1. (5pt) Prove that if $x, y$ are arbitrary integers then $x^2 y + xy^2$ is an even number.

   **Remark.** One option is to do a proof by cases.

2. (5pt) In the class we gave a proof that if $a$ is an integer then there are at most one integer $q$ and at most one integer $r$ such that
   
   (i) $0 \leq r < 5$, and  
   (ii) $a = 5 \cdot q + r$
We then formulated a general result with an integer \( b > 1 \) in place of 5.

Formulate this general result with \( b \) in place of 5 again, and then give a proof of the general result, which mimics the proof with number 5 in the lecture. Write the proof carefully. There are one or two spots in the proof which require a little bit more care than the special case with \( b = 5 \).

3. (5pt) In the lecture we proved, using induction on \( n \), that if \( n \geq 5 \) is an integer and \( A \) is a set with \( n \) elements, then

- there exist precisely \( \frac{n!}{5!(n-5)!} \) subsets of \( A \) which have exactly 5 elements.

Prove by induction on \( n \) the following generalization of this result, where number 5 is replaced with some arbitrary \( k \geq 0 \):

- there exist precisely \( \frac{n!}{k!(n-k)!} \) subsets of \( A \) which have exactly \( k \) elements.

Again, try to mimic the proof with number 5 from the lecture.

4. (5pt) In the lecture we proved, using strong induction, the following statement:

\((*)\) Every integer larger than 1 is divisible by a prime number.

We say that an integer \( n \) is a **product of primes** iff there exist some integer \( k > 0 \) and prime numbers \( p_1, \ldots, p_k \) such that \( n = p_1 \cdot p_2 \cdot \ldots \cdot p_k \). Notice in particular that it is allowed that \( k = 1 \), so every prime number is a product of primes in the above sense. Notice also that the primes \( p_1, \ldots, p_k \) are **not** required to be distinct.

Use strong induction to prove the following statement, which is a strengthening of \((*)\) above:

\((***)\) Every integer larger than 1 is a product of primes.