MATH 13 FALL 2019 HOMEWORK 3

Due: Thursday, October 31, 2019 Please turn in at the discussion.

If you work in a group of two please turn in only one paper, but put both names and Student ID's on the paper. If you work alone, turn in a paper with your name and Student ID only.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS:

- It is crucial that you write your arguments and explanations clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding.
- Try to write your arguments as efficiently as possible, meaning that you should judge what to write and what not. Writing irrelevant text makes the argument confused and difficult to understand. Leaving out important points makes the argument incomplete. So you need to judge what is relevant and what not. There is no recipe for this; the ability to recognize what to write and what not is a skill which needs to be developed, and this course (= Math 13) is intended to help you with this.
- When preparing the homeworks, please follow the **Rules for homeworks** on the course website under **Course information and policies**.

ALSO IMPORTANT: In arguments and proofs you are writing, **never** use symbols $\therefore$ and $\because$ but instead, write everything as a text.

Also avoid using the symbol $\nexists$.

1. (5pt) Prove that if x, y are arbitrary integers then

$$x^2y + xy^2$$

is an even number.

Remark. One option is to do a proof by cases.

2. (5pt) In the class we gave a proof that if a is an integer then there are **at most one** integer q and **at most one** integer r such that

(i) $0 \leq r < 5$, and
(ii) $a = 5 \cdot q + r$

We then formulated a general result with an integer $b > 1$ in place of 5.

Formulate this general result with b in place of 5 again, and then give a proof of the general result, which mimics the proof with number 5 in the lecture. Write the proof carefully. There are one or two spots in the proof which require a little bit more care than the special case with $b = 5$.

3. (5pt) In the lecture we proved, using induction on n, that if $n \geq 5$ is an integer and A is a set with n elements, then

- there exist precisely $\frac{n!}{5! \cdot (n-5)!}$ subsets of A which have exactly 5 elements.

Prove by induction on n the following generalization of this result, where number 5 is replaced with some arbitrary $k \geq 0$:

- There exist precisely $\frac{n!}{k! \cdot (n-k)!}$ subsets of A which have exactly k elements

Again, try to mimic the proof with number 5 from the lecture.

4. (5pt) In the lecture we proved, using strong induction, the following statement:

($*$) Every integer larger than 1 is divisible by a prime number.

We say that an integer n is a **product of primes** iff there exist some integer $k > 0$ and prime numbers $p_1, \dots, p_k$ such that $n = p_1 \cdot p_2 \cdot \dots \cdot p_k$. Notice in particular that it is allowed that $k = 1$, so every prime number is a product of primes in the above sense. Notice also that the primes $p_1, \dots, p_k$ are **not** required to be distinct.

Use strong induction to prove the following statement, which is a strengthening of ($*$) above:

($**$) Every integer larger than 1 is a product of primes.