MATH 13 FALL 2019 HOMEWORK 5

Due: Thursday, November 21, 2019 Please turn in at the discussion.

If you work in a group of two please turn in only one paper, but put both names and Student ID's on the paper. If you work alone, turn in a paper with your name and Student ID only.

Student name/id (include all students in the group):

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IMPORTANT INSTRUCTIONS:

- It is crucial that you write your arguments and explanations clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding.
- Try to write your arguments as efficiently as possible, meaning that you should judge what to write and what not. Writing irrelevant text makes the argument confused and difficult to understand. Leaving out important points makes the argument incomplete. So you need to judge what is relevant and what not. There is no recipe for this; the ability to recognize what to write and what not is a skill which needs to be developed, and this course (= Math 13) is intended to help you with this.
- When preparing the homeworks, please follow the **Rules for homeworks** on the course website under **Course information and policies**.

ALSO IMPORTANT: In arguments and proofs you are writing, **never** use symbols $\therefore$ and $\because$ but instead, write everything as a text.

Also avoid using the symbol $\nexists$.

1. (5pt) Consider sets $A, B, C, D \subseteq U$; here U is the background set. It is proved in the book that

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D). \tag{1}$$

(a) Consider $U = \mathbb{Z}$. Give examples of sets $A, B, C, D \subseteq U$ such that the inclusion in (1) cannot be reversed.
(b) Now again consider an arbitrary background set U. This time assume $A = C$. Can the inclusion in (1) be reversed?

In either case, support your conclusion by a proof.

2. (5pt) For each of the following binary relations decide whether it is reflexive, symmetric, antisymmetric, transitive. Then decide whether the relation in question is an equivalence relation or ordering relation, and if the latter is the case, decide whether the ordering relation is total or not. Support all your conclusions by proofs.

(a) R_1 is a binary relation on a set A defined as follows:
$$R_1 = \{\langle x, y\rangle \in A \times A \mid x \neq y\}$$
Also assume that A has more than one element.
(b) R_2 is a binary relation on $\mathbb{Z}$ defined by
$$R_2 = \{\langle k, \ell\rangle \in \mathbb{Z} \times \mathbb{Z} \mid k + \ell \text{ is even}\}$$
(c) R_3 is a binary relation defined on $(\mathbb{Q} \times \mathbb{Q}) \times (\mathbb{Q} \times \mathbb{Q})$ defined by
$$R_3 = \{\langle\langle q, r\rangle, \langle q', r'\rangle\rangle \mid q < q' \vee (q = q' \wedge r \leq r')\}$$
(d) R_4 is a binary relation on $\mathcal{P}(A)$ defined by
$$R_4 = \{\langle X, Y\rangle \in \mathcal{P}(A) \times \mathcal{P}(A) \mid X \cap Y = \varnothing\}$$

3. (5pt) Let $E = \mathbb{R} \times \mathbb{R}$ be the Euclidean plane. Prove that binary relations S_1, S_2 are equivalence relations. Also prove that T_1, T_2 are "very close to be" partial ordering relations on the respective sets, in the following sense: Istead of being antisymmetric, T_1 and T_2 satisfy the following properties, which resemble antisymmetricity:
$$\big(\langle X, Y\rangle \in T_1 \ \wedge \ \langle Y, X\rangle \in T_1\big) \quad \Longrightarrow \quad \langle X, Y\rangle \in S_1$$
and
$$\big(\langle\langle x, y\rangle, \langle x', y'\rangle\rangle \in T_2 \ \wedge \ \langle\langle x', y'\rangle, \langle x, y\rangle\rangle \in T_2\big) \quad \Longrightarrow \quad \langle\langle\langle x, y\rangle, \langle x', y'\rangle\rangle\rangle \in S_2$$
Think about it and try to understand why these properties resemble antisymmetricity.

(a) S_1 is the binary relation on $\mathcal{P}(\mathbb{N})$ defined by
$$\langle X, Y\rangle \in S_1 \iff X \bigtriangleup Y \text{ is finite}$$
(b) S_2 is the binary relation on E defined by
$$\langle\langle x, y\rangle, \langle x', y'\rangle\rangle \iff d(\langle x, y\rangle, \langle 0, 0\rangle) = d(\langle x', y'\rangle, \langle 0, 0\rangle)$$
Here $d(\langle a, b\rangle, \langle a', b'\rangle\rangle)$ is the distance between $\langle a, b\rangle$ and $\langle a', b'\rangle$, and recall that
$$d(\langle a, b\rangle, \langle a', b'\rangle\rangle) = \sqrt{(a - a')^2 + (b - b')^2}.$$
(c) T_1 is the binary relation on $\mathcal{P}(\mathbb{N})$ defined by
$$\langle X, Y\rangle \in T_1 \iff X \smallsetminus Y \text{ is finite.}$$

(d) T_2 is the binary relation on E defined by

$$\langle\langle x,y\rangle,\langle x',y'\rangle\rangle \iff d(\langle x,y\rangle,\langle 0,0\rangle) \leq d(\langle x',y'\rangle,\langle 0,0\rangle)$$

4. (5pt) Consider sets A, B.

(a) Prove that $\mathcal{P}(A)\cup\mathcal{P}(B) \subseteq \mathcal{P}(A\cup B)$. Also give examples of sets A, B which show that this inclusion cannot be reversed (give a proof!).

(b) Prove that $\mathcal{P}(A)\cap\mathcal{P}(B) = \mathcal{P}(A\cap B)$.