The following notation is used: Given a set A, we denote the set of all finite subsets of A by $\mathcal{P}_{\mathsf{fin}}(A)$. Also we denote the set of all positive prime numbers by P.

(a) $f_1 : \mathbb{N} \to \mathcal{P}_{\mathsf{fin}}(P)$ is defined by

$$f_1(n) = \text{the set of all } p \in P \text{ such that } p \mid n.$$

(b) $f_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ is defined by $f_2(\langle k, \ell \rangle) = \frac{k}{\ell}$.

(c) $f_3 : [0, \pi] \to [0, 1]$ is defined by $f_3(x) = \cos x$.

(d) $f_4 : (0, 1) \to (0, 1)$ is defined by $f_4(x) = \langle x^2, x^3 \rangle$.

(e) $f_5 : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is defined by $f_5(X) = X^c$.

(f) $f_6 : \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is defined by $f_6(X, Y) = X \setminus Y$.

(g) Given a set $X \subseteq \mathbb{N}$, define

$$X^e = \{n \in X \mid n \text{ is even}\}$$

and

$$X^o = \{n \in X \mid n \text{ is odd.}\}$$

Also denote by E the set of all even numbers in $\mathbb{N}$ and by O the set of all odd numbers in $\mathbb{N}$. Function $f_7 : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(E) \times \mathcal{P}(O)$ is defined by

$$f_7(X) = \langle X^e, X^o \rangle.$$