MATH 13 FALL 2019 HOMEWORK 6

Due: Wednesday, November 27, 2019 Please turn in at the lecture.

If you work in a group of two please turn in only one paper, but put both names and Student ID's on the paper. If you work alone, turn in a paper with your name and Student ID only.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS:

- It is crucial that you write your arguments and explanations clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding.
- Try to write your arguments as efficiently as possible, meaning that you should judge what to write and what not. Writing irrelevant text makes the argument confused and difficult to understand. Leaving out important points makes the argument incomplete. So you need to judge what is relevant and what not. There is no recipe for this; the ability to recognize what to write and what not is a skill which needs to be developed, and this course (= Math 13) is intended to help you with this.
- When preparing the homeworks, please follow the **Rules for homeworks** on the course website under **Course information and policies**.

ALSO IMPORTANT: In arguments and proofs you are writing, **never** use symbols $\therefore$ and $\because$ but instead, write everything as a text.

Also avoid using the symbol $\nexists$.

1. (5pt) Consider a binary relation R on A which is reflexive and transitive. Define binary relations E_R and P_R on A as follows.

$$\langle x, y\rangle \in E_R \iff (\langle x, y\rangle \in R \wedge \langle y, x\rangle \in R)$$

and

$$\langle x, y\rangle \in P_R \iff (\langle x, y\rangle \in R \wedge \langle y, x\rangle \notin R)$$

(a) Prove that E_R is an equivalence relation on A.
(b) Prove that P_R is a strict partial ordering on A.

Before starting the proofs, make sure you understand what is to be proved. It may be a good idea to write the goals down.

2. (5pt) Consider the following binary relations. In each case prove the relation in question is an equivalence relation and describe, in geometric terms, what the equivalence classes are.

(a) S_1 is a binary relation on $\mathbb{R}^2 \times \mathbb{R}^2$ defined by

$$\langle\langle x, y\rangle, \langle x', y'\rangle\rangle \in S_1 \iff |\,x\,| + |\,y\,| = |\,x'\,| + |\,y'\,|$$

Recall that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

(b) S_2 is a binary relation on $\mathbb{R}$ defined by

$$\langle x, y\rangle \in S_2 \iff x - y \in \mathbb{Z}$$

3. (5pt) Given a binary relation R, define the domain of R and range of R as follows.

- $\mathrm{dom}(R) = \{x \mid (\exists y)(\langle x, y\rangle \in R\}$.
- $\mathrm{rng}(R) = \{y \mid (\exists x)(\langle x, y\rangle \in R\}$.

In other words, $\mathrm{dom}(R)$ is the set of all first components of ordered pairs in R; $\mathrm{rng}(R)$ is the set of all second components of ordered pairs on R. More generally, given sets X, Y, we define

- $R[X] = \{y \mid (\exists x \in X)(\langle x, y\rangle \in R)$.
- $R^{-1}[Y] = \{x \mid (\exists y \in Y)(\langle x, y\rangle \in R)$.

In other words, $R[X]$ is the set of all second components of ordered pairs in R which have first component in X and $R^{-1}[Y]$ is the set of all first components of ordered pairs in R which have the second component in Y.

Example: If R is the relation $<$ on $\mathbb{Q}$ (that is, $R = \{\langle x, y\rangle \in \mathbb{Q}\times\mathbb{Q} \mid x < y\}$) then $\mathrm{dom}(R) = \mathbb{Q}$ because every $q \in \mathbb{Q}$ occurs as the first component of some ordered pair in R, for instance $\langle q, q + 1\rangle \in R$ because $q < q + 1$. Similarly $\mathrm{rng}(R) = \mathbb{Q}$ because every $q \in \mathbb{Q}$ occurs as the second component of an ordered pair in R, for instance $\langle q-1, q\rangle \in R$ because $q-1 < q$. If $X = \{0\} = Y$ then $R[X] = \mathbb{Q}^+$ because

$$\{y \in \mathbb{Q} \mid (\exists x \in X)(\langle x, y\rangle \in R\} = \{y \in \mathbb{Q} \mid \langle 0, y\rangle \in R\} = \{y \in \mathbb{Q} \mid 0 < y\} = \mathbb{Q}^+$$

and $R^{-1}[Y] = \mathbb{Q}^-$ because

$$\{x \in \mathbb{Q} \mid (\exists y \in Y)(\langle x, y\rangle \in R\} = \{x \in \mathbb{Q} \mid \langle x, 0\rangle \in R\} = \{y \in \mathbb{Q} \mid x < 0\} = \mathbb{Q}^-$$

(a) Prove that if R is any binary relation then $R[\mathrm{dom}(R)] = \mathrm{rng}(R)$ and $R^{-1}[\mathrm{rng}(R)] = \mathrm{dom}(R)$.

(b) Let S be the relation $<$ on $\mathbb{R}$, that is,
$$S = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} \mid x < y\}.$$
Also assume $X = Y =$ the open interval $(-\infty, 0)$. Prove that $S[X] = \mathbb{R}$. Also determine which set is $S^{-1}[X]$, and justify your conclusion by a proof.

(c) Let T be the divisibility relation on $\mathbb{N}$, so $T = \{\langle a, b\rangle \in \mathbb{N} \times \mathbb{N} : a \,|\, b\}$. Assume X is the set of all prime numbers. Prove that $T[X] = \mathbb{N} \setminus \{1\}$. Also determine which set is $T^{-1}[\{24, 25\}]$, and justify your conclusion by a proof.

4. (5pt) If R, S are binary relations then we can form the composition $S \circ R$ as follows:
$$\langle x, z\rangle \in S \circ R \iff (\exists y)(\langle x, y\rangle \in R \ \wedge \ \langle y, z\rangle \in S)$$
Caution: S is indeed written on the left, and R on the right! So the expression "$S \circ R$" is correct and **not** a typo. So if R is a binary relation from A to B and S is a binary relation from B to C then $S \circ R$ is a binary relation from A to C. Draw a picture!

(a) Assume R is a binary relation from A to B. Recall that E_A is the equality relation on A, that is, $E_A = \{\langle a, a\rangle \mid a \in A\}$. Similarly, E_B is the equality relation on B. Prove that $E_B \circ R = R = R \circ E_A$.

(b) Consider the following binary relations S, S' on $\mathbb{R}$:
$$S = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} \mid x < y\}.$$
and
$$S' = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} \mid y < x\}.$$
Prove that $S' \circ S = \mathbb{R} \times \mathbb{R}$.

(c) Assume T is a binary relation from A to B and T' is a binary relation from B to C such that $\text{rng}(T) \cap \text{dom}(T') = \varnothing$. Prove that $T' \circ T = \varnothing$.